3. Lecture 3

Abstract:
In this lecture I will cover LLV-decomposition and some known results on the bounds of the second Betti number.

In the second part of the lecture I will discuss Torelli theorem, in particular, defining Teichmüller space, moduli space of marked hyperkähler manifolds and discussing the issue of non-Hausdorffness.

I am also thankful to Pascale Voegtli who did a project on the topic of Bakker-Lehn proof of the general Torelli theorem ([P]).

3.0.1. *LLV-decomposition.* We are going to formulate some results on the algebra $\mathfrak{so}(\tilde{H}, \tilde{q})$ defined above.

Definition 3.1. *The Looijenga–Lunts–Verbitsky (LLV) algebra $\mathfrak{g}(X)$ of X is the Lie subalgebra of $\mathfrak{g}(H^*(X,\mathbb{Q}))$ generated by all formal Lefschetz and dual Lefschetz operators associated to almost all elements in $H^2(X,\mathbb{Q})$.*

Corollary 3.1. ***(of Theorem 2.6),*** [LL, V1] *Let X be a hyperkähler manifold, and $r = [b_2(X)/2]$. The LLV algebra $\mathfrak{g}$ is a simple Lie algebra of type B_{r+1} or D_{r+1}, depending on the parity of $b_2(X)$.*

Definition 3.2. *We have*

$$H^*(X,\mathbb{Q}) = \bigoplus_{\mu} V_{\mu}^{\oplus m_{\mu}},$$

with V_μ the irreducible $\mathfrak{g}$-module of highest weight μ. We call the decomposition above the LLV decomposition; it is a basic diffeomorphism invariant of X.

3.0.2. *Verbitsky representation.* There is one representation which appears for all hyperkähler manifolds, known and not discovered ([V1, ?].

Theorem 3.1. *Let X be a compact hyperkḧler manifold X of dimension $2n$. Then the subalgebra $SH^2(X,\mathbb{Q}) \subset H^*(X,\mathbb{Q})$ generated by $H^2(X,\mathbb{Q})$ is an irreducible $\mathfrak{g}$-module $V_{(n)} \subset Sym^n V$ of highest weight $\mu = (n)$.*

This representation was first described by Verbitsky, and later Bogomolov, sometimes it is also called $F-algebra$.

Here is the construction:

Let V be a vector space over an algebraically closed field of characteristic zero. Let q be a non-degenerate scalar product on V , and $S^\bullet V$ the symmetric algebra.

Multiplication by q^k[3] gives a natural embedding $S^iV \to^{q^k} S^{i+2k}V$. Denote by $R_{n,k}(V) \subset S^{n+k}V$ the orthogonal complement to the image of $S^{n-k}V \to S^{n+k}V$.

Definition 3.3. *Let $F_n^\bullet(V)$ be the quotient of $S^\bullet V$ by the ideal generated by $\cup_k R_{n,k}(V)$, with the grading multiplied by two, so that $F^{2i}(V)$ is the quotient of S^iV . The algebra $F_n^\bullet(V)$ is called **the n-th F-algebra** of V.*

Remark: This is an even-graded algebra, with $dimF_{4n}(V) = 1$.

Remark: By Bogomolov the F-algebra has the following description

$$F_n^\bullet(V) \simeq \frac{S^\bullet}{\langle x^{n+1} | x \in V,\ q(x,x) = 0\rangle}$$

[3]We may identify V and V^* via q and consider q as an element of S^2V

Idea of proof of Theorem 3.1: 1. We have the decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus (\mathfrak{so}(H, q) \oplus \mathbb{Q}\Xi) \oplus \mathfrak{g}_{-2}$ as in Exercise 3. It is enough to show $SH^2(X, \mathbb{Q})$ is closed under the $\mathfrak{g}_2$-action and $\mathfrak{g}_{-2}$-action.

2. Clearly $SH^2(X, \mathbb{Q})$ is closed under $\mathfrak{g}_2$. The vector space $\mathfrak{g}_{-2}$ is generated by the operators Λ_ω. To prove that $SH^2(X, \mathbb{Q})$ is closed we compute for any $x_1, ..., x_k \in H^2(X, \mathbb{Q})$, we have

$$\Lambda_\omega(x_1 x_2 \cdots x_k) = \Lambda_x(L_{x_1}(x_2 \cdots x_k)) = [L_{x_1}, \Lambda_\omega](x_2 \cdots x_k) - L_{x_1}(\Lambda_\omega(x_2 \cdots x_k)).$$

3. The first term is contained in $SH^2(X, \mathbb{Q})$. For the second one we use the induction on k.

Definition 3.4. *We call $SH^2(X, \mathbb{Q}) \simeq V_{(n)}$ the Verbitsky component of $H^*(X, \mathbb{Q})$.*

As we have seen above it looks in the following way

$$SH^2(X, \mathbb{Q})_{2k} = \begin{cases} Sym^k(H^2(X, \mathbb{Q})), \ 0 \leq k \leq n \\ Sym^{2n-k} H^2(X, \mathbb{Q}), \ n < k \leq 2n \end{cases}$$

3.0.3. *Kuga-Satake construction.* Recall that the Hodge structures of weight one are all of geometric origin. There is a natural bijection between the set of isomorphism classes of integral Hodge structures of weight one and the set of isomorphism classes of complex tori, and analogous bijection in the case of polarizable Hodge structures of weight one:

complex tori $\longleftrightarrow$ integral Hodge structures of weight one ,
abelian varieties $\longleftrightarrow$ polarizable integral Hodge structures of weight one

K3 surface has Hodge structure on the second cohomology, such that $h^{2,0} = 1$.

