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X CW complex $\longrightarrow H^{\bullet}_{sing}(X, \mathbb{Z})$

X smooth manifold, then $H^{\bullet}_{sing}(X, \mathbb{Z})$ has extra discrete structure (Poincare duality)

X smooth alg variety, then $H^{\bullet}_{sing}(X, \mathbb{Z})$ carries Hodge structure, some continuously varying structure

Why do we care it for?

- Topological classification (groups in $\pi_1(X)$)
- effective geometry (understanding subvarieties, Chow theory)
- Moduli (Hodge structures in families)
- Arithmetic (Galois reps, p-adic Hodge theory)

outline of course

1. Legendre families, period map
2. Hodge structure of curves and abelian surfaces
3. Hodge decomposition, Kähler manifolds
4. Hodge structures for hypersurfaces, K3, Kuga-Satake, period domains
5. Deformation theory intro and variations of the Hodge structure

bundles on $\mathbb{P}^1$

Let us start with a brief reminder
$\mathbb{P}^1 = (U_0 = (\mathbb{C}, z)) \sqcup_{\mathbb{C}^*} (U_1 = (\mathbb{C}, \xi)), z^-1 = \xi \in \mathbb{C}^*$

line bundles on $\mathbb{P}^1$

Definition

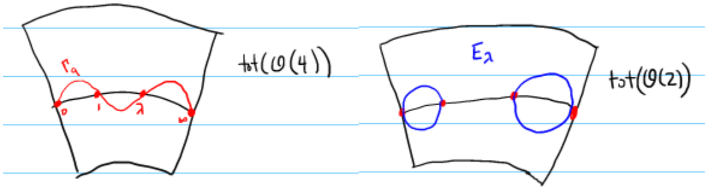
$\mathcal{O}(k)$ – line bundle w basis e_0, e_1 over U_0, U_1 s.t.
$e_0 = e_1 g_{01}, g_{01} : U_0 \cup U_1 \to \mathbb{C}^*, z \mapsto z^k, k \in \mathbb{Z}$
A **section** of $\mathcal{O}_K$ is therefore $(f_0, f_1) \in \mathcal{O}(U_0) \times \mathcal{O}(U_1) : f_0 - g_{01} f_1$.
Total space $X = Tot(\mathcal{O}(k))$ is a complex surface
$\left(U_0 \times \mathbb{C}) \sqcup_{(U_0 \cup U_1) \times \mathbb{C}}\right), (z, u) \sim (\xi, v) \Leftrightarrow z = \xi^{-1}, u = z^k v$

Tautological bundle on $\mathbb{P}^1$ is $\mathcal{O}(-1)$. And $T * \mathbb{P}^1 \simeq \mathcal{O}(-2)$.

Elliptic curves, Legendre family

Consider section $f_0 = z(z-1)(z-\lambda)$, where $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then $f_0 = z^3(1-\xi)(1-\lambda\xi)$ is the section of $\mathcal{O}(3)$, $z^4\xi(1-\xi)(1-\lambda\xi)$ is the section of $\mathcal{O}(4)$.

$q = (f_0, f_1) \in H^0(\mathbb{P}^1, \mathcal{O}(4))$ - a section with four distinct zeros $0, 1, \lambda, \infty$.



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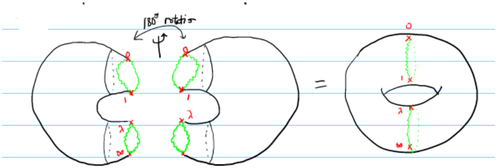
Elliptic curve, topology

Definition

The double branched cover of $\mathbb{P}^1$ along $\{0, 1, \lambda, \infty\}$, elliptic curve $E_\lambda = \{y \in Tot(\mathcal{O}(2)) : y^2 = q(\pi(y))\}$, or locally $= \{(z, y) : y^2 = z(z-1)(z-\lambda)\}$

For different λ it is a family of smooth curves.
Classically, $f(x) = \sqrt{x(x-1)(x-\lambda)}$ becomes single-valued on Riemann surface obtained after making cuts and glying 2 copies of $\mathbb{P}^1$. The result of this topological copy-paste is Riemann surface of degree 1.



