Torelli Theorems for irreducible symplectic manifolds

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Abstract

We provide an alternative proof of a well known Local and Global Torelli Theorem for irreducible symplectic manifolds (established originally by [3]) based on the proof philosophy of Bakker and Lehn [1]. The authors of loc.cit present a new proof of a generalized Torelli Theorem in the context of so called primitive symplectic varieties, a singular analogue of hyperkaehler manifolds. The present text contains a simplification of their ideas adapted to the smooth setting.

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2 Introduction and Motivation

A normal variety X is called symplectic if it admits a nondegenerate closed holomorphic two-form $\sigma \in H^0(X_{reg}, \Omega^2_{X_{reg}})$ on its regular part which extends holomorphically on some resolution of singularities $\pi : Y \to X$.
If furthermore X is compact, $H^1(X, O_X) = 0$, and σ is unique up to scaling, we say X is a primitive symplectic variety.
As the authors in [1] state, these varieties are to be considered as singular analogues of compact irreducible symplectic manifolds also known as hyperkaehler manifolds in the sense that at least in low dimensions, a smooth primitive symplectic variety can be shown to be a hyperkähler manifold (see [[1],Lemma 3.3]). Moreover, primitive symplectic varieties provide a natural setting in which a reasonable generalization of the well known global moduli theory for hyperkaehler manifolds is possible.
We will in the sequel mostly be interested in the case of irreducible symplectic manifolds or hyperkaehler manifolds (we use these terms interchangeably) but sometimes allude to their above defined singular analogues which form the natural setting in [1]. For convenience of the reader we first recall the definition of our objects of primary interest:

Definition 1. *A complex manifold X is called irreducible symplectic if*

- *X is compact and Kaehler*
- *X is simply connected*
- $H^0(X, \Omega^2_X)$ *is spanned by an everywhere non-degenerate two-form* σ.

Any holomorphic two-form σ induces a homomorphism $T_X \to \Omega_X$, which we also denote by σ. The two-form is everywhere non-degenerate if and only if this homomorphism is bijective. A crucial feature to observe is that condition 3 in the above definition implies that $h^{20}(X) = h^{02}(X) = 1$ and that the canonical divisor K_X is trivial. Using the isomorphism $T_X \to \Omega_X$ and the property $c_1(\wedge^r E) = c_1(E)$ for a rank r vector bundle, this in turn shows that the first Chern class vanishes, i.e $c_1(X) = 0$
This last property, the vanishing of the first Chern class, is worth noticing as it can be said to be the basic pre-requisite for the famous decomposition theorem first introduced by [7] which states that any compact Kaehler manifold with $c_1 = 0$ can be decomposed, up to a finite étale cover, into a product of complex tori, simply connected Calabi Yau manifolds and irreducible symplectic manifolds. In other words, hyperkaehler manifolds are one of the three main building blocks of Ricci-flat compact Kaehler manifolds.
The first investigation of higher dimensional irreducible symplectic manifolds was in great parts inspired and motivated by amazing findings for K3 surfaces which are the irreducible symplectic manifolds of complex dimension 2. In order to motivate the interest and relevance of (Global) Torelli theorems for hyperkaehler manifolds, we start with a brief review on the corresponding results for these surfaces.

3 K3 surfaces

A particular class of compact complex algebraic or Kaehler manifolds is said to satisfy a Global Torelli theorem if any two manifolds of the type under investigation can be told apart by their integral Hodge structures.
The historically first examples for which a Global Torelli theorem could be proven were complex tori and curves. Two complex tori T and $\tilde{T}$ are isomorphic iff their weight-one Hodge structures

are isomorphic, i.e. $H^1(T,\mathbb{Z}) \cong H^1(\tilde{T},\mathbb{Z})$. Similarly, two smooth compact complex curves C and $\tilde{C}$ are isomorphic iff there exists an isomorphism of weight-one Hodge structures which additionally respects the intersection paring [see [2]].
It turns out that a similar statement also holds for K3 surfaces with respect to their second cohomology [2]

Theorem 1. *Two K3 surfaces X and $\tilde{X}$ are isomorphic iff there is an isomorphism of Hodge structures $H^2(X,\mathbb{Z}) \cong H^2(\tilde{X},\mathbb{Z})$ which respects the intersection product.*

If one restricts attention to the generic case, an even stronger version of the theorem holds

Theorem 2. *For any Hodge isometry $\phi : H^2(X,\mathbb{Z}) \simeq H^2(\tilde{X},\mathbb{Z})$ there is an isomorphism $g : X \simeq \tilde{X}$ with $\phi = \pm g_*$*

In what follows we closely align our exposition at [2] and references therein. The above Global Torelli theorem can be rephrased using the framework that will be used later on in the case of irreducible symplectic manifolds of arbitrary dimension. We will rigorously define the concepts introduced here in these later sections.
The primary object of interest is the moduli space of marked K3 surfaces $\mathcal{M}$ together with the so called period map $p : \mathcal{M} \to \mathbb{P}(\Lambda \otimes \mathbb{C})$. A marked K3 surface S is a tuple (S,ϕ) consisting of a K3 surface S and an isomorphism of lattices $\phi : H^2(S,\mathbb{Z}) \cong \Lambda$, where Λ is the unique even unimodular lattice of signature (3,19) common to all K3 surfaces (see [6]). Two marked K3 surfaces (S,ϕ) and $(\tilde{S},\tilde{\phi})$ are called isomorphic if there is a biholomorphism $g : S \cong \tilde{S}$ such that $\phi \circ g^* = \tilde{\phi}$. We then define the moduli space of marked surfaces $\mathcal{M}$ by $\mathcal{M} = \{(S,\phi)\}/\cong$. It can be shown [2] that the Global Torelli theorem for K3 surfaces can be reformulated in the following equivalent manner:

