HYPERKÄHLER GEOMETRY

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1. Lecture 1

Abstract: In the first lecture I am going to cover definitions of hyperkähler manifolds, Beauville-Bogomolov theorem and examples of simple hyperkähler manifolds.

This notes don't exactly copy all things I told, there are some examples missed, however, there are also some details on the ideas of proofs written here, which I had no time to cover during the lecture.

The last section is construction of O'Grady's examples, which I will start with on the next lecture.

Lecture 2 will cover O'Grady's examples and cohomology of hyperkähler manifolds.

Please send all questions by email or ask in class!

1.1. **K3 surfaces.**

1.1.1. *Definition and geometry.* We start with an example in (complex) dimension two.Consider the following surface

Definition 1.1. *A K3 surface is a compact surface S such that $H^0(S, \Omega^2 S) = \mathbb{C}\omega$, where ω is a nowhere vanishing holomorphic 2-form on S, and $H^1(S, \mathcal{O}_S) = 0$.*

There are many examples of algebraic K3 surfaces like a smooth quartic in $\mathbb{P}^3$ (and in general complex smooth projective surfaces whose generic hyperplane section is a canonically embedded curve), double covers of $\mathbb{P}^2$ along smooth sextics, (2,3) complete intersections in $\mathbb{P}^4$, (2,2,2) complete intersections in $\mathbb{P}^5$.

1.1.2. *Torelli theorem.* The Torelli theorem (originally stated and proved for curves) answers the question as to whether a smooth Kähler complex manifold is determined (up to isomorphism) by (part of) its Hodge structure. In the case of polarized K3 surfaces, this property holds.

A polarization L on a K3 surface S is an isomorphism class of ample line bundles on S or equivalently, an ample class in $Pic(S)$, which is not divisible in $Pic(S)$.

Theorem 1.1. ***(Torelli theorem).*** *Let (S, L) and (S', L') be polarized complex K3 surfaces. If there exists an isometry of lattices*

$$\varphi : H^2(S', \mathbb{Z}) \tilde{\rightarrow} H^2(S, \mathbb{Z})$$

such that $\varphi(L') = L$ and $\varphi_{\mathbb{C}}(H^{2,0}(S')) = H^{2,0}(S)$, there exists an isomorphism $\sigma : S \tilde{\rightarrow} S'$ such that $\varphi = \sigma^*$.

We want to express this as the injectivity of a certain morphism, the period map, which we now construct.

Let S be a complex K3 surface. The lattice $(H^2(S,\mathbb{Z}),\cdot)$ is even unimodular with signature -16; it is therefore isomorphic to the rank-22 lattice

$$\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1) \oplus 2$$

We consider polarized K3 surfaces. For each $e \in \mathbb{Z}_+$, a primitive vector $h_{2e} \in \Lambda_{K3}$ with $h_{2e}^2 = 2e$. They are all in the same $O(\Lambda_{K3}$ by Eichler's criterion (Exercise 1).

Let now (S,L) be a polarized K3 surface of degree $2e$ and let $\varphi : H^2(S,\mathbb{Z}) \to \Lambda_{K3}$ be an isometry of lattices such that $\varphi(L) = h_{2e}$ (such an isometry exists by Eichler's criterion).

The "period" $p(S,L) := \varphi_{\mathbb{C}}(H^{2,0}(S)) \in \Lambda_{K3} \otimes \mathbb{C}$ is then in $h_{2e}^{\perp}$; it also satisfies the *Hodge– Riemann bilinear relations*

$$p(S,L)\cdot p(S,L) = 0, p(S,L)\cdot p(\bar{S},L) > 0.$$

Define the 19-dimensional (non-connected) complex manifold

$$\Omega_{2e} := \{[x] \in \mathbb{P}(\Lambda_{K3,2e} \otimes \mathbb{C}) | x\cdot x = 0, x\cdot \bar{x} > 0\},$$

so that $p(S,L)$ is in $2e$.

Remark: However, the point $p(S,L)$ depends on the choice of the isometry φ, so we would like to consider the quotient of Ω_{2e} by the image of the (injective) restriction morphism $\{\Phi \in O(\Lambda_{K3}) | \Phi(h_{2e}) = h_{2e}\} \to O(\Lambda_{K_{3,2e}}), \quad \Phi \mapsto \Phi|_{h_{2e}^{\perp}}$.

It turns out that this image is equal to the special orthogonal group $\tilde{O}(\Lambda_{K3,2e})$, so we set $\mathcal{P}_{2e} := \tilde{O}(\Lambda_{K_{3,2e}}) \setminus \Omega_{2e}$.

Let (S,L) be a polarized K3 surface of degree $2e$. Then there exists an irreducible 19-dimensional quasi-projective coarse moduli space $\mathcal{K}_{2e}$ for polarized complex K3 surfaces of degree $2e$.

Definition 1.2. *A period map is an algebraic morphism:*

$$\mathfrak{p}_{2e} : \mathcal{K}_{2e} \to \mathcal{P}_{2e}, \quad [(S,L)] \mapsto [p(S,L)]$$

Theorem 1.2. ***(Torelli theorem, revisited)*** *Let e be a positive integer. The period map*

$$\mathfrak{p}_{2e} : \mathcal{K}_{2e} \to \mathcal{P}_2 e$$

is an open embedding.

1.2. **Definition.** K3 is a modelling example of manifolds which we are going to talk about. Let us give first algebraic geometric definition.

Definition 1.3. *A (simple/irreducible) hyperkähler manifold is a simply connected compact Kähler manifold X such that $H^0(X,\Omega^2 X) = \mathbb{C}\Omega$, where Ω is a holomorphic 2-form on X which is nowhere degenerate (as a skew symmetric form on the tangent space).*

Remark: We prefer to call to the manifolds above IHS or simple hyperkähler as building blocks of hyperkähler geometry by Beauville-Bogomolov decomposition.

These properties imply that the canonical bundle is trivial, the dimension of X is even, say $2m$, and the abelian group $H^2(X,\mathbb{Z})$ is torsion-free.

If we omit the word Kähler above we end up with the definition of holomorphically symplectic manifold

Definition 1.4. *Complex manifold (M,I) is called holomorphically symplectic if it possess is a holomorphic 2-form Ω on X which is nowhere degenerate.*

1.3. **Differential geometric point of view.**

Theorem 1.3. ***(Berger, 1955)***. *If (M, g) is non-symmetric and irreducible then $Hol_0(g)$ is one of*

$$SO(n), U(2n), SU(2n), Sp(4n), Sp(4n) \times Sp(1), G_2, Spin(7).$$

The hyperkähler case is the case of $H = Sp(m)$ $(n = 4m)$. In this case we have a triple of Kähler forms $(\omega_1, \omega_2, \omega_3)$ modelled on $\sum_i d\bar{q}_i \wedge q_i$ on $\mathbb{H}^m$.