Definition 3.5. *We call V a Hodge structure of K3 type if V is a (rational or integral) Hodge structure of weight two with*

$$dim_{\mathbb{C}}(V^{2,0}) = 1, \quad V^{p,q} = 0 \ for \ |p - q| > 2.$$

Kuga-Satake construction is a way to get Hodge structure of weight 1 starting with K3-type Hodge structure V with quadratic form q.

Let $I := I(q) \subset T(V)$ be the two-sided ideal generated by the even elements $v \otimes v - q(v), v \in V$.

Recall that the Clifford algebra is the quotient algebra

$$Cl(V) := Cl(V, q) := T(V)/I(q).$$

The Clifford algebra has a natural $\mathbb{Z}/2\mathbb{Z}$-grading, so it consists of odd and even parts.

Now let us briefly recall the Kuga-Satake construction:

1.Let $\sigma = e_1 + ie_2$ be a generator of $V^{2,0}$with $e_1, e_2 \in V_{\mathbb{R}}$ and $q(e_1) = 1$.

2. Since $q(\sigma) = 0$ we have $e_1 \cdot e_2 = -e_2 \cdot e_1$ in $Cl(V_R)$, hence left multiplication with $J := e_1 \cdot e_2$ induces a complex structure on the real vector space $Cl(V_{\mathbb{R}})$.

3. Now one defines the *Kuga–Satake Hodge structure* as the Hodge structure of weight one on $Cl_+(V)$ given by

$$\rho : \mathbb{C}^* \to GL(Cl_+(V)_{\mathbb{R}}), \quad x + yi \mapsto x + yJ.$$

Remark: There is a natural bijection between rational Hodge structures of weight n on a rational vector space V and algebraic representations $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$ with $\mathbb{R}^*$ acting by $\rho(t)(v) = t^n \cdot v$. This leads to *Deligne torus* construction.

Definition 3.6. *The Kuga–Satake variety associated with the integral Hodge structure V of weight two is the complex torus.*

$$KS(V) := Cl_+(V_{\mathbb{R}})/Cl_+(V).$$

Remark: The dimension of KS variety is 2^{n-2}, where $n = dim_{\mathbb{C}} V_{\mathbb{C}}$.

Proposition 3.1. *The Kuga–Satake construction*

$$KS: \text{ Hodge structures of K3 type } \hookrightarrow \text{ Hodge structures of weight one}$$

is injective.

If we just apply the classical Kuga-Satake construction to $(H^2(M), q_M)$, we will lose all the information on the higher cohomology structure and LLV-algebra action. However, the classical Kuga-Satake construction has been generalized by the following theorem.

Theorem 3.2. ***(Kurnosov, Soldatenkov, Verbitsky,*** [KSV]***)***
Let M be a hyperkähler manifold. There exists an integer $l \geq 0$, a complex torus T, an embedding $\mathfrak{g}_{tot}(M) \hookrightarrow \}_{tot}(T)$ of Lie algebras, and an embedding Ψ : $H^\bullet(M,\mathbb{C}) \hookrightarrow H^{\bullet+l}(T,\mathbb{C})$ of $\mathfrak{g}_{tot}(M)$-modules. For each complex structure I of hyperkähler type on M there exists a complex structure on T such that Ψ is a morphism of Hodge structures.

3.1. **Rozansky-Witten invariants.** There is a deep theory of Rozansky-Witten invariants – developed by Rozansky, Witten, Kontsevich, Kapranov. In the hyperkähler case some computations were done in the works of Sawon and Hitchin. Namely, they computed RW invariants for the most simple graphs.

Let us recall the definition of the invariants in the case of hyperkähler manifolds and trivalent graphs and then state the result which later allowed Guan to get some bounds on b_2.

Sawon-Hitchin approach, [S, HS]**:**

Consider the Riemann curvature tensor R as a section $K \in \Omega^{1,1}(EndT)$ with components $K^i_{jk\bar{l}}$ relative to local complex coordinates. Using the non-degenerate holomorphic 2-form Ω to identify Θ_X and Ω_X, we can lower the first index and define

$$\Phi \in \Omega^{1,1}(\Omega_X \otimes \Omega_X) = \Omega^{0,1}(\Omega_X \otimes \Omega_X \otimes \Omega_X)$$

by

$$\Phi_{ijk\bar{l}} = \sum_m \Omega_{im} G K^m_{jk\bar{l}}$$

Remark: It is symmetrical in j, k because the connection is torsion-free and preserves the complex structure. It is also symmetrical in i, j because the the curvature takes values in the Lie algebra of $Sp(2k,\mathbb{C})$.