(δ, γ) give symplectic basis $H^1_{sing}(E, \mathbb{Z})$ s.t. $\delta \cdot \gamma = 1$.

Canonical bundle of E_λ

$\Omega = \frac{dz\wedge dy}{y^2 - f_0(z)}$, it is 2-form

Look how it is change while we come to U_1:

$= \frac{-\xi^{-2}d\xi\wedge z^2 dv}{z^4(v^2-f_1(\xi))} = \frac{-d\xi\wedge dv}{v^2-f_1(\xi)}$

Then Ω is the section of K_{X_2} with pole along E_λ, ie $\Omega \in H^0(X_2, \Omega^2(log E_\lambda)$.

Recall: If $D = \{z_1 = 0\}$ then $\Omega^k(log D)$ is $\Lambda^k(dz_1/z_1, dz_2, ..., dz_n)$

Idea

Any top degree form with log singularity on a hypersurface D determine top degree form on D itself (like the induced orientation)

Definition

The residue of $\Omega = f(z_1, z_2)\frac{dz_1}{z_1} \wedge dz_2$ is $f(0, z_2)dz_2 \in H^0(D, K_D)$

Example: In our case $(z, y) \in E_\lambda$

$Res_{z,y}\Omega = \frac{1}{2\pi i}\oint_\gamma \frac{dz\wedge dy}{y^2-f_0(z)} = \frac{dz}{2\sqrt{f_0(z)}}$

Hence, $\omega = \frac{dz}{\sqrt{f_0(z)}}$ is a well-defined nowhere vanishing 1-form on E_λ i.e $\omega \in H^0(E_\lambda, K) \Rightarrow K_{E_\lambda} = \mathcal{O}_{E_\lambda}$

global 1-form

Recall

$$H^1_{dR}(E,\mathbb{R}) \xrightarrow{\simeq} H_1^{sing}$$

which send form α to $\int_{(-)} \alpha$.

Form $\omega = \frac{dz}{y}$ is a global 1-form: it's fine away from points $0, 1, \lambda.\infty$. By differentiation we have $\frac{dz}{y} = 2\frac{dy}{f_0'(z)}$, y, f_0' have no common zeros.

Proposition

ω is global holomorphic 1-form $\Rightarrow$ closed

Proof.

locally $\omega = f(x)dx$ for some holomorphic f, so it is $(u+iv)(dz+udy)$. Thus,
$d\omega = ((-v_z - u_y) + i(u_z - u_y))dz \wedge dy = 0$ by Cauchy-Riemann. □

Then, $[\omega] \in H^1_{dR}(E,\mathbb{C})$, it is decomposed as $\left(\int_\delta \omega\right)\delta^\vee + \left(\int_\gamma \omega\right)\omega^\vee$. Coefficients are called **periods**.

Periods and $H^1_{dR}(E, \mathbb{C})$

Periods are integrals of holomorphic forms on singular cycles, they are essense of the Hodge structure.
Let $H^{1,0} = \mathbb{C}[\omega], H^{0,1} = \mathbb{C}[\overline{\omega}]$.

Lemma

$$H^1_{dR}(E, \mathbb{C}) = H^{1,0}(E) \oplus H^{0,1}(E)$$

Proof.

1. $[\omega] \cup [\overline{\omega}] = (A\overline{B} - B\overline{A})\delta^\vee \wedge \gamma^\vee$
2. Rescale $\omega = \delta^\vee + \tau\gamma^\vee$. Then $[\omega] \cup [\overline{\omega}] = (-2iIm\tau)[E]$
3. $\int_E i[\omega] \cup [\overline{\omega}] = 2Im\tau > 0$. Indeed,
$i\omega \wedge \overline{\omega} = |f(z)|^2 idx \wedge d\overline{x} = 2|f|^2 dz \wedge dy$ locally.
4. Then since $A \neq 0, B \neq 0$ we have $[\omega] \neq 0$. It implies that $dimH^{1,0} = 1$.
5. If $[\overline{\omega}] = \lambda[\omega] \Rightarrow \delta^\vee + \overline{\tau}\gamma^\vee = \lambda(\delta^\vee + \tau\gamma^\vee)$. So $\tau = \overline{\tau}$. □

An invariant

Definition

A marking of E_λ is a choice (δ, γ) of symplectic basis of $H_1^{sing}(E_\lambda, \mathbb{Z})$. Then $(E_\lambda, \delta, \gamma)$ is called **framed genus 1 curve**.

