2. Lecture 2

Abstract: In the second lecture I covered the definition of O'Grady's examples (see the lecture 1 notes) and started the discussion of the cohomology of hyperkähler manifolds.

In particular, I stated the local Torelli theorem to define the BBF-form., and described the structure of total Lie algebra acting on cohomology.

2.1. **Local Torelli theorem.**

2.1.1. *Deformations.*

Remark: Any smooth Kähler deformation of a hyperkähler manifold is again a hyperkähler manifold (any small deformation is Kähler, but there are large deformations of compact Kähler manifolds that cease to be Kähler).

Definition 2.1. *A deformation of a compact manifold X is a smooth proper holomorphic map $\mathcal{X} \to S$, where S is an analytic space and the fibre over a distinguished point $0 \in S$ is isomorphic to X.*

We will say that a certain property holds for the generic fibre, if for an open (in the analytic topology) dense set $U \subset S$ and all $t \in U$ the fibre $\mathcal{X}_t$ has this property. The property holds for the general fibre if such a set U exists that is the complement of the union of countably many nowhere dense closed (in the analytic topology) subsets.

Remark: One knows that for any compact Kähler manifold X there exists a semi-universal deformation $\mathcal{X} \to Def(X)$, where $Def(X)$ is a germ of an analytic space and the fibre $\mathcal{X}_0$ over $0 \in Def(X)$ is isomorphic to X. The Zariski tangent space of $Def(X)$ is naturally isomorphic to $H^1(X, \Theta_X)$.

Definition 2.2. *Deformation is call universal if for any deformation $\mathcal{X}_S \to S$ of X there exists a uniquely determined holomorphic map $S \to Def(X)$ such that $\mathcal{X}_S \simeq \mathcal{X} \times_{Def(X)} S$.*

Let X be a HK manifold. Let Ω be a symplectic holomorphic form on X. Contraction of tangent vectors with Ω defines an isomorphism of vector-bundles $\Theta_X \simeq \Omega_X$. Therefore, $H^0(X, \Theta_X) \simeq H^0(X, \Omega_X) \subset H^1(X, \mathbb{C})$ vanishes.

Remark: By Kuranishi's theorem, there is a universal local deformation. Its base is a germ of an analytic subspace of $H^1(X, \Theta_X)$ defined by $h^2(X, \Theta_X)$ equations.

Note that $h^2(X, \Theta_X) = h^2(X, \Omega_X) = h^{1,2}(X) = \frac{1}{2}b_3(X)$ can very well be nonzero.

However,

Theorem 2.1. ***(Bogomolov [B])*** *The deformation space of a HK manifold X is unobstructed.*

Remark: Explicitly Theorem 2.1 asserts the following: There exist a submersive map $f : \mathcal{X} \to U$ of complex manifolds and a point $0 \in U$ such that U is a polydisc, $F^{-1}(0) \simeq X$ and the Kodaira-Spencer map $\Theta_0 U \to H^1(\Theta_X)$ is an isomorphism.

Corollary 2.1. *The deformation space of a HK manifold X has dimension equal to $(b_2(X) - 2)$.*

Indeed, by Theorem 2.1 the deformation space of X has dimension $h^1(\Theta_X)$ and the latter equals $h^1(\Omega_X)$. Now $h^1(\Omega_X) = h^{1,1}(X)$ and by Hodge Theory $b_2(X) = 2h^{2,0}(X) + h^{1,1}(X)$ thus the corollary follows from $h^{2,0}(X) = 1$.

Remark: We have from above that if $n \geq 2$ then the generic deformation of $K3^{[n]}$ is not isomorphic to $K3^{[n]}$. In fact gives that a K3 surface has 20 moduli while $K3^{[n]}$ has 21 moduli because $b_2(K3^{[n]}) = 23$.