Thus

$$\Phi \in \Omega^{0,1}(Sym^3 \Omega_X)$$

and by Bianchi identity it is $H^1(M, Sym^2\Omega_X)$.

Let Γ be a trivalent (oriented)[4] graph with $2k$ vertices and no edges joining a vertex to itself.

[4]Two such orderings are equivalent if they differ on an even number of vertices

Choose an ordering of the vertices and consider the tensor of $2k$ copies of Φ. If vertex v_m and vertex v_n of the graph Γ are joined by an edge and $m < n$ then we contract with the skew form $\tilde{\Omega}$ on Ω_X dual to Ω:

$$c_{m,n}\Omega^{ij}\Phi \otimes ... \otimes \Phi_{i_m} \otimes ... \otimes \Phi_{i_n} \otimes ... \otimes \Phi$$

where $c_{m,n} = \pm$ depending if the orientation on the edge goes from v_m to v_n, or not. Continuing over all $3k$ edges, and projecting this to the exterior product, we obtain $(0, 2k)$-form $\Gamma(\Phi)$

Definition 3.7. *The Rozansky-Witten invariant of M of $dim_{\mathbb{R}} = 4k$ defined by the graph Γ is*

$$b_\Gamma(M) := \frac{1}{(8\pi^2)^k k!} \int_M \Gamma(\Phi)\Omega^k$$

Remark: The particular factor in the integral above is chosen to make the invariant have multiplicative properties.

There is exercise 2 which allows to compute RW invariant of graph θ^k. By the bubbling formula proved by Sawon one have

Theorem 3.3. ***(Sawon)***

$$-b_{\theta^l} \leq (b_2 + 2(k-1))b_{\theta^{k-2}\theta_2}$$

3.2. **Bounds on b_2.** We will review some particular know results on LLV-decomposition and discuss bounds on the second Betti number.

3.2.1. *Guan's results.*

Proposition 3.2. ***(Salamon,*** [Sa]***)***
Let X be a HK manifold of dimension $2n$. Then

$$nb_{2n}(X) = 2\sum_{i=1}^{2n}(-1)^i(3i^2 - n)b_{2n-i}(X).$$

Remark: This is the Riemann-Roch formula for irreducible compact hyperkähler manifolds of dimension $2n$.

Theorem 3.4. ***(Guan,*** [GKLR]***)***
Let X be hyperkähler fourfold then
(1) $b_2 \leq 23$.
(2)

$$b_3 \leq \frac{4(23 - b_2)(b - b_2}{b_2 + 1}$$

If $b_2 \geq 7$, then $(b_2, b_3) \in (8, 0), (23, 0)$ If $b_2 \leq 7$, then one of the following holds:

b_2	*3*	*4*	*5*	*6*	*7*
b_3	$4l, l \leq 17$	$4l, l \leq 15$	$4l, l \leq 9$	$4l, l \leq 4$	*0,8*

Idea of proof:

(1) From Salamon equation and Verbitsky's embedding $Sym^2H^2(X) \hookrightarrow H^4(X)$ it follows that $b_2 \leq 23$

(2) Guan proved inequality for b_3 using some direct computations with topological invariants. However, it is just corollary of Theorem 3.3

Remark: If $b_2 = 23$ the Hodge diamond of X is the same as the Hodge diamond of the Hilbert square of a K3 surface.When $b_2 = 7$, either $b_3 = 04$ or the Hodge diamond of X is the same of the Hodge diamond of a Kummer variety.

Remark: Results of Guan were generalized to low higher dimensions by Sawon, Kurnosov and by Fu-Menet to orbifolds of dimension four.

3.2.2. *Conjecture on b_2.* Moreover, Sawon-Kurnosov using Guan's motivation conjectured the following bound on b_2:

Conjecture 3.1.

$$b_2 \leq \frac{21 + \sqrt{96n + 433}}{2}, \quad if H^*_{odd}(X) = 0$$

Remark: This conjecture is related to so called Nagai's conjecture below. Let $\mathcal{X}/\Delta$ be a one-parameter projective degeneration of hyperkähler manifolds. Similar to the K3 case, it is natural to define the Type of the degeneration to be I, II, or III, in accordance to the index of nilpotence ν_2 of the log monodromy operator $N_2 = log(T_2)u$ on $H^2(X)$. As the hyperkähler manifolds are controlled by their second cohomology, one might expect some tight connection between the second monodromy and higher monodromies.

Conjecture 3.2. *(Nagai) The index of nilpotency ν_{2k} of log monodromy N_{2k} on $H^{2k}(\mathcal{X}_t)$ satisfies*

$$\nu_{2k} = k\nu_2$$

for $k = 1, ..., n$.

Green-Roblees-Kim-Laza have shown that all known examples satisfy the condition, which is even stronger then Nagai's conjecture

Theorem 3.5. ***(Green-Roblees-Kim-Laza,*** [?]***)***
Let X be a $2n$-dimensional hyperkähler manifold of $K3^{[n]}, K_n(T), OG_6$, or OG_10 type. Then any irreducible $\mathfrak{g}$-module component V_μ occurring in the LLV decomposition of $H^*(X)$ satisfies

$$\mu_0 + \;\; + \mu_{r-1} + |\mu_r| \leq n.$$

Remark: All known examples satisfy this conjecture. Furthermore, the equality holds for the Verbitsky component, an irreducible $\mathfrak{g}$-submodule with highest weight $\mu = (n, 0, ..., 0)$ that is always present in $H^*(X)$. In the other words inequality says that Verbitsky component is maximal one.

