Lecture 5

Abstract: In the last lecture we will talk about Lagrangian fibraions. I will review results on the base of fibration and the structure of fibers.

4.9. **Lagrangian fibrations.**

4.9.1. *Elliptic K3 surfaces.* Consider Fermat's quartic S (as we know it is K3 surface) in the following form

$$(x^2 + y^2)(x^2 - y^2) - (z^2 + t^2)(z^2 - t^2) = 0$$

Consider a point $p = [\lambda : \mu] \in \mathbb{P}^1$, then the following intersection is contained in S:

$$C_p := \begin{cases} \lambda(x^2 - y^2) = \mu(z^2 + t^2) \\ \mu(x^2 + y^2) = \lambda(z^2 - t^2) \end{cases}$$

Note that for the generic p this is a smooth elliptic curve (Exercise 1). Moreover, for $\lambda/\mu = 0, \pm 1, \pm i, \infty$ the fiber degenerates into a cycle of four lines. Therefore, we have 24 singularities.

Remark: For ones who is interested in mirror symmetry, look into paper of Auroux [Au]. In fact, the mirror of K3 surface is another K3 surface, carrying a special Lagrangian fibration whose base differs from B by an exchange of the two affine structures on the $B \simeq S^2 \setminus \{24 \text{ points }\}$.

4.9.2. *Matsushita theorem.* It appears that

Theorem 4.17. ***(Matsushita,*** [Mat1, Mat2, Mat3]***)***
Let $\pi : X \to B$ be a proper surjective holomorphic map with connected fibers from a hyperkähler manifold X to a projective variety B, with $0 < dim(B) < dim(X)$. Then
(1) $dim(B) = \frac{1}{2} dim(X)$, and
(2) the fibers of π are Lagrangian (this means that the holomorphic symplectic form vanishes on the fibers $\pi^{-1}(x)$).
(3) Moreover, B is a Fano variety with $\mathbb{Q}$-factorial singularities and Picard number 1.

We will discuss the proof of this theorem and most of known results about Lagrangian fibrations.

Example: Let S be a K3 surface with an elliptic fibration $\rho : S \to \mathbb{P}^1$. The composition $S^{[n]} \to S^{(n)} \to (\mathbb{P}^1)^{(n)} \simeq \mathbb{P}^n$ is a Lagrangian fibration $\pi : S^{[n]} \to \mathbb{P}^n$. A generic fiber of π is isomorphic to $C_1 \times \cdots \times C_n$, where $C_1, ..., C_n$ are generic distinct fibers of ρ. The generic deformation of the couple $(S^{[n]}, \pi)$ is not obtained by deforming (S, ρ). In fact by [**?**] the deformation space of $(S^{[n]}, \pi)$ is smooth and it has dimension one greater than the deformation space of (S, ρ).

4.9.3. *Fiber of the fibration.* We will start with the small remark.

Remark: The normal bundle of a Lagrangian submanifold $T \subset X$ is isomorphic to the cotangent bundle of T.

Let us state the following Theorem of Kollár and Saito ([Ko1, Ko2, Sai]):

Theorem 4.18. *Let $f : X \to B$ be a proper surjective morhpism from a smooth Kähler manifold X to a normal variety B. Then $R^i f_* \omega_X$ is torsion free, where ω_X is the dualizing sheaf of X.*

Proposition 4.6. *Let $f : X \to B$ be a proper Lagrangian fibration. Then every irreducible component of a fibre of f is a Lagrangian subvariety. In particular, f is equidimensional.Furthermore, every smooth fibre X_t of f is a complex torus and in fact an abelian variety.*

Idea of proof:

1. Let X_t be a general fiber. Then by adjunction, $K_F \sim 0$ and also the dimension of the general fiber is n.

2. Consider arbitrary $\omega \in H^2(X, \mathbb{R})$ and $\alpha \in H^2(B, \mathbb{R})$, then by Fujiki's equation we have

$$q(\Omega + \bar{\Omega} + x \cdot \omega + y \cdot f^*\alpha)^n = \lambda_X \int_X (\Omega + \bar{\Omega} + x \cdot \omega + y \cdot f^*\alpha)^{2n}$$