2.1.2. *The local period map.*

Definition 2.3. *Let X be a hyperkähler manifold and let $\pi : \mathcal{X} \to B$ be its universal local deformation, with $X = X_0$, the fiber at $0 \in B$.* [1]

The (local) period map is the map

$$\mathfrak{r} : B \to \mathbb{P}(H^2(X, \mathbb{C}), \quad b \mapsto [H^2(\mathcal{X}_b]$$

Remark: If B is simply-connected, then the family $\pi : \mathcal{X} \to B$ is differentially trivial and we may uniquely identify each $H^2(\mathcal{X}_b, \mathbb{Z})$ with $H^2(X, \mathbb{Z})$ (the local system $R^2\pi_*\mathbb{Z}$ is trivial because B is simply connected).

Proposition 2.1. ***(Griffiths)*** *The map $\mathfrak{r}$ is holomorphic and that its differential at the point $0 \in B$ is the composition*

$$T_{B,0} \tilde{\to} H^1(X, \Theta_X) \to Hom(H^{2,0}(X), H^1(X, \Omega_X)) \subset Hom(H^{2,0}(X), H^2(X, \mathbb{C})/H^{2,0}(X)) =$$
$$= T_{\mathbb{P}(H^2(X,\mathbb{C})), [H^{2,0}(X)]}$$

Indeed, the second arrow is the differential of period map.

The tangent space to $Def(X)$ is given by $H^1(X, \Theta_X)$, for the codomain we recall that $\mathbb{P}(H^2(X, \mathbb{C}))$ is parametrizing the 1-dimensional subspaces given by $H^{2,0}(\mathcal{X}_t)$. By the standard deformation theory for Grassmannians we have that the tangent space to $\mathbb{P}(H^2(X, \mathbb{C}))$ is given by

$$Hom(H^{2,0}(X), H^2(X, \mathbb{C})/H^{2,0}(X))) = Hom(H^{2,0}(X), H^{1,1}(X) \oplus H^{0,2}(X)$$

Theorem 2.2. ***(Local Torelli theorem)***

The local period map of a hyperkähler manifold is an embedding and its image is (a germ of) a smooth analytic hypersurface in $\mathbb{P}(H^2(X, \mathbb{C}))$.

Idea of proof: As we know the symplectic form Ω_X induces an isomorphism $\Theta_X \tilde{\to} \Omega_X$. The map g above is therefore an isomorphism, because it factors as

$$g : H^1(X, \Theta_X) \tilde{\to} H^1(X, \Omega_X) \tilde{\to} Hom(H^{2,0}(X), H^1(X, \Omega_X))$$

where the last isomorphism is given by $\alpha \mapsto (\Omega_X \mapsto \alpha)$

This finishes the proof of the theorem.

2.2. Cohomology of hyperkähler manifolds.

2.2.1. *Basic results.*

Proposition 2.2. *Let X be a hyperkähler manifold of dimension $2m$ and let ω be a generator of $H^0(X, \Omega_X^2)$. For each $r \in \{0, ..., 2m\}$, we have*

$$H^0(X, \Omega_X^r) = \begin{cases} \mathbb{C}\omega^{\wedge(r/2)} & r = 2l \\ 0 \quad r = 2l + 1 \end{cases}$$

In particular, $\chi(X, \mathcal{O}_X) = m + 1$.

[1] Upon shrinking B, we may assume that it is smooth and simply connected.

It follows from the classification of compact Kähler manifolds with vanishing real first Chern class ([Bea2]).

Proposition 2.3. *There are following equations on the Hodge numbers of hyperkähler manifold of complex dimension n:*

$$h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-p,q}$$

2.2.2. *BBF-form.*

Remark: One can be much more precise: we are going to show that the image of the local period map is an open subset of a smooth quadric Q in $\mathbb{P}(H^2(X, \mathbb{C}))$.

Theorem 2.3. ***(Fujiki,** [F]**)** Let M be a simple hyperkähler manifold, $\eta \in H^2(M)$ and $n = 2m = dimM$. Then*

$$\int_M \eta^{2m} = \lambda q(\eta, \eta)^m,$$

where q is a primitive integer quadratic form on $H^2(M, \mathbb{Z})$ and $\lambda > 0$ is a rational number.