Using this motivation Kim and Laza have deduced Conjecture 3.1 from the inequality from Theorem 3.5:

Theorem 3.6. *(Laza-Kim) Let X be a compact hyperkähler manifold of dimension $2n$. If the inequality in Theorem 3.5 holds for X, then*

$$b_2 \leq \begin{cases} \frac{21+\sqrt{96n+433}}{2}, & if H^*_{odd}(X) = 0 \\ 2k+1, & if H^k \neq 0 \; for \; odd \; k \end{cases}$$

3.3. Monodromy groups.

Definition 3.8. *Let M be a manifold, $Diff(M)$ its diffeomorphism group, and $Diff_0(M)$ the connected component of unity in $Diff(M)$. The quotient group $\Gamma = Diff(M)/Diff_0(M)$ is called the mapping class group*

Theorem 3.7. *The group Γ acts on the set of connected components of the set of all $Sp(n)$-metrics on M. The quotient by this action is finite.*

Definition 3.9. *The monodromy group $Mon(M)$ of a hyperkähler manifold M is a subgroup of $GL(H^*(M,\mathbb{Z}))$ generated by the monodromy of the Gauss- Manin local systems, for all holomorphic deformations of M over a connected complex analytic base.*

3.4. **Period map and Teichmuller spaces.**

Definition 3.10. *The period domain associated to Λ is the set*

$$\Omega_\Lambda := \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | q(x) = 0,\ q(x, \bar{x}) > 0\}.$$

3.4.1. *Teichmüller space $Teich$.* Now we introduce several spaces which serve as the source of Period map.

Definition 3.11. *Let (M, I) be a compact hyperkähler manifold, $\mathcal{I}$ the set of oriented complex structures of hyperkähler type on M, and $Diff_0(M)$ the group ofisotopies. The quotient space $Teich := \mathcal{I}/Diff_0(M)$ is called the Teichmüller space of (M, I), and the quotient of $Teich$ over a whole oriented diffeomorphism group the coarse moduli space of (M, I).*

Definition 3.12. *For any $J \in Teich$, (M, J) is also a simple hyperkähler manifold*

Remark: The fact that (M, J) is also simple hyperkähler manifold follows from the following statement due to Verbitsky: Let M be a compact hyperkähler manifold, which is homotopy equivalent to a simple hyperkähler manifold. Then M is also simple.

Remark: By assigning to a hyperkählerian complex structure on M the associated Hodge decomposition on H^2, we obtain the period map.

Definition 3.13. *Consider a map $\mathfrak{Per} : Teich \to \mathbb{P}H^2(M, \mathbb{C})$,sending J to the line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. Clearly, $\mathfrak{Per}$ maps $Teich$ into an open subset of the quadric, defined by*

$$\{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | q(x) = 0,\ q(x, \bar{x}) > 0\}$$

The map $\mathfrak{Per} : Teich \to \Omega$ is called the period map.

Definition 3.14. *Define the Torelli group $\mathcal{K}$ of M as the subgroup of all elements of the mapping class group Γ of M acting trivially on $H^2(M)$.*

Definition 3.15. *The marked moduli space of M is the quotient $Teich/\mathcal{K}$. is called the coarse, marked moduli space of complex structures, and its points – marked hyperkähler manifolds. To choose a marking it means to choose a basis in the cohomology of M. The period map is well defined on it as well.*

Remark: It follows that each connected component of $Teich/\mathcal{K}$ is diffeomorphic to the corresponding connected component of $Teich$, because $\mathcal{K}$ acts by permuting isomorphic connected components of $Teich$.

This theorem gives us a description of monodromy group in context of mapping class group

Theorem 3.8. ***(Verbitky,*** [V2]***)*** *Let (M, I) be a hyperkähler manifold, and $Teich^I$ the corresponding connected component of a Teichmüller space. Denote by Γ^I the subgroup of the mapping class group preserving the component $Teich^I$, and let $Mon(M)$ be the monodromy group of (M, I) defined above. Then $Mon(M)$ coincides with the image of Γ^I in $GL(H^*(M, \mathbb{Z}))$.*

Theorem 3.9. ***(Bogomolov)***
Let M be a simple hyperkähler manifold, and $Teich$ its Teichmüller space. Then the period map $\mathfrak{Per} : Teich \to \Omega$ is locally an unramified covering.

Remark: Bogomolov's theorem implies that $Teich$ is smooth. However, it is not necessarily Hausdorff (and it is non-Hausdorff even in the simplest examples). We will discuss non-Hausdorffness later.

