4. Lecture 4

Abstract: I will finish the idea of proof of Torelli theorem.

In the second part of lecture we will talk about automorphisms groups of hyperkähler manifolds, in particular about Kawamata-Morrison cone conjecture and finiteness results.

4.1. **Ideas of proving Torelli theorem.** Note that a point $z \in \Omega$ determines an oriented positive 2-plane Π_z in $H^2_{\mathbb{R}}$: for the associated Hodge decomposition, the sum $\Pi_z^{2,0} + \Pi_z^{0,2}$ is the complexification of a 2-plane Π_z in $H^2_{\mathbb{R}}$, which is indeed canonically oriented.

Therefore, the positive cone we have associated to Π_z is an open subset of the $Re(\Pi_z^{1,1})$. Conversely, an oriented positive 2-plane in $H^2_{\mathbb{R}}$ determines a point of Ω.

The main ingredient is the following result due to Huybrechts-Demaily-Paun ([H, DP]).

Theorem 4.1. *Let X be a hyperkähler manifold for which $H^{1,1}(X) \cap H^2(X;\mathbb{Z}) = \{0\}$, namely $\Pi_z^{\perp} \cap H^2 = \{0\}$). Then every element of the positive cone of X represents a Kähler class.*

Since period space is simply connected, Theorem 3.16 is equivalent to saying that $\mathfrak{Per}$ is a covering map.

It is sufficient to prove the following proposition.

Proposition 4.1. *Let $U \subset \Omega$ be a neighborhood of $\mathfrak{Per}(t)$ isomorphic to the open unit ball in $\mathbb{C}^{n+1}$. If $t \in \mathcal{J}_s$ lies over the center of this ball, then there exists a unique section σ of $\mathfrak{Per}$ over U which takes $\mathfrak{P}(t)$ to t.*

Definition 4.1. *If the rational closure of a linear subspace W in V for $\mathbb{Q}$-vector space is all of V , then we say that W is fully irrational.*

Remark: It is clear that in the Grassmannian of all linear subspaces of V that are not fully irrational form a countable union of proper subvarieties defined over $\mathbb{Q}$. In particular, the fully irrational subspaces are dense.

Idea of proof:

1.

Lemma 4.1. *Let W be a fully irrational, positive 3-plane in $H^2(\mathbb{R})$. Then $\mathfrak{Per}$ maps every connected component of $\mathfrak{Per}^{-1}D(W)$ isomorphically onto $D(W)$.*

Lemma follows from the Theorem 4.1.

2. Let us identify U with the open unit ball $B_{<1}$ in $\mathbb{C}^{n+1}$. Let r be the supremum of the $a \in (0,1]$ for which there exists a section over the open ball $B_{<a}$ that takes $\mathfrak{Per}(t)$ to t.

3. Suppose $r < 1$. Since $\mathfrak{Per}$ is a local homeomorphism between separated spaces, two sections defined on the same connected subset of Ω are equal when they are equal at some point. Hence, we have a section σ defined over its interior $B_{<r}$.

4. Let $z \in \partial B_r$. A positive line l in $\Pi_z^{\perp}$ determines a twistor conic $D(l + \Pi_z)$. Take l such that $D(l + \Pi_z)$ is transversal to the tangent space of ∂B_r at z and l is fully irrational.

Remark: First is an open, nonempty condition, second condition is dense.

5. By step 1 (Lemma 4.1) we have that $B_{<r} \cap D(l + \Pi_z)$ is nonempty and that the restriction of σ to this subset extends across z. Therefore, we obtain a section σ_z on an open ball neighborhood U_z of z in U such that σ and $\sigma_z$coincide at points of $U_z \cap B_{<r}$ by connectedness.

6. Moreover, σ and σ_z, $z \in \partial B_r$ all together define a section of period mp $\mathfrak{Per}$ on a neighborhood of B_r. It gives a contradiction, because it contains a ball of a larger radius.

4.2. **Cones.**

Definition 4.2. *Let X be an IHS. The Kähler cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is the open convex cone of all Kähler classes on X, i.e. classes that can be represented by some Kähler form.*

Definition 4.3. *$\tilde{\mathcal{K}}_M$ closure of Kähler cone in $H^{1,1}(M, \mathbb{R})$ is called the nef cone. A face of the Kähler cone is the intersection of the boundary of $\tilde{\mathcal{K}}_M$ and a hyperplane $V \subset H^{1,1}(M, \mathbb{R})$ which has non-empty interior.*

Definition 4.4. *The set of classes with positive square in $H^{1,1}(X, \mathbb{R})$ has two connected components. We call the positive cone the component containing the Kähler classes.*

The following analogue of Shafarevich and Pyatetski- Shapiro's/Looijenga and Peters' result is known.

Theorem 4.2. ***(Huybrechts, Boucksom; [Bo, H])***
The Kähler classes on an IHS X are the elements of the positive cone $Pos(X)$ which are strictly positive on all rational curves on X.