3. Hence, by comparing coefficients we have

$$\int_X (\Omega \wedge \bar{\Omega}) \wedge \omega^{n-2} \wedge f^*(\alpha^n) = 0$$

4. Restrict the equation above on X_t we have

$$\int_X (\Omega \wedge \bar{\Omega})|_{X_t} \wedge \omega^{n-2}|_{X_t} = 0$$

which implies $\Omega|_{X_t} = 0$.

5. Since X_t is Lagrangian we have $\Theta_{X_t} \simeq N^*_{X_t/X}$, normal bundle is trivial, therefore so is tangent bundle. Any compact Kähler manifold with trivial tangent bundle is a complex torus by Albanese morphism.

6. By the Theorem 4.18 we have $R^2 f_* \omega_X$ torsion-free, also we have $\omega_X \simeq \mathcal{O}_X$.

7. By Leray spectral sequence, there exists a morphism

$$\rho : H^2(X, \mathcal{O}_X) \to H^0(B, R^2 f_* \mathcal{O}_X)$$

so $\rho(\bar{\Omega})$ is a torsion element in $H^0(B, R^2 f_* \mathcal{O}_X)$ since general fiber is Lagrangian.

8. Let V be an irreducible component of a fibre. Let $\pi : \tilde{V} \to V$ be its resolution. Then $\pi^* \bar{\Omega}$ is trivial. Indeed, it is contained in the image of the following sequence

$$R^2 f_* \mathcal{O}_X \otimes k(p) \longrightarrow H^2(V, \mathcal{O}_V) \longrightarrow H^2(\tilde{V}, \mathcal{O}_{\tilde{V}})$$

9. The dimension of V is at least n because of semi-continuity of the dimension of the fibers. Therefore, V is Lagrangian.

Remark: If we drop the condition of properness, there exists a counter-example, see Exercise 3.

4.9.4. *Lagrangian tori.*

Proposition 4.7. ***(Voisin,** [Voi]**)***

Any Lagrangian submanifold $T \subset X$ of a hyperkähler manifold X is projective. In particular, any Lagrangian torus is an abelian variety.

Remark: As T is Lagrangian, the map $(H^{2,0} \oplus H^{0,2})(X) \longrightarrow H^2(T, \mathbb{C})$ is trivial, and $H^{1,1}(X) \to H^{1,1)}$ is certainly not trivial. Thus, $T \subset X$ deforms with X along a subset of codimension at least one.

Moreover, the restriction map

$$H^2(X, \mathbb{Z}) \to H^2(X_b, \mathbb{Z})$$

has rank 1, so that the fibers X_b are in fact canonically polarized by the restriction of any ample line bundle on X.

Question: Are every Lagrangian torus $T \subset X$ is the fiber of a Lagrangian fibration $X \to B$.

The question has been answered affirmatively by Greb-Lehn-Rollenske ([?]) for non-projective X and also in dimension four, it was also discussed in the works of Amerik-Campana ([AC]), the projective case was discussed by Hwang-Weiss.

4.9.5. *The base of a fibration.* In the Matsushita theorem 4.17 it is stated that the base is Fano variety with Picard 1 and $\mathbb{Q}$-factorial singularities. We refer to [Mat1] for the proof of it.

Here is a refined statement due to Matsushita for the smooth base. In that case

Theorem 4.19. ***(Matsushita)***

Assume $f : X \to B$ is a fibration with B smooth. Then B is a simply connected, smooth projective variety of dimension n satisfying

$$H^{p,q}(B) = 0 \text{ for all } p \neq q \text{ and } H^{p,p}(B) = H^{p,p}(\mathbb{P}^n)$$

for all $p > 0$ and $H^2(B, \mathbb{Q}) \simeq \mathbb{Q}$. In particular,

$$Pic(B) \simeq H^2(B, \mathbb{Z}) \simeq \mathbb{Z}$$

Moreover, B is a Fano variety.