Definition 1.5. *Riemannian manifold (M, g) is called hyperkähler if there exist three complex structures I, J, K compatible with metric, satisfying quaternionic relations and corresponding Kähler forms are closed.*

We can easily observe that any holomorphically symplectic manifold is hyperkähler.

Remark: Let (M, g) be a HK. The associated Kähler forms $\omega_I, \omega_J, \omega_K$ span a three- dimensional subspace $H^2_+(M, g) \subset H^2(M, \mathbb{R})$.If $X = (M, I)$, then $H^2_+(M, g) = (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}\omega_I$, where the orientation is given by the base $(Re(\Omega), Im(\Omega), \omega_I)$. In order to see this, one verifies that the holomorphic two-form Ω on $X = (M, I)$ can be given as $\Omega = \omega_J + i\omega_K$.

On the contrary we have the following consequence of the celebrated theorem of Calabi–Yau [Y]

Theorem 1.4. *Let X be an IHS. Then for any $\alpha \in \mathcal{K}_X$ there exists a unique hyperkähler metric g on M, such that $\alpha = [\omega_I]$ for $\omega_I = g(I(),)$.*

1.3.1. *BB-decomposition theorem.*

Theorem 1.5. *([Bea2, B]) Let X be a compact K̈hler manifold with $c_1(X) = 0$. There exists an étale finite cover $\Pi_{i=1} M_i \to X$ where each of the factors M_i is either a compact complex torus, a HK manifold or a Calabi-Yau variety i.e. a compact K̈hler manifold of dimension $n \geq 3$ with trivial canonical bundle and such that $h^0(\Omega^p M_i) = 0$ for $0 < p < n$.*

Remark: The above result follows from:

(a) Yau's Theorem (formerly Calabi's conjecture) on the existence of Ricci-flat metrics on compact Kähler manifolds with $c_1 = 0$,

(b) De Rham's decomposition Theorem for simply-connected complete riemannian manifolds, and

(c) Berger's classification of holonomy groups of complete riemannian manifolds.

Definition 1.6. *A Calabi-Yau manifold is a simply connected compact Kähler manifold X of dimension $n \geq 3$ with K_X trivial and $H^0(X, \Omega^p_X) = 0$ for all $0 < p < n$.*

Examples: One has in particular $\chi(X, \mathcal{O}_X) = 1 + (-1)^n$. There are some example which are easy to find: any smooth complete intersection of multidegree $(d_1, ..., d_r)$ in $\mathbb{P}^{n+r}$, with $d_1 + ... + d_r = n + r + 1$ and $n \geq 3$, is a Calabi–Yau manifold of dimension n.

1.4. **Examples.**

1.4.1. *Hilbert schemes.* We will start with Beauville example [Bea]. Consider zero-dimensional subschemes of smooth surfaces. Let S be a smooth complex projective surface and $S^{[n]}$ be the Hilbert scheme parametrizing length n subschemes of S. Then a point of $S^{[n]}$ is a subscheme $Z \subset S$ such that $H^0(\mathcal{O}_Z)$ is finite-dimensional of dimension n.

Remark: Well-known that the generic such Z is reduced i.e. it consists of n distinct points and that $S^{[n]}$ is a smooth complex projective variety of dimension $2n$.

Let $S^{(n)}$ be the symmetric n-th power of S. Elements of $S^{(n)}$ look like $\sum_i m_i p_i$ with $m_i \in \mathbb{N}$ and $\sum_i m_i = n$.

There is a regular **Hilbert-Chow map**

$$\gamma : S^{[n]} \to S^{(n)}$$

which sends Z into the formal sum $\sum_{p\in S} l(\mathcal{O}_{Z,p})p$, where $l(\mathcal{O}_{Z,p}$ is equal to the dimension of $\mathcal{O}_{Z,p}$ as $\mathbb{C}$-vector space.

Remark: Strictly speaking, $S^{[n]}$ is a scheme only if S is algebraic. In general, it is just a complex space.

Remark: It is possible to show that the morphism γ is a resolution of singularities. Note that γ is an isomorphism over $sm(X^{(n)})$, the smooth locus of $X^{(n)}$ (i.e. the subset parametrizing cycles $x_0 + ... + x_n$ with pairwise distinct x_i's. The fibers of γ over the singular locus $sing(X^{(n)})$ are positive dimensional.

Case $n = 2$**:** Since the singular locus of $X^{(2)}$ consists of double points, the map γ becomes the blow-up along the diagonal $sing(X(2)) = \Delta_2 = \{(x, x)|x \in X\}$, and one can define

$$X^{[2]} := Bl_{\Delta_2}(X^{(2)}).$$

The variety $X^{[2]}$ is stratified according to the dimensions of the fibers of γ. There are two strata: an open stratum isomorphic to $sm(X(2))$ (reduced length 2 subscheme), and a closed stratum isomorphic to the projectivization of the tangent bundle of X (nonreduced, length 2 subscheme, point $2x$ with a vector).

In general, let Δ_{ij} where the i-th and j-th components are equal. Then the action of the symmetric group is not free on such diagonals with stabilizer $1, (ij)$. Denote by D the image of $\cup_{i,j}\Delta_{ij}$ by quotient by symmetric group. It is irreducible.

Remark: Let $D_* \subset D$ be the open subset where exactly two coordinate are equal. Given $2x_1 + x_2 + ... + x_{r-1} \in D_*$, the datum of an Artinian subscheme of length r supported on $2x_1 + x_2 + ... + x_{r-1}$ is equivalent to the datum of a tangent line to S at x_1. Hence, the set of Artinian subschemes of length r supported on $2x_1 + x_2 + ... + x_{r-1}$ is naturally identified with $\mathbb{P}T_{x_1}S$.

Proposition 1.1. *The homomorphism $H_1(S;\mathbb{Z}) \to \pi_1(S^{[n]}; \{p_1, ..., p_n\})$ is an isomorphism.*

Idea of proof:

1. Consider n pairwise distinct points on S and map $h : S \setminus \{p_1, ..., p_{n-1}\} \to S^{[n]}$ given by $p \mapsto (p_1, ..., p_{n-1}, p)$. Then homomorphism induced on fundamental groups is surjective. Then we have the homomorphism above.