Remark: Hence, the Kähler cone is cut out within the positive cone by the orthogonal hyperplanes to the classes of the rational curves. Recall that, in the projective case, the ample classes are elements of the Neron-Severi group which have positive square and are positive on all the rational curves.

Assuming that the hyperkähler manifold X is projective, we now look at the intersections of the cones in $H^{1,1}(X)_{\mathbb{R}}$ that we have constructed with the real subspace $Pic(X)_{\mathbb{R}} := Pic(X) \otimes \mathbb{R} = <H^{1,1}(X)_{\mathbb{R}} \cap H^2(X, \mathbb{Q})>$.

The open cone

$$Pos^{alg}(X) := Pos(X) \cap Pic(X)_{\mathbb{R}}$$

is the component of $\{L \in Pic(X)_{\mathbb{R}} | L^2 > 0\}$ that contains the ample classes.

By exercise NN it makes sense to define the (possibly nonconvex) open bimeromorphic kähler cone

$$Bir\mathcal{K}(X) := \cup_u u^*(\mathcal{K}(X')) \subseteq Pos(X)$$

There might be infinitely many different cones $u^*(\mathcal{K}(X'))$.

The closed cone $Bir\overline{\mathcal{K}}(X)$ is convex. By Theorem 4.2 we have the following description:

$\mathcal{K}_X$=$\alpha \in Pos(X) | \alpha \cdot C > 0$ for all rational curves $C \subset X$
$\overline{\mathcal{K}_X}$=$\alpha \in \overline{Pos(X)} | \alpha \cdot C \geq 0$ for all rational curves $C \subset X$
$\overline{Bir(\mathcal{K})_X}$=$\alpha \in \overline{Pos(X)} q(\alpha, Y) \geq 0$ for all (uniruled/negative) irreducible hypersurfaces $Y \subset X$

The following result shows that these cones are either disjoint or equal.

Proposition 4.2. ***(Fujiki)***
Let X and X' be hyperkähler manifolds with a bimeromorphic isomorphism

$$u : X \dashrightarrow^{\sim} X' \iff u^*\mathcal{K}_{X'}) = \mathcal{X} \iff \mathcal{K}_X \cap u^*(\mathcal{K}_{X'} \neq \emptyset$$

4.2.1. *Ample/movable cone.*

Definition 4.5. *The intersection*

$$\mathcal{K}_X \cap Pic(X)_{\mathbb{R}} = Amp(X)$$

is the ample cone, generated by classes of ample line bundles.

Remark: We also have nef cone $\overline{\mathcal{K}_X} \cap Pic(X)_{\mathbb{R}} = \overline{Amp}(X) = Nef(X)$ generated by classes of nef line bundles.

By Theorem 4.2 we have

$$Amp(X) = \alpha \in Pos^{alg}(X) | \alpha \cdot C > 0 \text{ for all rational curves } C \subset X$$
$$Nef(X) = \alpha \in \overline{Pos}^{alg}(X) | \alpha \cdot C \geq 0 \text{ for all rational curves } C \subset X$$

Definition 4.6. *The "rational hull" of the ample cone $Nef^+(X)$ is the smallest convex cone containing the ample cone and all rational points of its boundary*

Remark: For some varieties these cones have a nice structure, e.g. for Fano varieties they are rational polyhedral. However in general they can be quite mysterious: they can have infinitely many isolated extremal rays or "round" parts. Both phenomena occur already for K3 surfaces.

Definition 4.7. *$Mov(X)$ is the convex hull in $NS(X) \otimes \mathbb{R}$ of all classes of movable*[5] *line bundles and $Mov^+(X)$ stands for keeping only the rationally defined part of the boundary.*

Proposition 4.3. *Let X be a projective hyperkähler manifold. We have*

$$\overline{Mov}(X) = \overline{Bir\mathcal{K}_X} \cap Pic(X)_{\mathbb{R}}$$

Example: Let S be a projective K3 surface and let $X = S^{[m]}$, with $m \geq 2$. We have

$$H^2(X, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta, \quad Pic(X) \simeq Pic(S) \oplus \mathbb{Z}\delta,$$

with $q_X(\delta) = -2(m-1)$. The irreducible hypersurface in X that parametrizes nonreduced subschemes has class 2δ; and it is therefore negative. The whole nef cone $Nef(S) \subset Pic(S)_{\mathbb{R}}$ corresponds to classes that are nef but not ample on X: it is contained in the boundary of $Nef(X) \subset Pic(X)_{\mathbb{R}}$. Hence,it is also contained in the boundary of the movable cone $Mov(X) \subset Pic(X)_{\mathbb{R}}$.

4.3. **(-2)-curves on K3 surfaces.** It is known since Shafarevich and Pyatetski-Shapiro's work [PS-S] in the algebraic case and Looijenga-Peters [**?**] in the compact Kähler case that some important aspects of the geometry of K3 surfaces are governed by smooth rational curves on these surfaces. By adjunction, the square of such a curve is always equal to -2.