Define $\tau(E_\lambda, \delta, \gamma) = \frac{\int_\gamma \omega_\lambda}{\int_\delta \omega_\lambda}$. Is it a good invariant?

Claim

If $f : E_\lambda \xrightarrow{\simeq} E_\mu$, then $\tau(E_\lambda, \delta, \gamma) = \tau(E_\mu, f_*\delta, f_*\gamma)$

Proof.

Since $f^*[\omega_\lambda] \in H^{1,0}(E_\mu)$ we have $f^*[\omega_\lambda] = c[\omega_\mu]$ (we use that holomorphic function on compact is constant)

Hence, $\int_{E_\lambda} f^*\omega_\mu \wedge f^*\overline{\omega}_m u = |c|^2 \int \omega_\lambda \wedge \overline{\omega}_\lambda = \int_{f_* E_\lambda} \omega_\mu \wedge \overline{\omega}_\mu = \int_{E_\mu} \omega_\mu \wedge \overline{\omega}_\mu \neq 0$. It follows that $c \neq 0$.

Then $\tau(E_\mu, f_*\delta, f_*\gamma) = \frac{\int_{f_*\gamma} \omega_\mu}{\int_{f_*\delta} \omega_\mu} = \frac{c \int_\gamma \omega_\lambda}{c \int_\delta \omega_\lambda} = \tau$ □

Computation of τ

Consider elliptic curve $y^2 = x^3 - x$. This curve has symmetry $\sigma : (x, y) \to (-x, iy)$.

Period map

Consider a small disk $\Delta \subset \mathbb{P}^1 \setminus (0,1,\infty)$, then the choice of δ, γ could be made so that it is valid for all $\lambda \in \Delta$.
It means that for the family $\{E_\lambda : \lambda \in \Delta\}$ the associated v/bundle of cohomological groups $H^1_{E/\Delta} \longrightarrow^{\pi}$ with $\pi^{-1}(\lambda) = H^1(E_\lambda, \mathbb{C})$ is trivialized by δ, γ. Hence, it has a flat connection $(\nabla\delta^* = \nabla\gamma^* = 0)$. It is called **Gauss-Manin connection**.
Over Δ $A(\lambda), B(\lambda)$ are holomorphic (check) and so is $P(\lambda) = \frac{B(\lambda)}{A(\lambda)}$.
By analytic continuation we get multiple-valued holomorphic $P : \mathbb{P}^1 \setminus (0,1,\infty) \longrightarrow \mathbb{C}$. P stands for **period map**.
Remark: It is multi-valued if λ crosses the contour while moving along the loop Γ.
Note, $P' = \frac{B'A - A'B}{A^2}$

Claim

$P' \neq 0$ for $\lambda \in \mathbb{P}^1 \setminus (0,1,\infty)$

We can observe that $[\omega_\lambda] = A\delta^\vee + B\gamma^\vee, [\omega_\lambda]' = A'\delta^\vee + B'\gamma^\vee$ so $[\omega_\lambda] \cup [\omega_\lambda]' = (AB' - BA')\delta^\vee \cup \gamma^\vee$
Goal: $\omega_\lambda, \omega'_\lambda$ are linearly independent in $H^1(E, \mathbb{C})$

Derivative of a period map

Claim

$\int [\omega_\lambda] \cup [\omega_\lambda]' = \frac{-4\pi i}{\lambda(\lambda-1)}$

Proof.

First it looks like $[\omega_\lambda]' = [\omega_\lambda'] = [\frac{1}{2}\frac{1}{z-\lambda}\omega_\lambda]$

However, this is only true in $H^1_{dR}(E \setminus p, \mathbb{C}), p = (\lambda, 0)$ since ω_λ' has a pole of order 2.