Remark: Of course if X is a K3 then q_X is the intersection form of X (and $\lambda = 1$). In general q_X gives $H^2(X; \mathbb{Z})$ a structure of lattice just as in the well-known case of K3 surfaces. Its signature is $(3, b_2 - 3)$ on $H^2(M, \mathbb{R})$ and $(1, b_2 - 3)$ on $H^{1,1}(M)$.

Idea of proof: 1. As we saw above, the image of the local period map is a smooth analytic hypersurface $Q \subset \mathbb{P}(H^2(X, \mathbb{C}))$ and

$$\forall \sigma \in Q \qquad \sigma^{m+1} = 0$$

Indeed, σ^{m+1} has a type $(n + 2, 0)$ in $H^{n+2}(\mathcal{X}_b, \mathbb{C})$.

2.Consider the polynomial $F(\alpha) := \int_X \alpha^n$. Its vanishing defines an algebraic hypersurface of degree n in $\mathbb{P}(H^2(X, \mathbb{C}))$

3. We can see that f vanishes along Q by taking derivatives. Hence f vanishes along its Zariski closure $\bar{Q}$. with multiplicity $\geq m$.

4. Suppose Q is contained in a hyperplane, then the hyperplane $H^{2,0}(\mathcal{X}_b) \oplus H^{1,1}(\mathcal{X}$ is the same fro all $b \in B$, that gives a contradiction.

5. Therefore, $\bar{Q}$ is a quadric with equation $q = 0$ and $f = q^m$. Since f has rational coefficients, one can choose q_X proportional to q integral and nondivisible, and λ as in the formula above. This formula determines q_X up to sign. We choose this sign so that q_X is positive on the convex cone of kähler forms.

We can define BBF-form by the following ([Bea2])

Definition 2.4. *The BBF(Beauville-Bogomolov-Fujiki)-form of an IHS X is the quadratic form on $H^2(X, \mathbb{R})$ given by*

$$q_X(\alpha) = \frac{n}{2} \int \alpha^2 (\Omega\bar{\Omega})^{n-1} + (1 - n)(\int \alpha\Omega^n \bar{\Omega}^{n-1})(\int \alpha\Omega^{n-1}\bar{\Omega}^n),$$

where $\Omega \in H^{2,0}(X)$ is chosen such that $q_X(\Omega\bar{\Omega})^n = 1$.

Remark: Using arguments of the deformation theory Beauville has shown that the form defined by the formula above is invariant under deformations.

Definition 2.5. *If ω is a kähler form on a compact manifold X of dimension n, one defines the primitive cohomology by*

$$H^2_{prim}(X, \mathbb{R}) := Ker(H^2(X, \mathbb{R}) \to^{\omega^{n-1}} H^{2n}(X, \mathbb{R}) \to \mathbb{R}$$

Remark: By Lefschetz theory the quadratic form $q_\omega(\alpha) := \int_X \alpha^2\omega^{n-2}$ is non-degenerate of signature $(2, b_2(X)-3)$ on $H^2_{prim}(X,\mathbb{R})$.

Proposition 2.4. *Let X be a hyperkähler manifold with symplectic form Ω_X. Its BBF-form q_X satisfies the following properties.*

(a) The quadric Q is defined as the set $q_X(\Omega_X) = 0, q_X(\Omega_X, \bar{\Omega}_X$

(b) One has

$$H^{1,1}(X) = (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp q_X}.$$

(d) The signature of q_X on $H^2(X,\mathbb{R})$ is $(3, b_2(X)-3)$.