Definition 4.8. *Curves with a square -2 are called (-2)-curves.*

[5] whose base locus has codimension ≥ 2 (no fixed divisors)

Remark: A fundamental theorem states that a line bundle on a K3 surface is ample if and only if it is of positive square and positive on all (-2)-curves. We also have an analogue in the non- algebraic case, with an ample line bundle replaced by a Kähler class. In other words, the orthogonal hyperplanes to the (-2)-curves bound the ample (resp. Kähler) cone inside the positive cone in $NS(X) \otimes \mathbb{R}$ (resp. $H^{1,1}(X))_{\mathbb{R}}$ The number of (-2)-curves on a K3 surface S is not necessarily finite.

Stark proved that there are finitely many of them up to the action of $Aut(S)$ [St].

Proposition 4.4. *For the projective K3 surface the group $Aut(S)$ has a polyhedral fundamental domain on $Nef^+(X)$.*

Remark: The Riemann-Roch formula implies that any divisor of square -2 on a K3 surface is effective, up to a $\pm$ sign. However not all Hodge classes of square -2 on a K3 surface are represented by (-2)-curves. Indeed the corresponding effective divisor can be reducible.

Observation: Nevertheless, for any (-2)-class z on a K3 surface X there is a deformation X' where $H^{1,1}(X') \cap H^2(X', \mathbb{Q}) = \langle z \rangle$. Indeed, the locus where a class α remains of type (1, 1) is equal to $\alpha^\perp$ in the space of local deformations $Def(X)$, identified with a neighbourhood of zero in $H^1(X, \Omega^1 X) \cong H^1(X, T_X)$, and a general element of the hyperplane $\alpha^\perp$ is not orthogonal to any other integral class.

4.4. **MBM classes.**

Definition 4.9. *Any birational map between hyperkähler manifolds $\varphi M \dashrightarrow M'$ is an isomorphism in codimension one, therefore induces an isomorphism on the second cohomology. We say that M and M' are birational models of each other.*

The observation from the previous section has the following generalization:

The space of deformations of X where a class $\alpha \in H^2(X, \mathbb{Z})$ is of type (1,1) can be described as the hyperplane $\alpha^\perp$. A general deformation X' of this type satisfies $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \langle \alpha \rangle$.

There are two cases: for a generic deformation X', some multiple of α is represented by a curve, or that no multiple of α is represented by a curve. By Theorem 4.2, in the second case every class in the positive cone of X' is Kähler.

In the first case, a multiple of α is represented by a rational curve and $\alpha^\perp$ defines the unique wall of the Kähler cone of X'. By the deformation theory this is also the case for any deformation x' with Picard group generated by α over the rationals.

Therefore, we have the following definition.

The MBM classes are defined as those classes whose orthogonal hy- perplanes support faces of the Kähler chambers.

Definition 4.10. *A negative integral cohomology class z of type (1,1) is called monodromy birationally minimal (MBM) if for some isometry $\gamma \in O(H^2(M, \mathbb{Z}))$ belonging to the monodromy group, $\gamma(z)^\perp \subset H^{1,1}(M, \mathbb{R})$. contains a face of the Kähler cone of one of birational models M' of M.*

Remark: Geometrically, the MBM classes are characterized among negative integral (1,1)-classes, as those which are, up to a scalar multiple, represented by minimal rational curves on deformations of M under the identification of $H^2(M, \mathbb{Q})$ with $H^2(M, \mathbb{Q})$ given by the BBF form. Roughly speaking, MBM classes are cohomology classes of negative BBF-square whose duals are represented by minimal rational curves on a deformation of M.

Remark: A face of Kähler cone by definition has dimension $h^{1,1}-1$. The z being MBM means that $\gamma(z)^{\perp} \cap \partial\mathcal{K}_{M'}$ contains an open subset of $\gamma(z)^{\perp}$. The MBM classes are analogues of "extremal rays" from projective geometry, up to monodromy and birational equivalence, that is the origin of a name. In fact when M is projective, those are exactly the monodromy transforms of extremal rays on birational models of M.

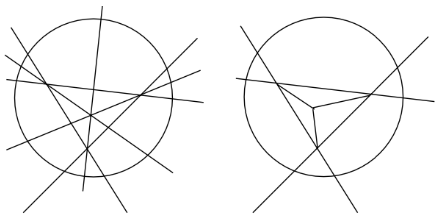
Example: MBM classes for K3 surfaces are (-2)-classes.

Theorem 4.3. ***(Amerik, Verbitsky,*** [AV1]***)***
Let M be a hyperkähler manifold, $z \in H^{1,1}(M)$ an integral cohomology class, $q(z,z) < 0$, and M' a deformation of M such that z remains of type (1,1) on M'. Assume that z is monodromy birationally minimal on M. Then z is monodromy birationally minimal on M'.