2. Let $Alb(S) = H^0(\Omega^1_S)^v/H_1(S,\mathbb{Z})$ be the Albanese variety of S and Albanese map $u : S \to Alb(S)$ given by $p \mapsto (\omega \mapsto \int_{p_0}^{p} \omega)$.

3. Define the map $Z \mapsto \sum_{p\in S} l(\mathcal{O}_{Z,p})u(p)$ which is considered as the sum in the group $Alb(S)$.

4. We have homomorphism $H_1(S,\mathbb{Z}) \to \pi_1(Alb(S); u(p_1)+...+u(p_n)) \simeq H_1(S,\mathbb{Z})$ which as identity. Therefore the desired map is injective, and hence an isomorphism.

Now we construct holomorphic 2-form on $S^{[n]}$ from the one on S.

From Exercise 4 we have the Cartesian diagram

$$\begin{array}{ccc} & \eta & \\ Bl_\Delta(S^n_*) & \to & S^n_* \\ {}^\rho\downarrow & & \downarrow^\pi \\ S^{[n]}_* & \to & S^{(n)}_* \\ & \gamma & \end{array}$$

Let $S^{(n)}_*$ be the open subset where at most two coordinates coincide and $S^{[n]}_*$ be the inverse image of $S^{(n)}_*$ in $S^{[n]}$. Let ω be a holomorphic symplectic form on S, one can define a two-form on the product S^n_* by

$$\tilde{\omega} = \sum pr_i^*\omega,$$

where $pr_i : S^n \to S$ is the projection to the i-th factor.

Remark: Such two-form is clearly invariant under the action of the symmetric group on S^n_*, and its pullback $\eta^*\tilde{\omega}$ to $Bl_\Delta(S^n_*)$ is an invariant under this action. Hence, there exists a holomorphic form τ on $S^{[n]}_*$ such that

$$\eta^*\tilde{\omega} = \rho^*\tau$$

Proposition 1.2. *If $K_S = \Omega^2 S$ is trivial, then $S^{[r]}$ admits a holomorphic symplectic form.*

Idea of proof:

1. Denote $E_{ij} = \eta^*\Delta_{ij}$. Then the divisors E_{ij} are exceptional divisors of the blow up η and the ramification divisors of the morphism ρ.

Hence,

$$K_{Bl_\Delta(S^n_*)} = \rho^* K_{S^{[n]}_*} + \sum E_{ij}$$

and for the divisor of zeros

$$Div(\rho^* \wedge^n \tau) = \rho^* Div(\wedge^n \tau) + \sum E_{ij}$$

2. However, since $\eta^*\tilde{\omega} = \rho^*\tau$ and
$tilde\omega$ is closed the the left hand-side is equal to

$$Div(\rho^* \wedge^n \tau) = Div(\eta^* \wedge^n \tilde{\omega}) = Div(\wedge^n \eta^*\tilde{\omega}) = \sum E_{ij}$$

3. Hence, $\rho^* Div(\wedge^n\tau) = 0$ so the form τ is a holomorphic symplectic form on $S^{[n]}$ which extends to $S^{[n]}$ by Hartog's theorem.

1.4.2. $K3^{[n]}$.

Proposition 1.3. *Let S be a projective K3 surface. Then $S^{[n]}$ is a smooth projective variety of dimension $2n$. We will prove that $S^{[n]}$ is hyperkähler and that $b_2(S^{[n]}) = 23$.*

Idea of proof: 1. $S^{[n]}$ is simply-connected by Proposition 1.1.

2. Let $\Omega \in H^0(\Omega^2 S)$ be non-zero. Then Ω is symplectic because S is a K3 surface and hence $\Omega^{[n]} \in H^0(\Omega^2)$ is symplectic by 1.2

3. Lastly Exercise 7 gives that $h^{2,0}(S^{[n]}) = 1$ and that (2.4.1) holds (the second Betti number of a K3 surface equals 22 by Noether's formula).

1.4.3. $K_n(T)$. Let T be an abelian surface. The Hilbert scheme $T^{[n+1]}$ carries a holomorphic symplectic form but it is not simple HK (Exercise **??**).

Indeed, the fibration $s_{n+1} : T^{[n+1]} \to T$ given by the same formula as for the Hilbert scheme and using that T is the group shows that $H_1(T^{[n+1]}, \mathbb{Q}) \neq 0$.

Moreover, $T^{[n+1]}$ carries non-zero holomorphic 2-forms which are not symplectic.

Definition 1.7. *Let $K_n(T) := s_{n+1}^{-1}(0)$. The variety $K_n(T)$ is known as a generalized Kummer variety.*

Remark: The name follows from the case $n = 1$, then it is isomorphic to the Kummer surface of T, which the minimal desingularization of the quotient $T/$(-1).

Proposition 1.4. *$K_n(T)$ is a hyperkähler variety and that $b_2(K_n(T)) = 7$.*

Idea of proof: 1. $K_n(T)$ is simply-connected because the long exact sequence associated to s_{n+1} gives an exact sequence

$$0 = \pi_2(T) \to \pi_1(K^{[n]}(T)) \to \pi_1(T^{[n+1]}) \to \pi_1(T)$$

the latter map is an isomorphism.

2. We have surjection given by restriction map

$$H^2(T^{[n+1]}, \mathbb{Q}) \twoheadrightarrow H^2(K_n(T), \mathbb{Q})$$

3. Now using the equation from Exercise 7 and simply-connectedness of $K_n(T)$ we have a surjection

$$H^2(T, \mathbb{Q}) \bigoplus \mathbb{Q}\xi_{n+1} \twoheadrightarrow H^2(K_n(T), \mathbb{Q})$$

4. The map above is an isomorphism, indeed, consider the regular map

$$f : K_n(T) \times T \to T^{[n+1]}, \quad (Z, a) \mapsto \tau_a(Z),$$

where $\tau_a : T \to T$ is a translation by a (Exercise 8).

5. $H^2(f)$ defines an injection $H^2(T^{[n+1]}, \mathbb{Q}) \hookrightarrow H^2(K_n(T), \mathbb{Q}) \oplus H^2(T, \mathbb{Q})$ by Künneth formula. By step 3 we have

$$2b_2 + 1 = b_2(T^{[n+1]}) \leq b_2(K_n(T)) + b_2(T) \leq 2b_2 + 1$$

6. Therefore $b_2(K_n(T)) = 7$ and $h^{2,0}(K_n(T)) = 1$

Remark: Map f is Beauville-Bogomolov decomposition of $T^{[n+1]}$.