Remark: The equalities $H^{p,0}(B) = H^{0,p}(B) = 0$ for all $p > 0$ and $H^2(B, \mathbb{Q}) = \mathbb{Q}$ are more easily to prove.

Remark: The proof of an isomorphism of rational Hodge structures $H^*(B, \mathbb{Q}) \simeq H^*(\mathbb{P}^n, \mathbb{Q})$ relies on the fact $H^*(X, \mathcal{O}_X) = H^*(\mathbb{P}^n, \mathbb{C})$. It was studied by Matsushita for projective X via higher direct images ($R^i f_* \mathcal{O}_X \simeq \Omega^i_B$) and then projectivity condition can be dropped by [?] where they used the deformation theory.

Nevertheless, we will proof couple of claims of the Proposition 4.19.

Idea of proof of the $Pic(B) = \mathbb{Q}$:

1. Since X is Kḧler, so is B.

2. Since $H^{2,0}(B) = H^{0,2}(B) = 0$, there exists a rational Kähler class on B implying that it is projective.

3. By results of Kollár the natural map $\pi_1(X) \to \pi_1(B)$ is surjective. Hence, B is simply-connected.

4. Then, by the universal coefficient theorem, $H^2(B, \mathbb{Z})$ is torsion-free.

5. The exponential sequence gives $Pic(B) \tilde{\to} H^2(B, \mathbb{Z})$.

Idea of proving that B is Fano[8]

1. X admits a Kḧler– Einstein metric. Hence, $\omega_B \simeq \mathcal{O}_B$ or ω_B^* is ample.

2.The case $\omega_B \simeq \mathcal{O}_B$ is excluded, since $H^{n,0}(B) = 0$.

[8] i.e. ω_B^* is ample

Flatness of f and smoothness of the base B.

Remark: In fact, f is flat if and only if B is smooth. Flatness of f follows from smoothness X and B by [**?**]

If B is not-smooth we can constract nonflat fibration.

Example: Let $X := A \times \mathbb{C}^3$, where A is a three-dimensional torus.

Let us define the following action of $\mathbb{Z}_2$ on X:

$$(x, y, z; u, v, w) \mapsto (-x, -y, z + \tau; -u, -v, w),$$

where τ is a 2-torsion element of A.

If we define the holomorphic symplectic form on X by $dx \wedge du + dy \wedge dv + dz \wedge dw$, the morphism $X/\mathbb{Z}_2 \to \mathbb{C}^3/\mathbb{Z}^2$ is a Lagrangian fibration.

One can check that it is nonflat Lagrangian fibration.

Later, Hwang [Hw] studied the smooth case and proved

Proposition 4.8. ***(Hwang)***
If the base B is smooth, then $B \simeq \mathbb{P}^n$.

Idea of proving this Proposition is based on interplay of the existence of two geometric structures on the base B: the theory of varieties of minimal rational tangents describes a certain geometric structure arising from minimal rational curves at general points of a Fano manifold B with $b_2(B) = 1$, on the other hand, the theory of Lagrangian fibrations (and related theory of completely integrable Hamiltonian systems) provides an affine structure at general points of the base manifold B via the classical action variables. If base is not $\mathbb{P}^n$ then the first structure is "non-flat", while the affine structure arising from the action variables is naturally "flat".

Remark: In dimension two, the result is immediate by Exercise NNN

Remark: However, it is still unclear if the base is always smooth.

Conjecture 4.2. *The base of Lagrangian fibration is always projective space $\mathbb{P}^n$.*

Dimension four

Theorem 4.20. ***(Ou,*** [Ou]***)***
Let $f : X \to B$ be a Lagrangian fibration from a projective irreducible symplectic manifold X of dimension 4 to a normal surface B. Then either $X \simeq \mathbb{P}^2$ or $X \simeq S_n(E_8)$.

The surface $S_n(E_8)$ is the unique Fano surface with exactly one singular point which is Du Val of type E_8, and two nodal rational curves in its anti-canonical system.