Using MBM classes we can describe Kähler cone explicitely.

Theorem 4.4. *Let M be a hyperkähler manifold, and $S \subset H^{1,1}(M)$ the set of all MBM classes of type (1,1). Consider the corresponding set of hyperplanes $S^{\perp} := \{z^{\perp} | z \in S\}$ in $H^{1,1}(M)$. Then the Kähler cone of M is a connected component of $Pos(M)\backslash \cup S^{\perp}$. Moreover, for any connected component K of $Pos(M)\backslash \cup S^{\perp}$, there exists $\gamma \in O(H^2(M,\mathbb{Z}))$ in the monodromy group of M and a birational model M' such that $\gamma(K)$ is the Kähler cone of M'.*

Remark: Note that for a negative integral class $z \in H^{1,1}(M)$, the orthogonal hyperplane $z^{\perp}$ either passes through the interiour of some Kähler-Weyl chamber and then it contains no face of a Kähler- Weyl chamber (it means that z is not MBM), or its intersection with the positive cone is a union of faces of such chambers (when z is MBM).



Allowed partition **Prohibited partition**

Hence, there are no "barycentric partitions" in the decomposition of the positive cone into the Kähler chambers.

4.5. **Kawamata-Morrison conjecture.**

Definition 4.11. *Let Γ be a group acting on a topological space X. A fundamental domain for the action of Γ is a connected open subset $D \subset X$ such that*

$$\cup_{\gamma \in \Gamma} \gamma \cdot \bar{D} = X$$

and the sets $\gamma \cdot D$ are pairwise disjoint.

Definition 4.12. *Let X be a subset of a metric space Y. A side of a convex subset $C \subset Y$ is a maximal nonempty convex subset of ∂C. A polyhedron in Y is a nonempty closed convex subset whose collection of sides is locally finite. A* *fundamental polyhedron for the action of a discrete isometry group Γ on X is a convex polyhedron D whose interior is a locally finite fundamental domain for Γ.* [6]

Definition 4.13. *Let V be a finite-dimensional real vector space equipped with a fixed $\mathbb{Q}$-structure. A rational polyhedral cone in V is a cone, which is an intersection of finitely many half spaces defined over $\mathbb{Q}$. In particular, such a cone is convex and has finitely many faces. For an open convex cone $\mathcal{C} \subset V$ we denote by $\mathcal{C}^+$ the convex hull of $\bar{\mathcal{C}} \cap V(\mathbb{Q})$.*

Conjecture 4.1. ***(Morrison-Kawamata cone conjecture)***
Let M be a Calabi-Yau manifold. Then the group $Aut(M)$ of biholomorphic automorphisms of M acts on the set of faces of $\mathcal{K}_M$ with finite number of orbits. And for the ample cone the automorphism group $Aut(M)$ has a finite polyhedral fundamental domain.

Remark: This conjecture is still wide open even for Calabi-Yau threefolds.

Remark: The statement about the finiteness of the orbits is often referred to as the *weak Morrison-Kawamata cone conjecture,* the strong one states that $Aut(M)$ has a finite polyhedral fundamental domain on ample cone, and on $Nef^+(M)$ The latter has a chance to be valid only in the algebraic setting, i.e. for the ample/movable cone rather than the Kähler/birational Kähler one. Indeed for a very general IHS X, the Kähler cone is round (it is equal to $Pos(X)$) whereas the group of (birational) automorphisms is trivial.

Theorem 4.5. ***(Sterk)***
The Cone Conjecture holds for any projective K3 surface S.

Remark: Let S is a very general K3 surface, then S is non-projective. Hence, the ample cone is empty but the Kähler cone is still rich. If S is very general, then $\rho(S) = 0$ and $Aut(S) = \{idS\}$. Moreover, S has no smooth rational curves and the Kähler cone of S coincides with the positive cone in $H^{1,1}(S)$, which is completely circular. This is also the unique fundamental domain as $Aut(S) = \{idS\}$. So the version of cone conjecture for the Kähler cone does not hold for a very general K3 surface.

Theorem 4.6. ***(Amerik-Verbitsky, Markman)*** *Let M be a projective simple hyperkähler manifold. Then*

(1) The group of automorphisms $Aut(M)$ acts with finitely many orbits on the set of faces of the Kähler cone $\mathcal{K}_M$.

(2) [AV2] *The group $Aut(M)$ has a finite polyhedral fundamental domain on $Nef^+(M)$.*

(3) ([M]*) The group $Bir(M)$ has a rational polyhedral fundamental domain on $Mov(M)^+$.*

MBM-classes have played the key role in study of Kawamata-Morrison conjecture, in particular, first Amerik-Verbitsky managed to prove the following theorem

Theorem 4.7. ***(Amerik, Verbitsky,*** [AV1]***)***
Let M be a simple hyperkähler manifold, and q the Bogomolov- Beauville-Fujiki form. Suppose that there exists $C > 0$ such that $|q(\eta, \eta)| < C$ for all primitive integral MBM classes. Then the Morrison-Kawamata cone conjecture holds for M.