Idea of proof:

1. If we differentiate Fujiki relation we will have

$$\forall \beta \in H^2(X,\mathbb{C}), \quad n\int_X \omega^{n-1} = m\lambda_X q_X(\omega)^{m-1} q_X(\omega,\beta)$$

2. Since $q_X(\omega_>0$ we have H^2_{prim} is the same as $\omega^{\perp q_X}$.

3. Taking another derivative, we ω is a kähler form on X, the restriction of q_X to $H^2(X,\mathbb{R})$ is a positive multiple of prim the form q_ω

Remark: Recall that if ω is a Kähler class on a hyperkähler manifold X, one has $q_X(\omega) > 0$. In particular, if L is an ample line bundle on X, one has $q_X(c_1(L)) > 0$. The following theorem states a sort of converse.

Theorem 2.4. ***(Huybrechts, Demailly–Boucksom).*** *A hyperkaähler manifold is projective if and only if there is a line bundle L on X such that $q_X(c_1(L)) > 0$.*

The Hirzebruch–Riemann–Roch theorem takes the following form on hyperkähler manifolds.

Theorem 2.5. *(Huybrechts, [H]). Let X be a hyperkähler manifold of dimension $2m$. There exist rational constants $a_0, a_2, ..., a_{2m}$ such that, for every line bundle L on X, one has*

$$\chi(X,L) = \sum_{i=0}^{m} a_{2i} q_X(L)^i$$

When X is the m-th Hilbert power of a K3 surface (or a deformation), we have (Ellingsrud-Götsche-M. Lehn)

$$\chi(X,L) = \binom{\frac{1}{2}q_x(L) + m + 1}{m};$$

When X is a generalized Kummer variety of dimension $2m$, we have (Britze)

$$\chi(X,L) = (m+1)\binom{\frac{1}{2}q_x(L) + 1}{m};$$

2.2.3. *Mukai extension.* Then we discussed moduli spaces of semistable bundles on K3 we introduced the notion of Mukai vector. We can extend the definition of Mukai extension to any quadratic vector space H, q over field k (algebraically closed).

Definition 2.6. *For a quadratic vector space (H, q) let $\tilde{H} = k \oplus H \oplus k$ be the graded vector space with direct summands of degree 0, 2 and 4. Define the quadratic form $\tilde{q}$ on $\tilde{H}$ as follows: let $\tilde{q}((a, x, b), (a', x', b')) = q(x, x') - ab' - a'b$ so that degree 0 and degree 4 summands make up a hyperbolic plane which is orthogonal to H, and the restriction of $\tilde{q}$ to H is q. Then vector space $(\tilde{H}, \tilde{q})$ is called Mukai extension of (H, q).*

Remark: For a K3 surface this q was the intersection form on the second cohomology, for higher-dimensional hyperkähler manifold we have $H^2(X)$ with BBF-form q_X.

2.2.4. *Lie algebra action.*

Definition 2.7. *Let (M, I, J, K, g) be a hyperkähler manifold, ω_I, ω_J, ω_K its Kähler forms. On $\Lambda^*(M)$, the following operators are defined:*

(1) *de Rham differential d, its adjoint d^* and the Laplacian Δ;*
(2) *The Lefschetz operators*

$$L_I(\alpha) = \omega_I \wedge \alpha, \qquad L_J(\alpha) = \omega_J \wedge \alpha, \qquad L_K(\alpha) = \omega_K \wedge \alpha$$

and their adjoints

$$\Lambda_I(\alpha) = * L_I * \alpha, \qquad \Lambda_J(\alpha) = * L_J * \alpha, \qquad \Lambda_K(\alpha) = * L_K * \alpha;$$

(3) *The Weil operators $W_I|_{\Lambda^{p,q}(M,I)} = \sqrt{-1}(p-q)$, $W_J|_{\Lambda^{p,q}(M,J)} = \sqrt{-1}(p-q)$, $W_K|_{\Lambda^{p,q}(M,K)} = \sqrt{-1}(p-q)$.*

Remark: One has $[L_I, \Lambda_J] = W_K$, $[L_J, \Lambda_K] = W_I$, $[L_I, \Lambda_K] = -W_J$

Proposition 2.5. ***(Verbitsky,** [V]**, Bogomolov,** [B2]**)***
Let M be a hyperkähler manifold, and $\mathfrak{a}$ be a Lie algebra generated by L_R and Λ_R for all induced complex structures R over M. Then the Lie algebra $\mathfrak{a}$ is isomorphic to $\mathfrak{so}(4, 1)$.