For a Fano surface B with canonical singularities and simpl-connected smooth locus and Picard number 1 there are following possibilities ([**?**]):

- If B is smooth, then $B \simeq \mathbb{P}^2$
- If there is one singular point, then possible type of singularities are $A_1, A_4, D_5, E_6, E_7, E_8$. There are isomorphic classes for E_8, and one for any other case.
- If there are two singular points, then one is of type A_1, and the other is of type A_2.

Let us describe the construction of the surfaces $S_c(E_8)$ and $S_n(E_8)$. Choose a singular cubic rational curve C in $B_1 = \mathbb{P}^2$. Let x be one of the smooth inflection points of C. Then we blow up the point x, and call resulting surface as B_2. Consider $B_3 \to B_2$ as the blow up of the intersection point of the strict transform of C in B_2 and the exceptional divisor of $B_2 \to B_1$. Then we can apply this procedure six more times and get B_9, which has eight (-2)-curves: the strict transform of tangent line to x, and the strict transforms of all the exceptional curves of $B_8 \to B_1$. Blow down all these curves to get surface B, then B is isomorphic to $S_c(E_8)$ if C is a cuspidal rational curve, or is isomorphic to $S_n(E_8)$ if C is a nodal curve.

The key theorem of Ou's proof is the following

Theorem 4.21. *Let $f : X \to B$ be a Lagrangian fibration from a complex projective irreducible symplectic manifold X to a normal projective variety B. Let H be a $\mathbb{Q}$-ample integral Weil divisor in B, and let $D = f^*H$. Then for all $j > 0$ and $i \geq 0$, we have*

$$h^j(X, R^i f_*(\mathcal{O}_X(D))) = 0 \text{ and } h^i(X, \mathcal{P}_X(D)) = h^0(B, \Omega^{[i]}[\otimes]\mathcal{O}_B(H)).$$

where $\Omega^{[i]} = (\Omega^i)^**$ for all $i > 0$, and for two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, let $\mathcal{F}[\otimes]\mathcal{G} = (\mathcal{F} \otimes \mathcal{G})^**$

As a corollary, we conclude that if $dimX = 4$, then

$$h^0(B, \mathcal{O}_B(H)) - h^0(B, \Omega^{[1]}[\otimes]\mathcal{O}_B(H)) + h^0(B, \mathcal{O}_B(H + K_X)) = 3$$

The theorem above follows from generalization of Kollár theorem: for a Lagrangian fibration f from a smooth complex projective symplectic variety X to a projective variety B. Let H be a Weil divisor on B, and let $D = f^*H$. We will show that the sheaf $R^i f_*\mathcal{O}_X(D)$ is reflexive and is isomorphic to $\Omega^i_B[\otimes]\mathcal{O}_B(H)$ for all $i \geq 0$.

Theorem 4.22. ***(Huybrechts-Xu, Bogomolov-Kurnosov,*** [HX, BK]***)***
The base of Lagrangian fibration of hyperkähler fourfold is $\mathbb{P}^2$.

For both papers main idea is that since the normal surface B is known to be $\mathbb{Q}$-factorial with at most log-terminal singularities, then, locally analytically, it is isomorphic to a quotient of the form $\mathbb{C}^2/G$ for some finite subgroup $G \subset GL(2, \mathbb{C})$ (see [**?**]). By Theorem 4.20 there is just on case left, this case leads to the non-trivial central extension of group A_5 by $\mathbb{Z}_2$ and was excluded by Bogomolov-Kurnosov. Huybrechts-Xi have proved the following theorem which is stronger than the anything left to prove after Ou:

Theorem 4.23. ***(Huybrechts,Xu)***
Assume $X \to B$ is a projective Lagrangian fibration of a quasi-projective symplectic fourfold over a normal algebraic surface B. If B is locally analytically at a point $0 \in B$ of the form $\mathbb{C}^2/G$ for a finite subgroup $G \subset GL(2, \mathbb{C})$, then G is not the binary icosahedral group.