[6]Local finiteness means that for each point $x \in X$ there is an open neighborhood U of x such that U meets only finitely many sets $\gamma \cdot \bar{D}, \gamma \in \Gamma$. Obviously, this also implies that every compact subset $K \subset X$ intersects only finitely many sets $\gamma \cdot \bar{D}$.

Definition 4.14. *Denote by $Mon^{Hdg}(M,I)$ the group of all oriented isometries of $H^2(M,\mathbb{Z})$ preserving the Hodge decomposition, we will call such isometries as Hodge isometries.*

In the other words, their complexifications preserve the line $H^{2,0}(M) \subset H^2(M,\mathbb{C})$.

Definition 4.15. *A Kähler-Weyl chamber of a hyperkähler manifold is the image of the Kähler cone of M' under some $\gamma \in Mon^{Hdg}(M,I)$, where M' runs through the set of all birational models of M.*

Then Theorem 4.4 says that the connected components of $Pos(M,I) \setminus S^\perp$ are Kähler-Weyl chambers of (M,I).

Theorem 4.8. *Let (M,I) be a hyperkähler manifold, and*

$$Mon(M,I) \subset O(H^2(M,\mathbb{Z}))$$

its monodromy group. Let G the image of $Aut(M)$ in $O(H^2(M,\mathbb{Z}))$. Then G is the set of all $\gamma \in Mon^{Hdg}(M,I)$ fixing the Kähler chamber.

Remark: The image of the mapping class group is a finite index subgroup in $O(H^2(M,\mathbb{Z}))$, and,then, $Mon^{Hdg}(M,I)$ is of finite index in the group of isometries of the Picard lattice.

Later they proved the boundedness assumption on squares of primitive MBM classes.

Theorem 4.9. *Let M be a simple hyperkähler manifold such that $b_2(M) \geq 6$. Then the primitive MBM classes of type (1,1) have bounded Beauville- Bogomolov square.*

4.6. **Finitness properties of automorphism groups.** By means of BBF-form, the signature of the Neron-Severi group $NS(M)$ is either $(1,0,\rho(M)-1), (0,1,\rho(M)-1), (0,0,\rho(M))$ where $\rho(M)$ is the Picard number of M. We call these three cases *hyperbolic, parabolic, elliptic* respectively.

Theorem 4.10. ***Huybrechts projectivity criterion,*** [H]
Let X be an IHS. Then X is projective iff $\mathcal{K}_X \cap H^2(X,\mathbb{Z}) \neq \emptyset$.

Remark: This follows from Demailly-Paun-Nakai-Moshezon theorem, which is quite hard to proof (using Demailly's regularization of currents).

Theorem 4.11. *Let X be a compact Kähler manifold. Then the Kähler cone of X is a connected component of the set of all classes $\alpha \in H^{1,1}(X,\mathbb{R})$ such that $\int_Y \alpha^d > 0$ for any irreducible analytic subset $Y \subset X$ of dimension d.*

By a theorem of Huybrechts, a holomorphic symplectic manifold M is projective if and only if it has an integral (1, 1)-class with strictly positive Beauville-Bogomolov square. In this case, the Picard lattice $H^{1,1}(M) = H^2(M,\mathbb{Z}) \cap_{\mathbb{Z}} H^{1,1}(M)$, equipped with the Beauville-Bogomolov form q, is a lattice of signature $(+,-,-,...,-)$. If M is not projective, the Picard lattice can be either negative definite, or degenerate negative semidefinite with one-dimensional kernel.

Let X be a hyperkähler manifold, let $Bir(X)$ be the group of its bimeromorphic automorphisms, and let $Aut(X)$ be the subgroup of its (biholomorphic) automorphisms. There are representations (by Exercise 5)

$$\Psi_B : Bir(X) \to O(H^2(X,\mathbb{Z}), q_X), \quad \Psi_A : Aut(X) \to O(H^2(X,\mathbb{Z}), q_X).$$

Proposition 4.5. ***(Huybrechts, Hassett–Tschinkel,*** [H]***)***
Let X be a hyperkähler manifold. The kernels of Ψ_A and Ψ_B are equal and finite, and they are invariant by deformation of X.

Idea of proof:
1. Consider bimeromorphic automorphism in the kernel of Ψ_B, it preserves a Kähler class, hence is biholomorphic (Proposition 4.2).
2. If an automorphism of X fixes a Kähler class, it fixes the unique Calabi–Yau metric in that class, therefore it is an isometry of the underlying Riemannian manifold.
3. We know that the group of isometries of a compact Riemanniann manifold is compact and the group $Aut(X)$ is discrete (since its Lie algebra is $H^0(X, T_X) = 0$), then kernel is finite.
4. The defromation invariance of the kernel is studied in [**?**]
Remark: In particular, the index of $\Psi_A(Aut(X))$ in $\Psi_B(Bir(X))$ is equal to the index of $Aut(X)$ in $Bir(X)$.