Remark: Using the boundedness results of Kollar and Matsusaka, D. Huybrechts has shown that the space $Teich$ has only a finite number of connected components:

Theorem 3.10. ***(Huybrechts,*** [H1]***)*** *Let M be a fixed compact manifold. Then there exist at most finitely many different deformation types of irreducible holomorphic symplectic complex structures on M.*

Remark: Note that the argument does not show that the number of connected components of $\mathcal{M}_\Lambda$ that parametrizes complex structures on a fixed manifold M is finite. A priori, this seems only to be the case modulo the action of $Aut(\Gamma)$ (as it was stated in Theorem 3.7.

Remark: To work over Tecihmüller spaces is rather complicated than over moduli of marked hyperkählers.

We will follow Loojenga approach ([**?**]) to the proof of Torelli theorem, so we will use his way to define (topology of) Teichmüller spaces.

3.4.2. *Teichmüller space $\mathcal{J}$.* From now on, we fix a compact simply-connected manifold M of dimension $4m$ which admits an irreducible hyperkähler structure.

We start with $Teich$ as the set, namely set the set of hyperkähler structures on M given up to isotopy.

Structure of manifold on $\mathcal{J}$: Consider an atlas whose charts are of the following type: Given an open subset U of Ω, then let us agree that a basic chart for $\mathcal{J}$ with domain U is given by a complex structure on $M \times U$ for which the resulting complex manifold $\mathcal{X}$ has the property that
(i) the projection $\mathcal{X} \to U$ is holomorphic,
(ii) the fibers of $\mathcal{X} \to U$ are hyperkähler manifolds and
(iii) its period map is given by the inclusion of U in Ω.

Remark: The basic charts cover all of $\mathcal{J}$. Indeed, it is clear that such an object defines an injection of U in $\mathcal{J}$. By the local Torelli theorem, every hyperkähler complex structure on M appears as a member of such a family.

Proposition 3.3. *Our atlas is complex-analytic and that it gives $\mathcal{J}$ the structure of a (non-separated) complex manifold for which $\mathfrak{Per}$ is a local isomorphism.*

3.4.3. *Non-separation for hyperkähler manifolds.* It is well-known for K3 surfaces that moduli space is not Hausdorff.

Example (Atiyah): Consider the following 1-parameter family S_t of quartic surfaces in $\mathbb{P}^3$, which is given in affine coordinates by

$$x^2(x^2-2)+y^2(y^2-2)+z^2(z^2-2)=2t^2.$$

Let the parameter t vary in a small neighbourhood B of the origin, for $t \neq 0$ these surfaces are smooth, whereas the surface S_0 has an ordinary double point at $(0,0,0)$, this singularity is the only one of the total space $\mathcal{S}$. After blow up of this node one obtains a smooth 3-fold $\tilde{\mathcal{S}}$ with a quadric $\mathbb{P}^1 \times \mathbb{P}^1$ as exceptional divisor E. Then the proper transform $\hat{S}_0$ of S_0 is a smooth K3 surface intersecting the exceptional divisor E in a rational curve of bidegree $(1,1)$. This is a nodal curve on $\hat{S}_0$. Both rulings on E can each be contracted giving rise to smooth 3-dimensional spaces $p_1 : \mathcal{S}_1 \to B$ and $p_2 : \mathcal{S}_2 \to B$, which coincide over $B \setminus \{0\}$.

Remark: They are not identical over all of B, since the identity on $\mathcal{S}\S_0$ would, otherwise, have to extend to an automorphism of the total space acting non-trivially on the tangent cone of the double point, which is clearly impossible.

Now choose a marking for p_1, it defines a marking for p_2. Period maps coincide away from 0, and differ on the central fibre.

This shows that $\mathcal{M}$ cannot be Hausdorff.

Remark: In fact all non-Hausdorff behaviour comes from the existence of different resolutions of double points in families.

Definition 3.16. *Let M be a topological space. We say that points $x, y \in M$ are inseparable (denoted $x \sim y$) if for any open subsets $U \ni x, V \ni y$, one has $U \cap V \neq \emptyset$.*

Theorem 3.11. ***(Huybrechts,** [H2]**)***

Let M be a hyperkähler manifold, $\mathcal{M}$ its marked moduli space, and $x, y \in \mathcal{M}$ points corresponding to hyperkähler manifolds M_x and M_y. Suppose that x and y are inseparable. Then the manifolds M_x and M_y are bimeromorphically equivalent. Conversely, if M_1 and M_2 are bimeromorphically equivalent, they can be realised as inseparable points on the Teichmüller space.

This inseparability condition is used everywhere to define a Hausdorff spaces and work with them in Verbitsky's original work and in Huybrecht's survey on Verbitsky's work to define "Hausdorff reduction" of moduli space $\mathcal{M}$.

Theorem 3.12. *Let $Teich$ be a Teichmüller space of a hyperkähler manifold, and $\sim$ the inseparability relation defined above. Then $\sim$ is an equivalence relation. Moreover, the quotient $Teich_b := Teich/\sim$ is a smooth, Hausdorff complex analytic manifold.*

Definition 3.17. *We call the quotient $Teich/\sim$ the birational Teichmüller space, denoting it as $Teich_b$.*

Clearly, the map $\mathfrak{Per} : Teich_b \to Per$ is well defined.

