LTCC: Hodge Theory

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Take-home exam

Submit your solutions via email. If you have any questions email me at n.kurnosov@ucl.ac.uk or use discord server. Exams starts at 12:01 am December 9th, and it ends at 23:59 pm January 8. You do not need to solve all problems for the highest grade, choose the problems you like. Any 5 problems give you 100% of a score. Good luck!

1. **(elliptic curves)** Consider the family of elliptic curves $E_{a,b,c}$ defined by $y^2 = (x-a)(x-b)(x-c)$.
   (a) What is the locus in $\mathbb{C}^3$ of the singular fibers? *(recall that only singular fibers for the Legendre family $y^2 = x(x-1)(x-\lambda)$ are at $\lambda = 0, 1$)*
   (b) Describe the monodromy representation.

2. **(hyperelliptic curves)** Let $y^2 = (z-a)(z-b)(z-c)(z-d)(z-e)(z-f)$ be a hyperelliptic curve.
   (a) Find the genus g of corresponding Riemann surface
   (b) Verify that the basis of holomorphic differentials is $z^i dz/y$, where $y = 0, ..., g-1$.

3. **(periods)** Let $E = \mathbb{Z}/\langle 1, \tau \rangle \mathbb{Z}$ be an elliptic curve. We know it has a period matrix

$$\begin{pmatrix} 1 & \tau \\ 1 & \bar{\tau} \end{pmatrix}$$

   where $\omega = \beta + \tau\alpha$ generates $H^{1,0}(E)$.
   (a) Write a period matrix for the Hodge structure on $H^1(E) \otimes H^1(E)$.
   (b) Is this Hodge structure irreducible (ie does not split into the direct some)?

4. **(period domains)** Consider the period domain $D_{(2,2)}$ of the integral Hodge structure for $H_{\mathbb{Z}} = \mathbb{Z}^4$ of weight 1 with the Hodge numbers (2,2) and polarization $q_{\mathbb{Z}} = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$. The Lie groups $G(\mathbb{R})$ and $G(\mathbb{C})$ are $Sp_4(\mathbb{R})$ and $Sp_4(\mathbb{C})$ respectively.
   (a) Fix a point $x_0 = \begin{pmatrix} i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in D_{(2,2)}$, where the left half corresponds to $H^{2,0}$ and the right one to $H^{0,2}$. Compute $Stab_{G(\mathbb{R})}(x_0)$ and $Stab_{G(\mathbb{C})}(x_0)$
   (b) Show that $D_{(2,2)} = Sp(4, \mathbb{R})/U(2)$ (recall from lectures that it is $\mathcal{H}_2$) and it is hermitian symmetric.
   (c) Show that $D_{(1,0,1,1,0,1)}$ fibers holomorphically over $D_{(2,2)}$ by redefining $H^{1,0}$ component of the Hodge structure.

5. (mixed Hodge structures) Let C be the nodal curve obtained by gluing together the points 0 and 1 of $\mathbb{A}^1_{\mathbb{C}}$. At the lecture 4 we have described the mixed Hodge structure of such curve: $W_1 = H^1(X, \mathbb{Q}), F^0 = H^1(X, \mathbb{C})$ and $H_{\mathbb{Z}} = H^1(X, \mathbb{Z})$. Suppose there is a map from C to the smooth projective X. There is the result of Deligne: *the homologies of complex algebraic varieties carry functorial mixed Hodge structures dual to the ones on cohomology.* Then using the result of Deligne show that the induced map on the first homologies must vanish.

6. (Kähler/non-Kähler) Consider the Heisenberg group $M = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}$, where $x, y, z \in \mathbb{C}$, and $\Gamma = M \cap GL_3(\mathbb{Z})$. Show that the Iwasawa manifold $\Gamma \backslash M$ is non-Kähler by exhibiting a non-closed holo-morphic form. *[Hint:consider $M^{-1}dM$].*

7. (Hodge diamond) Compute the Hodge number $h^{1,1}$ for the quintic surface in $\mathbb{P}^3$.