4.6.1. *Oguiso's results.*

Definition 4.16. *A group G is almost abelian of finite rank r if there are a normal subgroup $G' \lhd G$ of finite index and a finite group K which fit in the exact sequence*

$$id \to K \to G' \to \mathbb{Z}^r \to 0.$$

Then one has the following analogue of Tits' alternative for hyperkähler manifolds:

Theorem 4.12. ***(Oguiso,*** [Og] *Let M be a projective hyperkähler manifold and G be a subgroup of $Bir(M)$. Then G satisfies either:*
(1) G is an almost abelian group of finite rank, or
(2) G contains a non-abelian free group.

It is even simpler for non-projective case (recall that $NS(M)$ is not hyperbolic):

Theorem 4.13. ***(Oguiso,*** [Og]
Let M be a non-projective hyperkähler manifold. Then the group $Bir(M)$ (and hence $Aut(M)$) is almost abelian of rank at most $max(\rho(M) - 1, 1)$. In particular, these groups are finitely generated.
Moreover, (1) If $NS(M)$ is elliptic, then $Bir(M)$ falls into the exact sequence

$$1 \to N \to Bir(M) \to \mathbb{Z}^r \to 0$$

where N is a finite group and r is either 0 or 1. Moreover, $r = 0$ if $b_2(M) - \rho(M)$ is odd, i.e. the rank of transcendental lattice is odd. Moreover, in each dimension $2m$, both $r = 0$ and $r = 1$ are realizable.
(2) If $NS(M)$ is parabolic, then $Bir(M)$ is an almost abelian group of rank at most $\rho(M) - 1$. Moreover, in dimension 2, this estimate is optimal.

4.6.2. *Projective hyperkähler manifolds.*

Theorem 4.14. ***(Boissiere, Sarti, Cattaneo,Fu, Kurnosov,Yasinsky,*** [BS, CF, KY]***)***
For any projective hyperkähler manifold X, the automorphism group $Aut(X)$ and the birational automorphism group $Bir(X)$ are finitely presented

Boissier-Sarti has studied $Aut(X)$ via Torelli theorem. Cattaneo-Fu, and Kurnosov-Yasinsky proved finiteness property of $Bir(X)$.

Remark: Using Global Torelli Theorem, S. Boissiere and A. Sarti proved that $Bir(M)$ is finitely generated. This does not imply that $Aut(M)$ is finitely generated since $Aut(M)$ is not necessarily of finite index in $Bir(M)$.

Both proofs are based on Amerik-Verbitsky results on Kawamata-Morrison conjecture. The Cattaneo-Fu proof uses Kleinian groups, the proof of Kurnosov-Yasinsky based on the hyperbolic nature of the action. More precisely, it is proved that both groups are so-called CAT(0)-groups. From this the following strong form of Tits' alternative for hyperkähler manifolds was deduced:

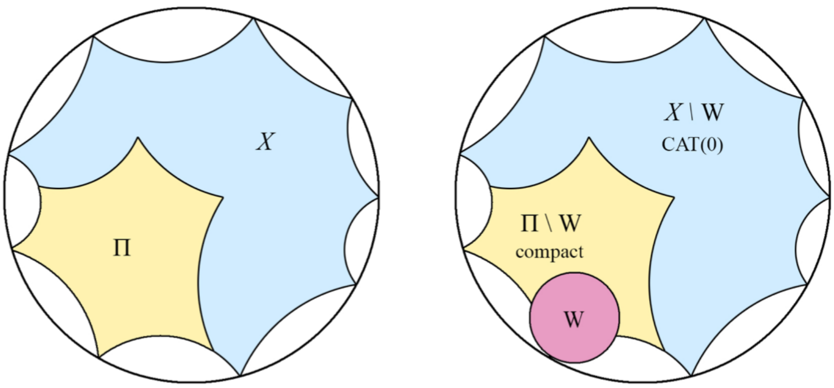
Theorem 4.15. *Let M be a projective hyperkähler manifold, and $G \subseteq Bir(M)$ be a subgroup. Then*

(1) either G contains a finite index subgroup isomorphic to $\mathbb{Z}^n$;

(2) or G contains a non-commutative free group. In particular, so are $Bir(M)$ and $Aut(M)$.

4.6.3. *A bit of hyperbolic geometry.* We give a idea of proving Theorem 4.14 following [KY].