Remark: However, it is not straightforward that the inseparability relation defined in 3.16 is transitive. For the moduli spaces it is much easier and it was done in [H2]. This problem brings us back to Loojenga's work on Torelli theorem and his approach of defining Teichmüller space $\mathcal{J}$.

Defining separated Teichmüller space $\mathcal{J}_s$

Definition 3.18. *Define separated Teichmüller space $\mathcal{J}_s$ as follows: two hyperkähler complex structures on M which give complex manifolds X and X', define the same point of $\mathcal{J}_s$ if and only if there exist basic charts $\mathcal{X}/U, \mathcal{X}'/U$ containing X resp. X' over the same open subset $U \subset \Omega$, and a sequence $(z_i \in U)_{i=1}^{\infty}$ converging to some $\wr \in U$ such that X_{z_i} and X'_{z_i} differ by a C^∞-isotopy and $X_\wr = X$ and $X'_\wr = X'$, or equivalently, if there exists a bimeromorphic equivalence $f : X \dashrightarrow X'$ whose associated isotopy class $[f]$ is that of the identity of M.*

Remark: The space $\mathcal{J}_s$ is indeed a separated complex manifold and the period map factors through the separated period map

$$\mathfrak{Per}_s : \mathcal{J}_s \to \Omega$$

which of course is still a local isomorphism.

Remark: With abuse of notation we will use $\mathfrak{Per}$ for both $\mathcal{J}$ and $\mathcal{J}_s$

3.4.4. *Teich vs $\mathcal{J}$.* $Teich$ is defined as the orbit space of the space of hyperkähler structures on M with respect to the action of the group of diffeomorphisms isotopic to the identity.

Atlas of $\mathcal{J}$ consists of certain sections to orbits, and so we have a priori a map $T \to Teich$ for which $Teich$ has the quotient topology.

Proposition 3.4. *This map is homeomorphism.*

Indeed, this map clearly is a bijection.

Equivalence relation for $Teich$ for $\mathcal{J}$ this is just the relation that says that the two points lie in the same fiber of $\mathcal{J} \to \mathcal{J}_s$. Hence it is an equivalence relation, so that our $\mathcal{J} \to \mathcal{J}_s$ can be identified with Verbitsky's $Teich \to Teich_b$.

3.4.5. *Teichmüller spaces vs Moduli spaces of marked HK.* A finitness result (Theorem 3.7) implies that $\mathcal{K}$ acts properly on the connected component set $\pi_0\mathcal{J}$ of $\mathcal{J}$ and has only finitely many orbits in $\pi_0\mathcal{J}$.

Example:

3.5. **Surjectivity of the period map.**

Theorem 3.13. ***(Huybrechts, Surjectivity of the period map)***

Let $\mathcal{M}_\Lambda^0$ be a connected component of the moduli space $\mathcal{M}_\Lambda$ of marked hyperkähler manifolds. Then the restriction of the period map

$$\mathfrak{Per} : \mathcal{M}_\Lambda^0 \to \Omega_\Lambda$$

is surjective.

Any two points $x, y \in \Omega$ are (strongly) equivalent.

Idea of proof:

1. It is enough to show that $x \in \mathfrak{Per}(\mathcal{M}^0)$ iff $y \in \mathfrak{Per}(\mathcal{M}^0)$ for any two points $x, y \in D(W) \subset \Omega$.

2. Indeed, the generic twistor line $D(W)$ can be lifted through any preimage (X, φ) of x, then y is contained in the image of the lift.

3.5.1. *Twistor line.*

Definition 3.19. *A subspace $W \subset \Lambda \otimes \mathbb{R}$ of dimension three such that $q|_W$ is positive definite is called a positive three-space.*

Definition 3.20. *For any positive three-space W one defines the associated twistor line $D(W)$ as the intersection*

$$D(W) := \Omega_\Lambda \cap \mathbb{P}(W \otimes \mathbb{C}).$$

Remark: For W a positive three-space, $\mathbb{P}(W \otimes \mathbb{C})$ is a plane in $\mathbb{P}(\Lambda \otimes \mathbb{C})$ and $D(W)$ is a smooth quadric in $\mathbb{P}(W \otimes \mathbb{C}) \simeq \mathbb{P}^2$. Thus, as a complex manifold $D(W)$ is simply $\mathbb{P}^1$. Two distinct points $x, y \in \Omega_\Lambda$ are contained in one twistor line if and only if their associated positive planes $P(x)$ and $P(y)$ span a positive three-space $\langle P(x), P(y)\rangle \subset \Lambda \otimes \mathbb{R}$.

Remark: In Loojenga's paper they are called twistor conics. It does make sense since twistor line is also used for a section of a twistor deformation.

3.5.2. *Twistor deformation.* Hyperkähler metric comes with a sphere of complex structures $L = aI + bJ + cK$, each (M, L) is again a complex manifold of hyperkähler type.