For the proof of this theorem they needed to study G-action on the essential skeleton of the one-dimensional degeneration over $Spec(R)$. The study of degenerations and essential skeleton leads to many other results on the geometry of Lagrangian fibrations (see Section 4.11.2).

4.10. **SYZ conjecture.** Holomorphic Lagrangian fibrations are important in Mirror Symmetry. In [SYZ] Strominger, Yau and Zaslow conjectured that Mirror Symmetry of Calabi-Yau manifolds comes from real Lagrangian fibrations. In particular, any Calabi-Yau manifold which admits mirror must have a Lagrangian fibration and the dual fibration corresponds to the mirror dual Calabi-Yau manifold.

Definition 4.20. *Let L be a line bundle on a compact complex manifold M. Then L is called semiample if there exists a holomorphic map $\pi : M \to X$ to a projective variety X, and $L^N \simeq \pi^*(\mathcal{O}(1))$, for some $N > 0$.*

Conjecture 4.3. ***(The hyperkähler SYZ conjecture,*** [V4]***)***

Let L be a line bundle on a hyperkähler manifold, with $q(c_1(L), c_1(L)) = 0$, and $c_1(L)$ nef[9]*. Then L is semiample.*

By Theorem 4.17 for sufficiently big N, the corresponding projective morphism $M \to \mathbb{P}(H^0(M, L^{\otimes N})^*)$ is a Lagrangian fibration onto its image.

Remark: Semiampleness of L implies that the Kodaira dimension and numerical dimension of L are equal, this condition is often called "the abundance conjecture" [DPS].

The Conjecture 4.3 is known for all known examples of hyperkähler manifolds. And we can state it in more weak and popular form

Conjecture 4.4. *Let M be a hyperkähler manifold. Then M can be deformed to a hyperkähler manifold admitting a holomorphic Lagrangian fibration.*

Remark: In the hyperlähler case this conjecture was stated by Hassett-Tschinkel, and then by Huybrechts, Sawon, who studied them. In his review of SYZ conjecture Verbitsky says that conjecture was stated even earlier by Bogomolov and Tyurin.

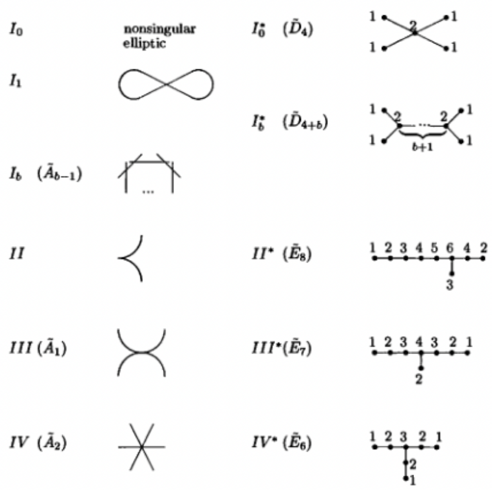
Remark: The weakened form of Matsushita's conjecture have been proved in the work of Geemen-Voisin [GV]

4.10.1. *Special fibers.* We have proved that the general fiber is Lagrangian tori. The structure of special fibers is also very interesting question. It was first studied by Kodaira, who gave a classification of singular fibers for elliptic surfaces. In the hyperkähler case this question was studied by Hwang, Oguiso, Sawon [HO, S1] and others.

Theorem 4.24. ***(Kodaira)***

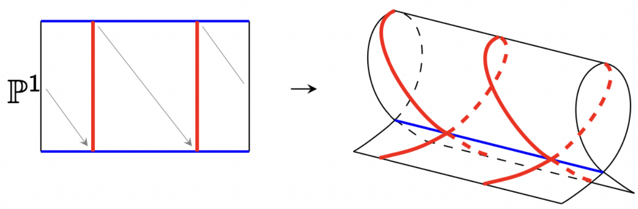
Singular fibres of elliptic surfaces look like:

[9] i.e. it belongs to the closure of the Kähler cone.