Definition 4.17. *A horosphere on a hyperbolic space is a sphere which is everywhere orthogonal to a pencil of geodesics passing through one point at infinity, and a horoball is a ball bounded by a horosphere. A cusp point for an n-dimensional hyperbolic manifold $\mathbb{H}/\Gamma$ is a point on the boundary $\partial\mathbb{H}$ such that its stabilizer in Γ contains a free abelian group of rank $n-1$. Such subgroups are called maximal parabolic. For any point $p \in \partial\mathbb{H}$ stabilized by $\Gamma_0 \subset \Gamma$, and any horosphere S tangent to the boundary in p, Γ_0 acts on S by isometries. In such a situation, p is a cusp point if and only if $(S \setminus p)/\Gamma_0$ is compact.*



By Amerik-Verbitsky results there is fundamental domain of the action of automorphism group.

The group Γ acts on the image of ample/movable cone in hyperbolic space (since we are in the projective case) with a fundamental domain Π, which moreover has finitely many sides. There exists a finite family W of horocusp regions with disjoint closures such that $\Pi \setminus W$ is compact, we can make this collection such that they have disjoint closures and $X \setminus W$ is a CAT(0) space[7]. And action of Γ on $X \setminus W$ is properly discountinuous and cocompact.

4.7. **Symplectic automorphisms.** I would like to mention few results on the symplectic automorphisms.

Definition 4.18. *Symplectic automorphisms are automorphisms which preserve the holomorphic symplectic structure.*

In fact, the symplectic automorphisms of hyperkähler manifolds can be classified in the same way as automorphism of the hyperbolic plane (see below). There are hyperbolic, or, more precisely, loxodromic automorphisms (ones which act on $H^{1,1}(M)$ with two real eigenvalues of absolute value $\neq 1$), elliptic ones (automorphisms of finite order) and parabolic (quasiunipotent with a non-trivial rank 3 Jordan cell).

Recall that the BBF form has signature $(1, b_2-3)$ on $H^{1,1}(M)$. An automorphism of a hyperkähler manifold (M, I) is called elliptic (parabolic, hyperbolic) if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M, \mathbb{R})$. Namely, we have the following cases for an automorphism α.

Definition 4.19. *Let $n > 0$, and $\alpha \in SO^+(1, n)$ a non-trivial oriented isometry acting on $V = R^{1,n}$. Then one and only one of these three cases occurs:*
(i) α has an eigenvector x with $q(x, x) > 0$ ("an elliptic isometry")
(ii) α has an eigenvector x with $q(x, x) = 0$ and real eigenvalue λ_x satisfying $|\lambda_x| > 1$ ("hyperbolic isometry").
(iii) α has a unique eigenvector x with $q(x, x) = 0$ and eigenvalue 1, and no fixed points on $\mathbb{P}V$ ("parabolic isometry").

Remark: All eigenvalues of elliptic and parabolic isometries have ab- solute value 1. Hyperbolic and elliptic isometries are semisimple (i.e. the corresponding linear operators are diagonalizable over $\mathbb{C}$), parabolic are not.

Remark: When (V, q) has some underlying integral structure the elliptic isometries preserving the integral structure are of finite order. Parabolic ones are of infinite order, since any linear homomorphism of finite order is semisimple.

In our case $(V, q) = (H^{1,1}(M), q_{BBF})$.

Elliptic:
I am not going to talk about them in these Lectures. There are series of works of Mongardi and others. They have constructed some specific automorphisms and studied them for known examples of hyperkähler manifolds.

Hyperbolic:
By Exercise 7 we have the following theorem:

Theorem 4.16. ***(Amerik, Verbitsky,*** [AV3]***)***
Let M be a hyperkähler manifold with $b_2(M) \geq 5$. Then M has a deformation admitting a hyperbolic automorphism.

[7] i.e. that geodesic triangles are "not thicker" than Euclidean ones.

Parabolic:

Amerik-Verbitsky have proved that any hyperkähler manifold with $b_2 \geq 14$ admits a projective deformation with a parabolic automorphism.

Remark: The parabolic automorphisms are those which fix an isotropic vector. This vector must be rational, namely BBF-isotropic line bundle. Such line bundle is nef on a birational model of M, and there is conjecture that sections of this line bundle give a Lagrangian fibration. The interest to study parabolic automorophisms is that they preserve such fibrations which we will study in the Lecture 5.