Structure of $\mathfrak{so}(4, 1)$:
(1) Operators W_R $(R = I, J, K)$ generate a 3-dimensional Lie algebra, which is isomorphic to $\mathfrak{su}(2)$.
(2) The basis of $\mathfrak{so}(4, 1)$ is given by L_R, Λ_R, W_R and the element $\Xi = [L_R, \Lambda_R]$, it is the standard Hodge operator acting on r-forms as multiplication by a scalar $n - r$, where $n = dim_{\mathbb{C}} M$.
(3) The triple (L, Λ, Ξ) is called Lefschetz triple and it $\mathfrak{sl}(2)$ algebra.
Moreover, this action coincide with the Hodge decomposition, namely

Proposition 2.6. *The $\mathfrak{sl}(2)$-action of (L, Λ, Ξ) and the action of the Weil operator W commute with Laplacian, hence preserve the harmonic forms on a Kähler manifold.*

The triple (L, Λ, Ξ) is called Lefschetz triple and it can be defined in a very general setting

Definition 2.8. *Let $A^\bullet = \bigoplus_{i=0}^{r} A^i$ be a graded-commutative algebra with* $\dim A^r = 1$*. Consider the A^r-valued form on $A^\bullet$ mapping $x, y \in A^\bullet$ to the A^r-component of xy. The algebra $A^\bullet$ is called* **degree r graded Frobenius algebra** *if this pairing is non-degenerate.*

Example: The basic example of a Frobenius algebra is the cohomology algebra of a compact manifold.

Definition 2.9. *A* **Lefschetz triple** *in a Frobenius algebra* $A = \bigoplus_{i=0}^{2n} A^i$ *is a triple of operators* $L_\eta, \Xi, \Lambda_\eta \in A^\bullet$ *where* $\eta \in A^2$ *is a fixed element,* $L_\eta(x) := \eta x$, $\Xi|_{A^i} = i - n$ *and* Λ_η *is an element such that* $L_\eta, \Xi, \Lambda_\eta$ *form an* $\mathfrak{sl}(2)$*-triple.*

Remark: It is easy to see that such Λ_η is uniquely determined by Ξ and η (this statement is sometimes called "Morozov's lemma", and sometimes included in the statement of Jacobson-Morozov theorem). Existence of one Lefschetz triple is a non-trivial condition; however, the space of $\eta \in A^2$ for which the Lefschetz triple exists is Zariski open.

Definition 2.10. *The* **Frobenius-Lefschetz algebra** *is a Frobenius algebra admitting a Lefschetz* $\mathfrak{sl}(2)$*-triple.*

Remark: The Kähler assumption is needed only to conclude that the set of $\omega \in H^2(X)$ for which Λ_ω is defined is a non-empty (and thus dense) open Zariski set.

Loojienga-Lunts ([LL]) have studied this Lie algebra generated by all Lefschetz triples in the case of hyperkähler manifolds and for other geometric examples of Frobenius algebras (flag varieties, Hodge classes on an abelian variety). We will restrict ourself from now on to the the hyperkähler case.

Remark: A Cartan subalgebra of $\mathfrak{so}(4,1)$ can be given as $(\Xi, \sqrt{-1}W_I)$. The weight decomposition on $H^*(M)$ associated with this Cartan algebra action coincides with the Hodge decomposition.

From this exercise 2, the following structure theorem can be deduced ([V1, V2, LL]).