Remark: The whole Hodge diamond of generalized Kummer varieties has been computed by Göttsche and Sorgel [GS]

1.4.4. *Geometric constructions of Beauville's examples.* As for K3-surface there are many geometric constructions of simple hyperkähler manifolds deformationally equivalent to the Hilbert scheme of point on K3 surface. Among them most famous are the following:

(1) Fano variety of lines on cubic fourfold.

Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface and let $X := F(Y)$ be the Fano variety of lines on Y. It is a smooth 4-dimensional subvariety of the Grassmannian $Gr(2,6)$.

All cubics are deformation equivalent and, hence, so are the corresponding Fano varieties. Beauville and Donagi showed that for a special cubic Y the Fano variety X is isomorphic to the Hilbert scheme $K3^{[2]}$ of a special K3 surface of degree 14 in $\mathbb{P}^8$. Hence, for an arbitrary cubic Y the Fano variety X is a deformation of $K3^{[2]}$ and, therefore, irreducible symplectic.

(2) Twisted cubics on cubic fourfold

Definition 1.8. *A rational normal curve of degree 3, or twisted cubic for short, is a smooth curve $C \subset \mathbb{P}^3$ that is projectively equivalent to the image of $\mathbb{P}^1$ under the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$ of degree 3.*

Lehn, Lehn, Sorger and van Straten constructed a hyperkähler eightfold out of twisted cubics on cubic fourfolds. It is deformationally equivalent to the $K3^{[4]}$.

1.4.5. *O'Grady sporadic examples.*

Moduli of semistable sheaves

We need a brief reminder on the theory of semistable sheaves, we mostly follow notation from [HL].

Definition 1.9. *Let X be a complex projective variety and H an ample Cartier divisor on X. We let $\mathcal{O}_X(1) := \mathcal{O}_X(H)$. Let F be a coherent sheaf on X, and $Ann(F) \subset \mathcal{O}X$ be the annihilator of F.*

Note: The $Ann(F)$ is an ideal sheaf; the support of F is the subscheme of X defined by $supp(F) := V(Ann(F))$. The dimension $dim(F)$ of F is equal to the dimension of $supp(F)$.

Definition 1.10. *The sheaf F is pure if any non-zero subsheaf $G \subset F$ has dimension equal to $dim(F)$.*

Example: If $dimX = 1$, then a sheaf is pure iff it is torsion-free. For $dimX = 2$ look in Exercises.

Remark: Recall that for any X and ample divisor H, we can associate to a coherent sheaf Hilbert polynomial $P(F,n) = \chi(F \otimes \mathcal{O}_X(nH) = \int_X ch(E)ch(\mathcal{O}_X(nH))Td(X)$

Example: 1. Let X be a surface. Then $ch(F) = (r, c_1, ch_2(F))$. For the Hilbert polynomial we have

$$P(F,n) = \int_X (r, c_1, ch_2(E)) \cdot (1, nH, \frac{n^2H^2}{2}) \cdot (1, \frac{c_1(X)}{2}, \frac{c_1(X)^2 + c_2(X)}{12}) =$$

$$= r\frac{H^2}{2}n^2 + c_1Hn + ch_2(E) + \frac{c_1c_1(X)}{2} + r\frac{Hc_1(X)}{2}n + r\frac{c_1(X)^2 + c_2(X)}{12}$$

2. Let X be a K3 surface now, then $c_2(X) = 24, c_1(X) = 0$. Hence,

$$P(F,n) = r\frac{H^2}{2}n^2 + c_1Hn + ch_2(E) + 2r$$

Definition 1.11. *Let F be a sheaf on X we let $F(n) := F \otimes \mathcal{O}_X(n)$. Suppose that F is non-zero and let $d := dim(F) \geq 0$. The Hilbert polynomial $\chi(F(n))$ is integer-valued, it follows that there exists a unique sequence of integers a_i for $0 \leq i \leq d$ such that*

$$\chi(F(n)) = \sum_{i=0}^{d} a_i \binom{n}{i}$$

Example: Suppose that $dim(F) = dimX$. Then F is locally-free on an open dense subset $X_0 \subset X$ and the rank of F, denoted by $rk(F)$, is equal to the rank of the vector-bundle $F|_{X_0}$. Then $a_d(F) = rk(F)\int_X c_1(H)^d$.

Definition 1.12. *Let F be a sheaf on X. Let $d := dim(F)$. The reduced Hilbert polynomial of F, denoted by p_F is defined by*

$$p_F(n) := \frac{\chi F(n))}{a_d(F)}$$

Definition 1.13. *Let X be a smooth irreducible projective variety equipped with an ample divisor H. A non-zero pure sheaf F on X is H-semistable if for every non-zero subsheaf $E \subset F$ we have*

$$p_E(n) \leq p_F(n) \quad \forall n \gg 0.$$

If strict inequality holds whenever $E \neq F$ then F is H-stable.

Examples:

$d = 0$: $p_F(n) = 1$, F is pure, semistable. F is stable iff $F \simeq k(x),\ x \in X$.

$d = 1$: $F = i_*E$, where E is locally free on $i : C \hookrightarrow X$, then μ-stability of $E \Leftrightarrow$ stability of F (Exercise **??**).

$d = 2$: Assume X is a K3, so $c_2(X) = 24$:

$P(F,n) = a_0(F) + a_1(F)n + a_2(F)\frac{n^2}{2} = 2r_F + ch_2(F) + c_1(F)Hn + r_FH^2\frac{n^2}{2}$

Then $p_F(n) = \frac{a_0(F)}{r_FH^2} + \frac{c_1(F)H}{r_FH^2}n + \frac{n^2}{2}$

Therefore, stability criteria of F are

(1) $\frac{c_1(F)H}{r_FH^2} > \frac{c_1(E)H}{r_EH^2}$, or

(2) if they are equal, then $\frac{a_0(E)}{r_EH^2} > \frac{a_0(F)}{r_FH^2}$

Remark: For pure sheaves of dimension equal to $dimX$ there is the notion of μ-(slope)-semistability: one replaces the reduced Hilbert polynomial by the slope. The slope of a sheaf F of dimension equal to$dimX$ is

$$\mu(F) := \frac{1}{rk(F)}\int_X c_1(F)c_1(H)^{dimX-1}$$

Definition 1.14. *F is μ-semistable if for every non-zero subsheaf $E \subset F$ we have $\mu(E) \leq \mu(F)$.*

Remark: The usual semistability/stability imply μ one.