Key point: ω_λ' has no residue at this pole.

We have Gysin sequence:

$$0 \longrightarrow H^1_{dR}(E) \xrightarrow{Res} H^1_{dR}(E \setminus p) \longrightarrow H^2(\mathbb{C})$$

The third term above is $H^0(\{p\}) = \mathbb{C}$. So if $Res\omega_\lambda' = 0$ it comes from a class on $E : y^2 = p(x)$.

Let's do short computation:

$\omega_\lambda' = \frac{dx}{2y(x-\lambda)} = \frac{dy}{p'(x)(x-\lambda)} = \frac{dy}{y^2}\frac{x(x-1)}{p'(x)} \backsim \frac{dy}{y^2} +$ reg.stuff

□

P', continuation

smoothing: cut-off smooth function $\rho(y)$ which is 1 for close neighborhood of λ and then monotonously decreasing till ε-neighborhood (on annulus $Delta'$) and zero after.
Denote $[\theta'_\lambda] = [\omega'_\lambda + d(\rho/y)] \in H^1(E \setminus p)$. Then $[\omega'_\lambda] = [\omega_\lambda]'$.
Now we are able to compute numerator of P' using integration and Stokes' theorem:
$\int_E [\omega_\lambda] \cup [\omega_\lambda]' = \int_E \omega_\lambda \wedge \theta'_\lambda = \int_{\Delta'} \omega_\lambda \wedge d(\rho/y) =$
$-\int_{\Delta'} d(\rho/y\omega_\lambda \overset{Stokes}{=} -\int_{\partial\Delta'} \rho/y\omega_\lambda = -\oint_{\partial\Delta'} \frac{\omega_\lambda}{y} = -2\pi i(\frac{2}{p'(x)} =$
$4\pi i \frac{1}{\lambda(\lambda-1)}$

Corollary

The map $P : \mathbb{P}^1 \setminus (0, 1, \infty) \longrightarrow \mathbb{H} := \{\tau \in \mathbb{C} | Im\tau > 0\}$ is a local isomorphism everywhere, in particular, **period is locally a complete invariant**.

Global period map

From above: For a given $\lambda \in \Delta$ $(\delta.\gamma)$ is framing of E_λ

$\rightsquigarrow \quad P : \Delta \longrightarrow \mathbb{H}$ st $\lambda \mapsto \tau(\underbrace{E_\lambda, \delta, \gamma})$, this map is holomorphic and P' nowhere zero $\rightsquigarrow \tilde{P} : (X := \mathbb{P}^1 \setminus (0, 1, \infty)) \longrightarrow \mathbb{H}$ by analytic continuation.

Question

How does $P(\lambda)$ change as we continue around loop $\alpha \in \pi_1(X, \lambda_0)$

- ω_λ does not change
- framing changes to $(\delta', \gamma') = (a\delta + b\gamma, c\delta + d\gamma)$

So, $\tau' = \frac{c+d\tau}{a+b\tau}$.
Namely, any matrix from $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformation (and factors through $PSL_2(\mathbb{R})$).

description of $\tilde{P}$

From above $\tilde{P}$ is equivariant and $\rho : \pi_1(X, \lambda_0) \longrightarrow SL_2(\mathbb{Z})$ (monodromy representation)

Or, in the other words, $P : \mathbb{P}^1 \setminus (0, 1, \infty) \longrightarrow img\,\rho \backslash \mathbb{H}$, where $img\,\rho$ acts properly discontinuously.

Picard-Lefschetz transformation

Let us study the monodromy of Gauss-Manin connection.

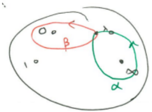
Quote

In mathematics, **monodromy** is the study of how objects from mathematical analysis, algebraic topology, algebraic geometry and differential geometry behave as they "run round" a singularity.