Definition 3.21. *The twistor space associated to Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$ is the complex manifold $\mathcal{X}$ described by the complex structure $\mathrm{I} \in End(T_m M \oplus T_L \mathbb{P}^1)$, given by $(v, w) \mapsto (L(v), I_{\mathbb{P}^1}(w))$ on the differentiable manifold $M \times \mathbb{P}^1$, where $I_{\mathbb{P}^1}$ is the standard complex structure on $\mathbb{P}^1$.*

Clearly, we have holomorphic map $\mathcal{X} \to \mathbb{P}^1$. By construction, the twistor space is a family of complex structures on a fixed manifold M. The period map is a holomorphic map $\mathfrak{Per} : \mathbb{P}^1 \to \Omega$. Namely, $\mathfrak{Per}$ identifies $\mathbb{P}^1$ with the twistor line $D(W_\alpha)$, which is associated to the positive three-space $W_\alpha := \langle [\omega_I], [Re(\sigma_I)], [Im(\sigma_I)]\rangle = \mathbb{R}\alpha \oplus (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$.

3.6. Torelli Theorem.

3.6.1. *Torelli theorem for K3.* Let us first recall the global Torelli theorem for K3:
By definition $\mathfrak{M} = \{(S, \varphi)\}/_{\simeq}$

Theorem 3.14. ***(Global Torelli for K3 surfaces)*** *The moduli space $\mathfrak{M}$ has two connected components interchanged by $(S, \varphi) \mapsto (S, -\varphi)$ and the period map*

$$P : \mathfrak{M} \to \Omega_\Lambda, \quad (S, \varphi) \mapsto [\varphi(H^{2,0}(S))]$$

is generically injective on each of the two components.

Remark: Injectivity really only holds generically, i.e. for (S, φ) in the complement of a countable union of hypersurfaces.

Let us now consider the natural action

$$O(\Lambda) \times M \to M, \quad (\varphi, (S, \psi)) \mapsto (S, \varphi \circ \psi).$$

The transformation $-id \in O(\Lambda)$ induces the involution $(S, \varphi) \mapsto (S, -\varphi)$ that interchanges the two connected components and, it is an unique one with a such property. This becomes part of the following reformulation of the Global Torelli theorem for K3 surfaces:

Each connected component $\mathcal{M}^o \subset \mathcal{M}$ maps generically injectively into Ω and for any K3 surface S one has $O(H^2(S, \mathbb{Z}))/Mon(S) = \{\pm\}$.

3.6.2. *Global Torelli theorem.* Standard form of Torelli theorem was proven by Verbitsky:

Theorem 3.15. *(global Torelli theorem) Let Λ be a lattice of signature $(3, b_2 - 3)$ and let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space $\mathfrak{M}_\Lambda$ of marked compact hyperkähler manifolds $(X, \varphi : H^2(X, \mathbb{Z}) \tilde{\to} \Lambda)$. Then the period map*

$$P : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda \subset \mathbb{P}(\Lambda \otimes \mathbb{C}), \quad (X, \varphi) \to [\varphi(H^{2,0}(X))]$$

is generically injective.

This theorem might be reformulated in terms of Teichmüller spaces as it was done in Verbitsky's paper [V] (for $Teich_b$) and Loojenga's survey [?] (for $\mathcal{J}_s$)

Theorem 3.16. *The period map $\mathfrak{Per} : \mathcal{J}_s \to \Omega$ maps every connected component of $\mathcal{J}_s$ isomorphically onto Ω. In particular, the Γ- stabilizer of a component acts with finite kernel on $H^2(M; \mathbb{Z})$.*

Later, Bakker-Lehn ([BL]) extended global Torelli theorem for singular holomorphic symplectic manifolds and even get a different proof without notion of twistor lines. Amarked moduli space $\mathcal{M}^{lt}$ obtained by gluing the universal locally trivial deformation spaces together,

Theorem 3.17. *Let X be a symplectic variety with $b_2(X) > 4$ that admits an irreducible symplectic resolution. Let $\mathcal{N}^l t \subset \mathcal{M}$.*

(1) The period map $\mathfrak{Per} : \mathcal{N}^{lt} \to \Omega$ is surjective, generically injective, and the points in any fiber are pairwise nonseparated. Moreover, varieties underlying points in the same fiber are birational.

(2) The points in the fiber containing (X, ν) are in bijective correspondence with the cones obtained by restricting the Kähler chambers of a resolution to $H^{1,1}(X, \mathbb{R})$.

(3) The locally trivial weight two monodromy group $Mon^2(X)^{lt}$ is a finite index subgroup of $O(H^2(X, \mathbb{Z}))$.

3.7. **Exercises 3.**

(1) Let X be a hyperkähler manifold of dimension $2m$. The kernel of the canonical morphism

$$\mu : Sym^{\bullet} H^2(X,\mathbb{C}) \to H^{2\bullet}(X,\mathbb{C})$$

is generated by the relations α^{m+1} for all $\alpha \in H^2(X,\mathbb{C})$ such that $q_X(\alpha) = 0$. In particular, μ is injective in degrees $\leq m$.
[Hint: $\forall \alpha \in H^2(X,\mathbb{C}) q_X(\alpha) = 0 \Leftrightarrow \alpha^{m+1} = 0 \ in \ H^{2m+2}(X,\mathbb{C})$]

(2) Check that Kuga-Satake construction (subsection 3.0.3) for K3-type Hodge structure gives a complex torus

(3) Prove that

$$b_{\theta^k} = \frac{k!}{(4\pi^2 l)^k} \frac{||R||^{2k}}{(vol M)^{k-1}}$$

using the fact that on Ricci-flat Kähler manifold of complex dimension n the $\mathcal{L}^2$-norm of the curvature can be expressed in terms of c_2 and the Kähler class:

$$||R||^2 = \frac{8\pi^2}{(n-2)!} \int_M c_2 \omega^{n-2}$$

(4) Check that inseparability is indeed an equivalence condition on $Teich$.