Definition 1.15. *Consider an equivalence relation which is weaker than isomorphism. Let F be a pure H-semistable sheaf on X. There exist a Jordan-Hoölder (J-H) filtration of $F0 = F_0 \subset F_1 \subset ... \subset F_t P$. with the property that each quotient F_i/F_{i-1} is pure, H-stable with reduced Hilbert polynomial equal to P_F .*

Examples:
(1) If F is H-stable, then a JH-filtration is necessarily trivial.
(2) Let $F = L \otimes_{\mathbb{C}} V$ where L is a line-bundle on X and V is a vector space of dimension r. Then

$$\text{Set of JH-filtrations of } F \longleftrightarrow \text{set of complete flags on } V.$$

Remark: The JH-filtration is not unique. However, the associated graded sum

$$gr^{JH}(F) := \oplus_{i=1}^{l} F_i/F_{i-1}$$

is unique up to isomorphism.

Definition 1.16. *Let F and G be pure H-semistable sheaves on X. Then are called S-equivalent if $gr^{JH} \simeq gr^{JH}(G)$.*

Remark: If F is H-stable then F is S-equivalent to G if and only if $F \simeq G$.

Definition 1.17. *Moduli space of pure semistable sheaves on X with a given Hilbert polynomial modulo S-equivalence is*

$$\mathcal{M}_X(P) := \{F \text{ pure } H\text{-semistable sheaf on } X | \chi(F(n)) = P(n)\}/S\text{-equivalence}$$

Notation: Consider a pure H-semistable sheaf with Hilbert polynomial P we denote by $[F] \in \mathcal{M}_X(P)$ to be the point corresponding to the S-equivalence class of F.

Definition 1.18. *Let $\mathcal{M}_X(P)^s \subset \mathcal{M}_X(P)$ be the subset parametrizing stable sheaves.*

Remark: The space $\mathcal{M}_X(P)^s$ is open.
Local structure of $\mathcal{M}_X(P)$:

Proposition 1.5. *Let $[F] \in \mathcal{M}_X(P)^s$. There is a natural isomorphism*

$$\Theta_{[F]}\mathcal{M}X(P) \simeq Ext^1(F, F).$$

Idea of proof: Let $[F] \in \mathcal{M}_X(P)^s$, then there is a natural identification between the germ of $\mathcal{M}_X(P)$ at $[F]$ and the universal deformation space of F. Indeed, assume $\mathcal{M}_X(P)$ is fine. Consider $[F] \in \mathcal{M}_X(P)$, it corresponds to a stable sheaf F, one has

$$T_{[F]}M = Hom_{[F]}(Spec\mathbb{C}[t]/t^2, \mathcal{M}_X(P)) = \mathcal{M}(Spec\mathbb{C}[t]/t^2) = Ext^1(F, F)$$

Remark: It works even if $\mathcal{M}_X(P)$ is not fine (Exercise 9). The real question now is smoothness of this space. The answer is given by the following theorem
Example: Suppose $d = dimF = 0$, then $P(F, n) \equiv n$. Therefore, $\mathcal{M}_X(P) = Sym^n X$

Remark: It might happen that the moduli space defined above behave quite bad (like $M_{\mathbb{P}^1}(2,0)$ from exercises).

In particular, problem is that of we find a strictly semistable sheaf, then $\mathcal{M}$ cannot be represented, i.e. $\mathcal{M}$ is not fine.

Definition 1.19. *Let F be a (coherent) sheaf on X; one can define a trace map*

$$Tr^i : Ext^i(F,F) \to H^i(\mathcal{O}_X)$$

Then let $Ext^i(F,F)^0 := kerTr^i$.

Theorem 1.6. ***(Mukai, Artamkin).*** *Suppose that $[F] \in \mathcal{M}_X(P)^s$ and that $Ext^2(F,F)^0 = 0$. Then $\mathcal{M}_X(P)$ is smooth at $[F]$ and its tangent space is canonical identified with $Ext^1(F,F)$.*

Geometric example of rank 2 moduli space

Let X be a degree 8 hypersurface in $\mathbb{P}^5$, given by the complete intersection of three quadrics Q_0, Q_1, Q_2. Let H denote its hyperplane class in $\mathcal{O}_X(1)$.

Consider the locus

$$\lambda_0 Q_1 + \lambda_1 Q_1 + \lambda_2 Q_2 = 0$$

If we denote by $[X_0, ..., X_5]$ the coordinates of $\mathbb{P}^5$, and suppose that the equations of the quadrics are given by

$$Q_i = \sum a_{jk} X_j X_k, \quad i = 1,2,3$$

then the zero locus of the determinant of the matrix

$$A = (a_{jk}) = (\lambda_1 a_1^{jk} + \lambda_2 a_2^{jk} + \lambda_3 a_3^{jk})$$

is the vanishing locus of a degree six polynomial in λ_i hence it is a sextic curve $C = V(detA) \subset \mathbb{P}^2$ and it parametrize the degenerate quadrics in the locus.

Remark: If $rk(A) = 5$, moreover, such sextic is smooth.

One denote by $\varphi : M \to \mathbb{P}^2$ the degree two branched cover of $\mathbb{P}^2$ ramified along C: M is a K3 surface. Next we show that under additional assumptions on X, the K3 surface M is in fact naturally isomorphic to a moduli space $\mathcal{M}$ of degree two sheaves on X.

Proposition 1.6. *Then the moduli space $M(2,H,2) \simeq M$ is a fine moduli space and one has $M(2,H,2) \simeq M \simeq X$.*

Mukai lattice.

Let S be a symplectic projective surface.

Consider

$$\tilde{H}(S) := H^0(S) \oplus H^2(S) \oplus H^4(S)$$

It has an integral Hodge structure of weight 2 as follows:

$$\tilde{H}(S)^{2,0} = H^{2,0}(S), \tilde{H}(S)^{0,2} = H^{0,2}(S), \tilde{H}(S)^{1,1} = H^0(S) \oplus H^{1,1}(S) \oplus H^4(S).$$

Definition 1.20. *The Mukai lattice of S is the group $\tilde{H}(S;\mathbb{Z})$ equipped with the symmetric bilinear form*

$$(\sum_{i=0}^{2} \alpha_i, \sum_{i=0}^{2} \beta_i) := \int_S (-\alpha_0\beta_2 - \alpha_2\beta_0 + \alpha_1 \wedge \beta_1)$$

where $\alpha_i, \beta_i \in H^{2i}(S;\mathbb{Z})$.