Example

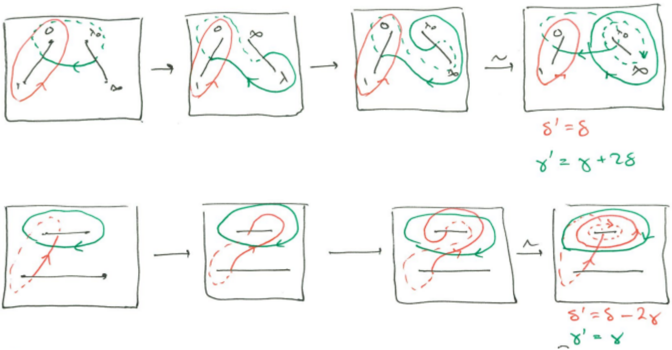
Consider $\Gamma \in \pi_1(\mathbb{C} \setminus \{0,1\})$ around 1. If $\lambda \to 1$ then torus degenerates to the pinched torus, and cycle γ is getting killed. It is called **vanishing cycle**

Let's define α and β, the generators of $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, \lambda)$, as the equivalence classes of loops shown below



Then we can see that singularity points arise only when $\lambda \to 0$ and $\lambda \to \infty$.

Monodromy



Note: No canonical choice of slits, they all give the same Riemann surface (torus in that case).

Monodromy representation

$$img\,\rho = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \rangle$$

It is called **monodromy group of the Legendre family**.

Definition

Group $\Gamma(2) := \{M \in SL_2(\mathbb{C}) | M \equiv id(mod 2)\}$

structure of $img\,\rho$

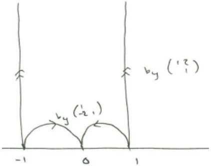
$\bullet\Gamma(2)$ has index 2 at $SL_2(\mathbb{C})$

$\bullet img\,\rho = \Gamma(2)$ in $PSL_2(\mathbb{Z})$

$\bullet\rho$ is injective

Proof.

Compare the fundamental domains of $img\,\rho$ on $\mathbb{H}$ and of $\Gamma(2)$, they are the same.



Then $\pi_1(\Gamma(2)\backslash\mathbb{H}) \simeq \mathbb{Z}^{*2}$ generated by $\rho(\alpha), \rho(\beta)$ □

Hence, $P : \mathbb{P}^1 \setminus (0, 1, \infty) \longrightarrow img\,\rho\backslash\mathbb{H}$ is $\simeq$ on π_1

Claim

P is isomorphism

Proof.

First, $img\,\rho\backslash\mathbb{H}$ is quasiprojective curve and P is algebraic map (check). Then recall that proper étale map which is isomorphism on π_1 is isomorphism. □

Reconstructon of H.str via local periods

Input: τ and period domain $\mathbb{H}$
Goal: Obtain pure Hodge structure
We need for this to reconstruct ω by these data. For each λ fix basis $[\delta_\lambda], [\gamma_\lambda]$ for each $H_1(E_\lambda)$ with $[\delta \cup \gamma]$ being the fundamental class.
We may define $\omega = \delta_\lambda^\vee + \tau(\lambda, \gamma, \delta)\gamma^\vee$
Note: It is not canonical.

General remarks

1. $E = \mathbb{C}/\Lambda$, where lattice $\Lambda \simeq \mathbb{Z}^2$ and $\iota : \Lambda \hookrightarrow \mathbb{C}$.
2. $\Lambda \simeq H_1^{sing}(E, \mathbb{Z})$, oriented integral basis (framing)
3. dz is translation-invariant so it gives the form on E. Moreover, $[dz] = [\omega]$ up to scale. So $\int_\delta dz = \iota(\delta)$ for $\delta \in \Lambda$.
4. In particular, if $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ then $\tau(E, 1, \tau) = \tau$ (!!)
5. So period is determind uniquely up to $SL_2(\mathbb{Z})$
6. Recall $E[2] \simeq \Lambda/2\lambda$, so $\Gamma(2)\backslash\mathbb{H}$ parametrizes $E[2] \to^{\sim} (\mathbb{Z}/2)^2$

In general

Derivative of period map generalizes to the deformation theory, and monodromy to Picard-Fuchs equations

Thanks!