Remark: All these singular fibres can occur for elliptic K3s except multiple fibres and I_b, I_b^* for $b \gg 0$. A very general elliptic K3 has 24 singular fibres of type I_1.

In higher dimensions there is not much known. Below there is on of possible singular fibers. Unlike elliptic K3 surfaces, the singular fibres of Lagrangian fibrations can have multiplicities



Theorem 4.25. ***(C. Lehn, Sawon)***

For a general Lagrangian fibration $X \to \mathbb{P}^n$ a general singular fibre has characteristic cycle of type I_b, I_∞, II (cuspidal), III (tacnode), or IV (triple intersection).

4.11. **Degeneration of hyperkähler manifolds.**

4.11.1. *$P = W$ conjecture.*

Definition 4.21. *Let $\pi : X \to \Delta$ be a projective degeneration of hyperkähler manifolds over the unit disk which we assume to be semistable, namely the central fiber X_0 is reduced with simple normal crossings. For $t \in \Delta^*$, let N denote the logarithmic monodromy operator on $H^*(\mathcal{X}_t, \mathbb{Q})$. The weight filtration of N centered at d on $H^d(\mathcal{X}_t, \mathbb{Q})$, denoted by $W_k H^d(\mathcal{X}_t, \mathbb{Q})$, is the weight filtration of the limit mixed Hodge structure associated to π. The degeneration $\pi : \mathcal{X} \to \Delta$ is called of type III if $N^2 \neq 0$ and $N^3 = 0$ on $H^2(\mathcal{X}_t, \mathbb{Q})$. In this case, the limit mixed Hodge structure is of Hodge–Tate type by result of Soldatenkov* [Sol].

Theorem 4.26. ***(Harder,Li, Shen, Yin)***

For any Lagrangian fibration $f : X \to B$, there exists a type III projective degeneration of hyperkähler manifolds $\pi : \mathcal{X} \to \Delta$ with $\mathcal{X}_t$ deformation equivalent to X for all $t \in \Delta^*$, together with a multiplicative isomorphism $H^*(X, \mathbb{Q}) \simeq H(\mathcal{X}_t, \mathbb{Q})$, such that

$$P_k H^*(X, \mathbb{Q}) = W_{2k} H^*(\mathcal{X}_t, \mathbb{Q}) = W_{2k+1} H^*(\mathcal{X}_t, \mathbb{Q}$$

4.11.2. *Mazzon-Brown result.*

Definition 4.22. *Let $\mathcal{X}^*/\Delta^*$ be a projective degeneration of hyper-Kähler manifolds. Let $\nu \in \{1, 2, 3\}$ be the nilpotency index for the associated monodromy operator N on $H^2(\mathcal{X}_t)$ (i.e. $N = logT_u$, where T_u is the unipotent part of the monodromy $T = T_s T_u$). We say that the degeneration is of Type I, II, or III respectively if $\nu = 1, 2, 3$ respectively.*

Definition 4.23. *Any projective 1-parameter degeneration $\mathcal{X}/\Delta$ of K3 surfaces can be arranged to be semistable with trivial canonical bundle; such a degeneration is called a Kulikov degeneration of K3s.*

For a Kulikov degeneration, one can give a rather precise description of the possible central fibers X_0 of the degeneration.

Theorem 4.27. ***(Kulikov, Persson, Roan,*** [Ku]

Let $\mathcal{X}/\Delta$ be a Kulikov degeneration of K3 surfaces. Then, depending on the Type of the degeneration (or equivalently, the nilpotency index of N) the central fiber X_0 of the degeneration is as follows:

i) Type I: X_0 is a smooth K3 surface.

ii) Type II: X_0 is a chain of surfaces, glued along smooth elliptic curves. The end surfaces are rational surfaces, and the corresponding double curves are smooth anticanonical divisors. The intermediary surfaces (possibly none) are (birationally) elliptically ruled; the double curves for such surfaces are two distinct sections which sum up to an anticanonical divisor.

iii) Type III: X_0 is a normal crossing union of rational surfaces such that the associated dual complex is a triangulation of S^2. On each irreducible component V of X_0, the double curves form a cycle of rational curves giving an anticanonical divisor of V.