Remark: Later we will define Mukai lattice for any (simple) hyperkähler manifold, and using it we will prove existence of some huge Le algebra acting on cohomology of hyperkähler manifolds.

Remark: This lattice is even unimodular of signature (4,$b_2(S)-2$), i.e. just second cohomology lattice with hyperbolic lattice added.

Notation: We denote elements of $\tilde{H}(S)$ by (r,l,s) where $l \in H^2(S)$ and $r,s \in \mathbb{C}$, where we identify $H^4(S)$ with $\mathbb{C}$ via the orientation class η of S.

Definition 1.21. *A Mukai vector is a*

$$v = r + l + s\eta \in \tilde{H}^{1,1}(S)$$

such that $r \geq 0$ *and such that* l *is effective if* $r = 0$.

Definition 1.22. *A Mukai-vector* $v \in \tilde{H}(X,\mathbb{Z})$ *is indivisible if there is a vector* v' *with Mukai-pairing* $(v,v') = 1$.

Definition 1.23. *Let* F *be a coherent sheaf on* S*, then Mukai vector associated with* F *is*

$$v(F) := ch(F)\sqrt{Td(S)} = ch(F)(1+\epsilon\eta) = (r, c_1, \frac{c_1^2}{2} - c_2 + r$$

where $\eta \in H^4(S,\mathbb{Z})$ *is the orientation class, and* ϵ *is equal to 1 if* S *is a K3 surface, and 0 if* S *is an abelian surface, and* c_1, c_2 *are the Chern classes of* F *and* r *its rank.*

Remark: As a reminder here are the first few terms of the Chern character and Todd class for reference: $ch(F) = rk(F) + c_1(F) + \frac{1}{2}(c_1(F)^2 - 2c_2(F)) + ...$ and $td(F) = 1 + \frac{1}{2}c_1(F)^2 + \frac{1}{12}(c_1(F)^2 + c_2(F)) + ...$

Definition 1.24. *As* X *is a K3 surface,* $tdX = 1 + \eta$ *where* $\eta \in H^4(X,\mathbb{Z})$ *is the fundamental class of* X*. So* $v(F) = (rk(F), c_1(F), rk(F) + ch_2(F))$*. We call a Mukai vector* $(rk, c_1, a\eta)$ *primitive if* $gcd(rk, c_1, a) = 1$.

Proposition 1.7. *We have*

$$dimExt^1(F,F) = 2dimHom(F,F) + (v(F), v(F))$$

Idea of proof:

1. $v(F) \in \tilde{H}^{1,1}_{\mathbb{Z}}(S)$
2. By Hirzebruch-Riemann-Roch we have

$$(v(E), v(F)) = -\chi(E,F) := -\sum_{i=0}^{2} dimExt^i(E,F)$$

3. Serre duality gives us $Ext^2(F,F) \simeq Hom(F,F)^\vee$.

Remark: Notice that if F is a pure sheaf of dimension 2 or 1 then $v(F)$ is a Mukai vector.

Let $v \in \tilde{H}^{1,1}$ be a Mukai vector: the Hilbert polynomial $\chi(F(n))$ of a sheaf F such that $v(F) = v$ is independent of F, call it P.

Definition 1.25. *Let H be an ample divisor on S. Define the moduli space of H-semistable pure sheaves on S parametrized by Mukai vectors:*

$$\mathcal{M}_S(v) := \{[F] \in \mathcal{M}_S(P_v) | v(F) = v\}.$$

Remark: The space $\mathcal{M}_S(v)$ is open and closed in $\mathcal{M}_S(P)$. Moreover, the rank of sheaves parametrized by $\mathcal{M}_S(P)$ is constant and the Chern classes of sheaves parametrized by $\mathcal{M}_S(P)$ are locally constant.

Remark:

Theorem 1.7. *(Mukai, [M]) Let S be a projective symplectic surface and H an ample divisor on S. Let v be a Mukai vector. Then*

(1) $\mathcal{M}_S(v)$ is a projective scheme.

(2) Suppose that $[F] \in \mathcal{M}_S(v)^s$. Then $\mathcal{M}_S(v)$ is smooth at $[F]$ and

$$dim_{[F]}\mathcal{M}_S(v) = 2 + (v, v).$$

An important example is covered by Exercise 13.

Theorem 1.8. *Let (r, l, s) be primitive, either $r \geq 1$ or $s \neq 0$, and $(v, v) = 0$. Then, for generic H, $\mathcal{M}(v)$ is a K3 surface.*

Idea of proof:

1. We know that $M := \mathcal{M}^s$ is a smooth projective surface with form $\Omega \in \Gamma(\Omega^2_M)$. Then $\omega_M \simeq \mathcal{O}_M$. Our goal is to prove $H^1(\mathcal{O}_M) = 0$.

2. Consider universal sheaf E on $X \times M$ with p, q be projections on X and M respectively. Then by Bondal-Orlov criterion for surfaces with $\omega_M \simeq \mathcal{O}_M$ we have Fourier-Mukai transform $FM_E : D^b(M) \to D^b(X)$ which is equivalence of categories.

3. We have $H^i(M, Ext^i_p(E, E))$ and $H^i(X, Ext^i_q(E, E))$ both giving $Ext^{i+j}(E, E)$.

4. Thus embeddings $H^1(M, \mathcal{O}_M) \hookrightarrow Ext^1(E, E)$ and$H^1(X, \mathcal{O}_X) \hookrightarrow Ext^1(E, E)$. Also we know that $H^1(X, \mathcal{O}_X) = H^0(X, \Theta_X) = 0$, and $\Theta_X = Ext^1_q(E, E)$. Therefore $H^1(\mathcal{O}_M) = 0$ and we are done with our goal.

Example: Let S be a K3 surface, and $u = (1, 0, 0)$. Then $\mathcal{M}_S(u) \simeq S$ and the tautological sheaf is I_Δ where $\Delta \subset S \times S$ is the diagonal.

Example: Let $X \subset \mathbb{P}^3$ be a quartic and $v = (2, \mathcal{O}_X(-1), 1)$, then we have $\mathcal{M}(v) \tilde{\to} X$ given by $[E] \mapsto x$. Indeed, we have $E \hookrightarrow \mathcal{O}_X^{\oplus 3} \twoheadrightarrow I_x$.