(5) Two points $x, y \in \Omega$ are called equivalent if there exists a chain of twistor lines $D(W_1), ..., D(W_k)$ and points $x = x_1, ..., x_{k+1} = y$ with $x_i, x_{i+1} \in D(W_i)$.
Show that any two points $x, y \in \Omega$ are equivalent.

(6) Consider a marked hyperkähler manifold $(X, \varphi) \in \mathcal{M}$ and assume that its period is contained in a generic twistor line $D(W) \subset \Omega$. Then there exists a unique lift of $D(W)$ to a curve in $\bar{\mathcal{M}}$ through (X, φ).

References

[Bea] A. Beauville, Variétes Kähleriennes dont la premiére classe de Chern est nulle, J. Differential geometry 18 (1983), pp. 755-782.

[Bea2] Some remarks on Kähler manifolds with $c_1 = 0$. Classification of algebraic and analytic manifolds (Katata, 1982), 1–26, Progr. Math., 39, Birkhauser Boston, Boston, MA, 1983.

[B] Bogomolov, F., *On the decomposition of Kähler manifolds with trivial canonical class*, Math. USSR-Sb. **22** (1974) 580 - 583.

[B2] Bogomolov, F., *On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky)* Geom. Funct. Anal., Vol. 6, No. 4, 1996, pp. 612–618.

[BL] Bakker,B.,Lehn,C.:The global moduli theory of symplectic varieties. arXiv:1812.09748

[F] Fujiki, A. *On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold*, Adv. Stud. Pure Math. 10 (1987) 105-165.

[H] D. Huybrechts, Compact Hyperkaehler Manifolds: Basic Results, arXiv:alg-geom/9705025 (alg-geom)

[H1] Huybrechts,D.:Finiteness results for compact hyperkählermanifolds.J.Reine Angew.Math.558,15-22 (2003)

[H2] Huybrechts, D.: A global Torelli theorem for hyperkähler manifolds (after M. Verbitsky), Seminaire Bourbaki Exp. 1040. Astérisque 348, 375–403 (2012)

[HS] N. Hitchin, J. Sawon, Curvature and characteristic numbers of hyperkähler manifolds, arxiv:9908114v1

[G] D. Guan, On the Betti Numbers of Irreducible Compact Hyperkähler Manifolds of Complex Dimension Four, Mathematical Research Letters, 2001

[GKLR] reen,M.,Kim,Yoon-Joo,Laza R.,Robles,C.:The LLV decomposition of hyper-Kähler cohomology. arXiv:1906.03432

[P] P. Voegtli, Bakker-Lehn proof of Torelli theorem for the smooth case, LSGNT project (will be on my webpage as well)

[LL] Looijenga, E., Lunts, V., A Lie algebra attached to a projective variety, Invent. Math. 129 (1997), no. 2, 361-412.

[L] Looijenga, E. Teichmüller spaces and Torelli theorems for hyperkähler manifolds. Math. Z. 298, 261–279 (2021).

[K] M. Kapranov, Rozansky-Witten invariants via Atiyah classes, Compositio Math- ematica 115 (1999) 71-113.

[Kon] M. Kontsevich, Deformation quantization of Poisson manifolds, I preprint q- alg/9709040

[KS] Kreck, M., Su, Y.: Finiteness and infiniteness results for Torelli groups of (hyper-)Kähler manifolds. arXiv:1907.05693

[KSV] Kurnosov,N.,Soldatenkov,A.,Verbitsky,M.:Kuga–Satake construction and cohomology of hyperkähler manifolds. Adv. Math. 351, 275–295 (2019)

[RW] L. Rozansky, E. Witten, Hyperkähler geometry and invariants of three-manifolds, Selecta Mathematica, New Series, 3 (1997) 401-458

[S] J. Sawon, PhD Thesis, Rozansky-Witten invariants of hyperkähler manifolds, arxiv: 0404360

[Sa] S. Salamon, On the cohomology of Kähler and hyper-Kähler manifolds, Topology 35 (1996), 137-155.

[V1] Verbitsky, M., Cohomology of compact hyperkähler manifolds. alg-geom electronic preprint 9501001, 89 pages, LaTeX.

[V2] Verbitsky, M.: Mapping class group and a global Torelli theorem for hyperkähler manifolds, Appendix A by Eyal Markman. Duke Math. J. 162, 2929–2986 (2013) (Erratum: Duke Math. J. 169, 1037–1038 (2020)), and Erratum to mapping class group and a global Torelli. arXiv:1908.11772.pdf