4.8. **Exercises 4.**

(1) Using the Cauchy-Schwarz inequality check that the positive cone is convex.
(2) Describe a Kähler cone of complex torus
(3) Show that Kähler cone coincides with the positive cone if hyperkähler manifold is projective and the Picard number is one.
(4) If (S, L) is a very general polarized K3 surface, $X = S^{[2]}$, with L is ample of degree $2e$, so that $Pic(S) = \mathbb{Z}L$ and $Pic(X) = \mathbb{Z}L \oplus \mathbb{Z}\delta$, we have $q_X(aL + b\delta) = 2ea^2 - 2b^2$.
    Describe $Pos^{alg}(X)$.
(5) Let X and X' be hyperkähler manifolds with a bimeromorphic isomorphism $u : X \dashrightarrow^{\sim} X'$. There exist open subsets $U \subset X$ and $U' \subset X'$ (with codimension of the complement at least two, such subsets are called *big*) such that u induces an isomorphism $U \dashrightarrow^{\sim} U'$ and a Hodge isometry $u^* : (H^2(X, \mathbb{Z}), q_{X'}) \tilde{\to} (H^2(X, \mathbb{Z}), q_X)$.
(6) Let X be a very general compact hyperkähler manifold. The Kähler cone coincide with the set of all classes $\alpha \in H^{1,1}(X, \mathbb{R})$ such that $\int_Y \alpha^d > 0$ for any irreducible analytic subset $Y \subset X$ of dimension d.
(7) Let L be a non-degenerate indefinite lattice of rank ≥ 5, and N a natural number. Then L contains a primitive rank 2 sublattice Λ of signature (1,1) which does not represent numbers of absolute value less than N.
(8) Let Λ' be a lattice containing a sublattice spanned by two (-2)-classes c_1, c_2 with $(c_1, c_2) = 2n$. We let M' denote the Γ-orbit of $\{c_1, c_2\}$. If $N = \mathbb{Z}c_2$ and $\Lambda = N^{\perp}$, then $c_1 + nc_2$ has square $2(n^2 - 1)$ and is the orthogonal projection of c_1. Note that this situation is realized geometrically by K3 surface with two smooth rational curves intersecting in $2n$ points.

References

[AV] Amerik, E., Verbitsky, M., Rational curves on hyperkähler manifolds, Int. Math. Res. Not. IMRN 23 (2015), 13009–13045.

[AV1] Amerik,E.,Verbitsky,M.,Morrison–Kawamata cone conjecture for hyperkähler manifolds,Ann.Sci.Ec Norm. Sup.. 50 (2017), 973–993.

[AV2] Amerik E., Verbitsky, M., Rational curves and MBM classes on hyperkähler manifolds: a survey. arxiv: 2011.08727v1

[AV3] E. Amerik, M. Verbitsky, Collections of parabolic orbits in homogeneous spaces, homogeneous dynamics and hyperkahler geometry to appear in International Mathematics Research Notices, arxiv: 1604.03927.

[Bo] Boucksom,S.: Le cone kählerien d'une variete hyperkählerienne. C.R.Acad.Sci.Paris Ser.I Math.333, 935–938 (2001)

[BS] S. Boissiere, A. Sarti, A note on automorphisms and birational transformations of holomorphic symplectic manifolds, Proc. Amer. Math. Soc., 140(12): pp. 4053–4062, 2012.

[CF] A. Cattaneo, L. Fu Finiteness of Klein actions and real structures on compact hyperhahler manifolds, arxiv:1806.03864.

[DP] Demailly,J.-P.,Paun,M.:Numerical characterization of the Kähler cone of a compact Kähler manifold. Ann. Math. (2) 159, 1247–1274 (2004)

[H] D. Huybrechts, Compact Hyperkaehler Manifolds: Basic Results, arXiv:alg-geom/9705025 (alg-geom)

[H2] Huybrechts, D.: A global Torelli theorem for hyperkähler manifolds (after M. Verbitsky), Seminaire Bourbaki Exp. 1040. Astérisque 348, 375–403 (2012)

[L] Looijenga, E. Teichmüller spaces and Torelli theorems for hyperkähler manifolds. Math. Z. 298, 261–279 (2021).

[KS] Kreck, M., Su, Y.: Finiteness and infiniteness results for Torelli groups of (hyper-)Kähler manifolds. arXiv:1907.05693

[KY] N. Kurnosov, E. Yasinsky, Automorphisms of hyperkähler manifolds and CAT(0)-groups, arxiv: 1810.09730

[M] Markman, E., A survey of Torelli and monodromy results for holomorphic-symplectic varieties, in Complex and differential geometry, 257–322, Springer Proc. Math. 8, Springer, Heidelberg, 2011.

[Og] K. Oguiso, Bimeromorphic automorphism groups of non-projective hyperkähler manifolds – a note inspired by C. T. McMullen, J. Diff. Geom., 78, 1 (2008), 163–191.

[P] P. Voegtli, Bakker-Lehn proof of Torelli theorem for the smooth case, LSGNT project (will be on my webpage as well)

[PS-S] I. Pyatetski-Shapiro, I. Shafarevich: Torelli's theorem for algebraic surfaces of type K3. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 35 1971 530–572

[St] H. Sterk, Finiteness results for algebraic K3 surfaces. Math. Z. 189 (1985), no. 4, 507–513.

[V2] Verbitsky, M.: Mapping class group and a global Torelli theorem for hyperkähler manifolds, Appendix A by Eyal Markman. Duke Math. J. 162, 2929–2986 (2013) (Erratum: Duke Math. J. 169, 1037–1038 (2020)), and Erratum to mapping class group and a global Torelli. arXiv:1908.11772.pdf