Semistable sheaves on symplectic surfaces

A lot of things in this section (and some of previous ones) were done by Mukai first. [M]

Example: Let S be a K3 surface. We have an isomorphism

$$S^{[n]} \tilde{\to} \mathcal{M}_S(1 - (n-1)\eta)$$

$$[Z] \mapsto [I_Z]$$

A torsion free sheaf F with Mukai vector $v(F) = (1, 0, 1-n)$ will have rank one, $c_1 = 0$, and $c_2 = n$. The cokernel of the inclusion $F \hookrightarrow F^{\vee\vee}$ will be the structure sheaf $\mathcal{O}_Z$ of a zero-dimensional subscheme $Z \subset X$ of length n. Moreover, we have the following:

$$0 \to E \simeq I_Z \to E^{\vee\vee} \simeq \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

Theorem 1.9. ***(Mukai, Göttsche - Huybrechts, O'Grady, Yoshioka).***
Let S be a projective K3 surface. Let v be Mukai vector, and suppose that
(1) v is indivisible,
(2) $-2 \leq (v, v)$,
(3) $(r, s) \neq (0, 0)$.
Let H be a v-generic ample divisor on S. Then $\mathcal{M}_S(v)$ is an irreducible symplectic variety deformation equivalent to $S^{[n]}$ where $2n = 2 + (v, v)$.

Remark: In general the moduli space $\mathcal{M}_S(v)$ is not isomorphic to a Hilbert scheme $S^{[n]}$, not even birational. So the proof of theorem above rely on the following idea:

(1) the moduli of stable sheaves $\mathcal{M}_H(v)$ on an arbitrary K3 surface S is deformation equivalent to a Hilbert scheme of points, to showing that $\mathcal{M}_{H'}(v')$ on some fixed K3 surface S' is (for appropriate H' and v'). We can do this for two reasons: (1) the moduli space of polarized K3 surfaces of fixed degree is connected, and (2) we can construct the moduli of stable sheaves in the relative setting.

(2) Then for some particular Mukai vector we finish the proof using the following results of Huybrechts:

Proposition 1.8. . *Let X be a projective symplectic variety and Y an irreducible holomorphic symplectic variety. If X is birational to Y , then X is irreducible holomorphic symplectic as well.*

Proposition 1.9. *Two irreducible holomorphic symplectic varieties which are birational are deformation equivalent.*

Remark: There is an analog of Theorem 1.9 which has been proved by Mukai-Yoshioka, I will not state it here, just mention that kernel of a map $\mathcal{M}_T(v) \to T \times Pic(T)$ for some values of Mukai-vector v is an an irreducible symplectic variety deformation equivalent to $K_n(T)$ where $2n = (v, v) - 2$.

Construction of O'Grady examples

Let S be a projective symplectic surface and v a Mukai vector for S which is divisible. Thus

$$v = mv_0,\ v_0 \in H^{1,1}(S) \text{ indivisible}, m \in \mathbb{N}, m \geq 2.$$

This theorem summarizes results of many mathematicians, and answers to the question which types of hyperkähler manifolds we can obtain by this construction

Theorem 1.10. ***(O'Grady, Kiem, Rapagnetta, Kaledin, Lehn, Sorger, Perego).*** *Let S be a symplectic projective surface. Let v be a divisible Mukai vector as above. Suppose that $v_0^2 \geq 2$ and that $(r, s) \neq (0, 0)$. Let H be a v-generic ample divisor on S. Then $\mathcal{M}_S(v)$ is non-empty, irreducible of dimension $(2 + v^2)$ and its smooth locus is equal to $\mathcal{M}_S(v)^{st}$. There exists a symplectic desingularization $\tilde{\pi} : \tilde{M}(v) \to M(v)$ if and only if $m = 2$ and $v^2 = 2$. Now suppose that $m = 2 = v^2$.*

(1) If S is a K3 surface then $\tilde{M}(2v_0)$ is a 10-dimensional HK variety and $b_2(\tilde{M}_S(v_0)) = 24$.

(2) If S is an abelian surface let $\tilde{M}(2v_0)^0 := f^{-1}(\tilde{M}(2v_0)^0)$. Then $\tilde{M}(2v_0)^0$ is a 6-dimensional HK variety and $b_2(\tilde{M}(2v_0)^0) = 8$.

(3) Let S and S' be K3 surfaces, v_0 and v_0' Mukai vectors for S and S' with $2 = v_0^2 = (v_0')^2$ and H, H' ample divisors on S and S' respectively which are $2v_0$

and $2v_0'$ generic respectively. Then $\tilde{M}_S(2v)$ is deformation equivalent to $\tilde{M}_{S'}(2v_0')$. A similar statement holds for abelian surfaces.

Remark:
The second Betti number of $\tilde{M}_S(2v)$ is different from that of $K3^{[n]}$ and of a generalized Kummer, so it is not a deformation of Beauville examples. A similar is true for $\tilde{M}_S(2v)^0$.

Remark: If we consider an H-stable sheaf F with a primitive Mukai vector v_0, then for $m \geq 2$, the sheaf $F^{\oplus m}$ is strictly H-semistable. Hence if we set $v = mv_0$, this sheaf determines a singular point of the moduli space $\mathcal{M}_v(X, H)$, whose smooth locus still carries a holomorphic symplectic form. O'Grady considered the case of $v_0 = (1, 0, -1)$ and $m = 2$(a sheaf F parametrized by $\mathcal{M}_S(2v_0)$ has rank 2,$c_1 = 0$ and c_2 equal to 4 if S is a K3 and 2 if S is an abelian surface.), and showed that the singular symplectic variety $\mathcal{M}_v(X, H)$ admits a symplectic resolution $\tilde{\mathcal{M}}(X, H)$.

Definition 1.26. *Holomorphically symplectic manifolds constructed in the theorem above are called (sporadic) O'Grady's examples, and denoted as OG_6 and OG_10.*

1.4.6. *Geometric construction of O'Grady's.* In the case of abelian surface if we construct moduli space, and if $v_0^2 \geq 4$, then there is a non-trivial Albanese variety, so, in order to get an irreducible holomorphic symplectic manifold, one needs to consider a fiber

$$K_v(A, H) := alb^{-1}(0),$$

where $alb : \mathcal{M}_v(A, H) \to A \times A^\vee$ is Albanese morphism.

Let us consider a principal polarization $\Theta \subset A$ on abelian surface A. The Mukai vector $v_0 = (0, \Theta, 1)$ satisfies $v_0^2 = 2$, and hence, if we set $v = 2v_0$, there is symplectic resolution $\tilde{K}_v \to K_v$ that is deformation equivalent to OG_6. There is a natural support morphism $K_v \to |2\Theta| = \mathbb{P}^3$, realizing K_v as a Lagrangian fibration.