Definition 4.24. *Let R be a discrete valuation ring with quotient field K and residue field k, and let X be a smooth proper variety over K. By resolution of singularities guarantees that we can always produce an R-model $\mathcal{X}$ where the special fiber $\mathcal{X}_k$ is a strict normal crossings (snc) divisor. Given such a model, we associate the dual complex $D(\mathcal{X}_k)$, which is the dual intersection complex of the components of the special fiber.*

Remark: The dual complex of the special fiber of a such degeneration reflects the geometry of the generic fiber. If the generic fiber is rationally connected, then the dual complex of the special fiber is contractible. For Calabi-Yau varieties, degenerations are classified by the action of monodromy on the cohomology. The principle is that the degenerations with maximally unipotent actions have the most rich combinatorial structure in the dual complex.

Just as for Calabi-Yau varieties, degenerations of hyper-Kähler varieties can be understood in terms of the monodromy operator on cohomology, with classification into types I, II, III. Type I is the case where the dual complex is just a single point, but types II and III have more interesting structure. In [KLSV] was show that in the type II case the dual complex has the rational homology type of a point, and in the type III case of a complex projective space. Gulbrandsen, Halle, and Hulek use GIT to construct a model of the degeneration of n-th order Hilbert schemes arising from some type II degenerations of K3 surfaces, and show that the dual complex is an n-simplex. There are considerations from mirror symmetry that suggest that for a type III degeneration the dual complex should be homeomorphic to $\mathbb{CP}^n$

Remark: The dual complex of the special fiber of a Kulikov degeneration of a K3 surface is a point, a closed interval or the sphere S^2 according to the respective type.

Definition 4.25. *For complete K with respect to the valuation induced by R, we get non- archimedean norm on K. We associate a K-analytic space to X; each point corresponds to a real valuation on the residue field of a point of X, extending the discrete valuation on K. This space, denoted by X^{an}, is called the Berkovich space associated to X. From any snc model $\mathcal{X}$ of X one can construct a subspace of X^{an}, called the Berkovich skeleton of $\mathcal{X}$ and denoted by $Sk(X)$: it is homeomorphic to the dual intersection complex of the divisor $\mathcal{X}_k$.*

Theorem 4.28. ***(Mazzon-Brown,** [BM]**)***

If the essential skeleton of S is PL homeomorphic to a point, a closed interval or the 2-dimensional sphere, then the essential skeleton of $S^{[n]}$ is PL homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.

If the essential skeleton of A is PL homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$, then the essential skeleton of $K_n(A)$ is PL homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.

Conjecture 4.5. *For type III degenerations of $2n$-dimensional hyper-Kähler manifolds, the base of the SYZ fibration is $\mathbb{CP}^n$.*

4.12. **Exercises 5.**

(1) Show that generic fiber of the elliptic K3 surface from Lectures is a smooth elliptic curve. Also show that special fibers degenerates into a cycle of four lines.
Hint: Show that its tangent bundle fits in some exact sequence.
(2) Construct Lagrangian fibrations for $K3^{[n]}$ and $K_n(T)$.
(3) Let us consider the morphism $\pi : \mathbb{C}^4 \to \mathbb{C}^2$ which is defined by $(x, y, z, w) \mapsto (xy, y)$. Define a symplectic form on $\mathbb{C}^4$ by $dx \wedge dz + dy \wedge dw$. Show that π is a Lagrangian fibration and the fibre $\pi^{-1}(0, 0)$ is not a Lagrangian subvariety.
(4) Any smooth projective surface B with ω_B^* ample and $H^2(B, Q) = Q$ is isomorphic to $\mathbb{P}^2$.
(5) For a degeneration of abelian surfaces, the dual complex of the special fiber is homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$ according to the three types of degeneration.

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