Theorem 2.6. *The algebra* $\mathfrak{g}$ *generated by all* $\mathfrak{so}(4,1)$ *for all hyperkähler triples on a given hyperkähler manifold* M *of maximal holonomy is isomorphic to* $\mathfrak{so}(4, b_2(M) - 2)$*.*

1. Consider the action of $\mathfrak{g}$ on the Mukai extension of $H^2(M,\mathbb{R})$

$$\tilde{H} := \mathbb{R}\cdot x \oplus H^2(M,\mathbb{R}) \oplus \mathbb{R}\cdot y,$$

where x has degree 0, y has degree 4, $H^2(M,\mathbb{R})$ is in degree 2. On $\tilde{H}$ we have the extended quadratic form $\tilde{q}$. The action of $\mathfrak{g}$ is determined by the following properties:

(1) It is compatible with the grading;
(2) For all $\alpha, \beta \in H^2(M,\mathbb{R})$, one has $L_\alpha x = \alpha$, $L_\alpha\beta = q(\alpha,\beta)y$, where q is the BBF form.
(3) $\Lambda_\alpha y = \alpha$, $\Lambda_\alpha \beta = q(\alpha,\beta)x$.

To see that this action is well-defined, we need to check that commutator relations hold. This follows from commutator relations in $\mathfrak{so}(4,1)$ and Zariski density of pairs $\alpha, \beta \in \langle \omega_I, \omega_J, \omega_K\rangle$ in the set of all pairs $\alpha, \beta \in H^2(M,\mathbb{R})$. [2]

[2]To obtain that the set of such pairs is Zariski dense, we use Torelli theorem

2. We have constructed a homomorphism $\mathfrak{g} \to End(\tilde{H})$. By construction, it preserves the Mukai pairing $\tilde{q} \in S^2(\tilde{H}^*)$. This defines a homomorphism $\Psi: \mathfrak{g} \to \mathfrak{so}(\tilde{H}, \tilde{q}) = \mathfrak{so}(4, b_2(M) - 2)$.

3. Ψ is surjective, because it is surjective on generators, and injective, because the relations in $\mathfrak{so}(4, b_2 - 2)$ can be obtained from relations in $\mathfrak{so}(4, 1)$.

Remark: As we have seen from the proof Lie algebra acting on cohomology is isomorphic to $\mathfrak{so}(\tilde{H}, \tilde{q})$ of Mukai extension.

Corollary 2.2. *The cohomology algebra $H^*(X)$ of hyperkähler manifold is decomposed by irreducible representations of $\mathfrak{so}(\tilde{H}, \tilde{q})$.*

Remark: Since $\mathfrak{so}(\tilde{H}, \tilde{q})$ consists of only even degree operators, then its action preserves the even and odd cohomology. Since $\mathfrak{so}(\tilde{H}, \tilde{q})$ is semisimple, we have decomposition provided by Corollary 2.2.

We are going to study this decomposition in the next subsection.

The induced by Mukai extension decomposition of $\mathfrak{g}$ is described in Exercise 3.

2.3. **Exercises 2.**

(1) Check that the Lie algebra formed by Lefschetz triples associated with I, J, K is isomorphic to $\mathfrak{so}(4,1)$.

(2) Let $(L_\omega, \Xi, \Lambda_\omega)$ and $(L_{\omega'}, \Xi, \Lambda_{\omega'})$ be two $\mathfrak{sl}(2)$-triples on a hyperkähler manifolds. Then $[\Lambda_{\omega'}, \Lambda_\omega] = 0$.

(3) We have $\mathfrak{g}_2 \simeq \mathfrak{g}_{-2} \simeq H$ as vector spaces, $\mathfrak{g}_0 \simeq k \oplus \mathfrak{so}(H,q)$ as subalgebra (the first summand is the center of $\mathfrak{g}_0$). The action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ is via the standard representation of $\mathfrak{so}(H,q)$. The Lie bracket of two elements $x \in \mathfrak{g}_{-2}$ and $y \in \mathfrak{g}_2$ is given by $[x,y] = (q(x,y), x \wedge y) \in \mathfrak{g}_0$, where we use the natural isomorphism $\mathfrak{so}(H,q) \simeq \Lambda^2 H$.

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