By definition of K_v, the fiber over a smooth curve $C \in |2\Theta|$ is the kernel of the natural morphism $Pic^6(C) \to A$ (which is also the restriction of alb to $Pic^6(C) \subset \mathcal{M}_v(A, H)$).

Remark: The morphism associated to the linear system $|2\Theta|$ is the quotient morphism $q : A \to A/\pm 1 \subset \mathbb{P}^3$ onto the singular Kummer surface of A. Let $p : S \to A/\pm 1$ be the minimal resolution of A, it is a K3 surface.

Remark: S come naturally equipped with the degree 4 nef line bundle D obtained by pulling back the hyperplane section of $A/\pm 1 \subset \mathbb{P}^3$.

There is $\tilde{A}$, the blow up of A at its 16 2–torsion points or, equivalently, the ramified cover of S along the exceptional curves $E_1, ..., E_{16}$ of p. Consider the moduli space $\mathcal{M}_w(S)$ of sheaves on S with Mukai vector $w = (0, D, 1)$ that are stable with respect to a choosen, sufficiently general, polarization.

Fact: This is an IHS manifold birational to the Hilbert cube of S and it has a natural morphism $\mathcal{M}_w(S) \to |D| = \mathbb{P}^3$ realizing it as the relative compactified Jacobian of the linear system $|D|$ (also a Lagrangian fibration).

We have a rational generically $2:1$ map $q^*p_* : \mathcal{M}_w(S) \dashrightarrow \mathcal{M}_v(A, H)$.

Remark: Since $\mathcal{M}_w(S)$ is simply connected, the image of this map lies in a fiber of alb, giving a $2:1$ morphism $\Phi : \mathcal{M}_w(S) \dashrightarrow K_v(A, H)$.

On the smooth fibers, this maps restricts to the natural $2:1$ pull back morphism $Pic^3(C') \to Pic^6(C)$, whose image is precisely $ker[Pic^6(C) \to A]$. Recall that $\sum_i E_i$ is divisible by 2 in $H^2(S, \mathbb{Z})$ and that the line bundle $eta := \mathcal{O}_S(\frac{1}{2}\sum E_i)$ determines the double cover q. Then the involution on $\mathcal{M}_w(S)$ corresponding to

Φ is given tensoring by η and $K_v(A, H)$ is a birational model of the "quotient" of $M_w(S)$ by the birational involution induced by tensorization by η.

Theorem 1.11. ***(Sacca, Rapagnetta, Mongardi)*** *This construction is well-determined and the resulting manifold is indeed an irreducible holomorphic symplectic of type OG_6.*

1.5. **Exercises.**

(1) **(Eichler's criterion)** Let Λ be an even lattice that contains at least two orthogonal copies of U. The $O(\Lambda)$-orbit of a primitive vector $x \in \Lambda$ is determined by the integer $q(x)$ and the element x^* of discriminant group $D(\Lambda) = \Lambda^\vee/\Lambda$.

(2) Prove that by the Baily–Borel theory, $\mathcal{P}_{2e}$ is an irreducible quasi-projective normal non-compact variety of dimension 19.

(3) Let (M, g) be a HK. Then for any $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$ the complex manifold $(M, aI + bJ + cK)$ is an IHS. Thus, for any HK (M, g) there exists a two-sphere $S^2 \subset \mathbb{R}^3$ of complex structures compatible with the Riemannian metric g.

(4) (a) The complex analytic pair $(S^{(n)}_*, D_*)$ is locally isomorphic to $(B\times C, B\times O)$, where B is a ball,C is a cone with vertex O over a smooth conic in $\mathbb{P}^2$.
(b)The complex manifold $S^{[n]}$ is the blow up of $S^{(n)}$ along D.
(c) If we denote $Bl_\Delta(S^n_*)$ the blow up of S^n_* along the union of its diagonals, then the action of symmetric group Σ_n lifts to $Bl_\Delta(S^n_*)$ and

$$S^{[n]}_* = Bl_\Delta(S^n_*)/\Sigma_n$$

(5) Show that $H_1(T^{[n+1]}, \mathbb{Q}) \neq 0$.

(6) Show the surjectivity of $H^2(T^{[n+1]}, \mathbb{Q}) \twoheadrightarrow H^2(K_n(T), \mathbb{Q})$ using the irreducibility of $\Delta_{n+1}|_{K_n(T)}$ $(n \geq 2)$

(7) Let S be a smooth complex projective surface. Assume that $H^*(S, \mathbb{Z})$ has no torsion. Then

$$H^2(S^{[n]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \bigoplus \Lambda^2 H^1(S, \mathbb{Z}) \bigoplus \mathbb{Z}$$

(8) Prove that the map $K_n(T) \times T \to T^{[n+1]}$, $(Z, a) \mapsto \tau_a(Z)$, where $\tau_a : T \to T$ is a translation by a is Galois with the group $T^{[n+1]}$.

(9) Recall that rank 1 bundles on $\mathbb{P}^1$ are parametrized by Picard scheme. Prove that $\mathcal{M}_{\mathbb{P}^1}(2, 0)$ (where $r = 2$ is rank, $c_1 = 0$ is chern class) is not of finite type.

(10) Let X be a complex projective surface.
(a) Then a sheaf is pure of dimension 2 if and only if it is torsion-free.
(b) Prove that the sheaf $F := i_*V$, is pure of dimension 1 where $i : C \hookrightarrow X$ is the inclusion of an irreducible curve, and V is a torsion-free sheaf on C.

(11) Let S be a symplectic projective surface with an ample divisor H. Suppose that F is an H-stable sheaf on S.
(a) Then $Hom(F, F) = \mathbb{C}Id_F$
(b) Prove that

$$-2 \leq v(F)^2.$$

(12) Let S be a K3 surface and $\varphi \in H^0(\Omega^2 S)$. The Hilbert scheme $S^{[n]}$ is identified with the moduli space $\mathcal{M}_S(1 - (n-1)\eta)$ as in the example 1.8. Define 2-form $\tau(\varphi)$ on $\mathcal{M}_S(v)^2$ by setting $\tau(\varphi)(\alpha, \beta) := \int_S \varphi \wedge Tr^2(\alpha \cup \beta)$. So we have the holomorphic 2-forms $\varphi^{[n]}$ and $\tau(\varphi)$. Prove that the relation between the forms is the following:

$$\tau(\varphi) = -4\pi^2\varphi^{[n]}.$$

(13) If $(v, v) = -2$ then $M_H(v)$ is empty or a single point $(Spec\mathbb{C})$.

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