Theorem 3. *The moduli space $\mathcal{M}$ has two connection components interchanged by $(S,\phi) \mapsto (S,-\phi)$. Furthermore, the period map is given by*

$$\begin{aligned} p : \mathcal{M} \to D_\Lambda := &\{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | x^2 = 0, (x.\bar{x}) > 0\} \\ (S,\phi) \mapsto &[H^{20}(S)] \end{aligned} \tag{1}$$

and is generically injective on each of the components.

The moduli space of marked K3 surfaces is moreover equipped with an action

$$\begin{aligned} &O(\Lambda) \times \mathcal{M} \to \mathcal{M} \\ &(\gamma.(S,\phi)) \mapsto (S,\gamma \circ \phi) \end{aligned} \tag{2}$$

Then for any $(S,\phi) \in \mathcal{M}'$, where $\mathcal{M}'$ denotes a component of $\mathcal{M}$, the subgroup of $O(\Lambda)$ fixing $\mathcal{M}'$ is given by $\phi \circ Mon(S) \circ \phi^{-1}$. Herein the monodromy group $Mon(S) \subset O(H^2(S,\mathbb{Z}))$ is generated by all monodromies $\pi_1(X,t) \to O(H^2(S,\mathbb{Z}))$ induced by any smooth proper family $\mathcal{Y} \to X$ which satisfies $\mathcal{Y}_t = S$.
Furthermore,it can be shown [2] that the transformation -id acts by exchanging the two connection components of the moduli space $\mathcal{M}$ by sending $(S,\phi) \mapsto (S,-\phi)$ and is the only element of $O(\Lambda)$ with this property.
This last insight partially leads to the following rephrasing of the Global Torelli theorem for K3 surfaces:

Theorem 4. *Each connection component $\mathcal{M}' \subset \mathcal{M}$ maps generically injectively into the period domain D_Λ and for any K3 surface S one has $O(H^2(S,\mathbb{Z}))/Mon(S) = \{\pm 1\}$.*

An essential ingredient in the proof of the above version of the Global Torelli theorem is the pleasant fact that all K3 surfaces are deformation equivalent, which means by the theory of variation of Hodge structures that they can all be realized by complex structures of the same parametrizing differentiable manifold.
We draw the reader's attention to this deformation equivalence of K3 surfaces as it is in contrast to the situation in the case of higher dimensional irreducible symplectic manifolds where we will have to restrict to a certain deformation type in order to obtain analogous Torelli theorems. Besides this, the main construction of the Moduli space of marked K3 surfaces carries over to higher dimensions almost literally. However, parts of the statements of the Torelli theorems will have to be weakend as we will see in later sections.

4 Irreducible symplectic manifolds of higher dimensions

4.1 Recollections on the structure of hyperkähler manifolds

In order to be able to outline the generalizations of the above motivated approach to Local rsp. Global Torelli theorems for higher dimensional hyperkaehler manifolds, we first need to introduce and recall some classical results concerning these manifolds. Our exposition in the next few subsections closely follows [2] and [4].

4.2 The Beauville-Bogomolov Form

One of the most characteristic properties of hyperkaehler manifolds is undoubtedly the existence of a primitive integral quadratic form q_X on the second cohomology $H^2(X,\mathbb{Z})$, the Beauville-Bogomolov form. In the context of our report, the following properties of this quadratic form will be of foremost interest:

- q_X is non-degenerate and of signature $(3, b_2(X)-3)$
- There is a positive constant c such that $q_X^n(\alpha) = c\int \alpha^{2n}$ for all classes $\alpha \in H^2(X,\mathbb{Z})$. I.e. up to scaling, q_X is a root of the top intersection product on second cohomology.
- The Hodge decomposition of the complex second cohomology of X,

$$H^2(X,\mathbb{C}) = (H^{20} \oplus H^{02})(X) \oplus H^{11}(X)$$

is orthogonal with respect to the $\mathbb{C}$-linear extension of the BB-form. Moreover, $q_X(\sigma) = 0$ and $q_X(\sigma,\bar{\sigma}) > 0$ for σ the non-degenerate 2-form spanning $H^0(X,\Omega^2_X)$

Remark 1.
- *In the last statement we draw on the common abuse of notation of not distinguishing notationally between the quadratic form and its underlying symmetric bilinear form.*
- *Notice that the second bullet point above implies that q_x is invariant under deformations. We will need this feature in our proof of the Global Torelli theorem.*
- *From the third listing point we furthermore learn that the (polarized) weight two Hodge structure on $H^2(X,\mathbb{Z})$ endowed with q_X is uniquely determined by $\sigma \in H^{20}$. Furthermore, (see [4]), for any line bundle L on a hyperkaehler manifold X the following reformulation of the holomorphic Euler characteristic is possible*

$$\mathscr{X}(L) = \sum \frac{a_i}{2i!} q_X(c_1(L))^i \tag{3}$$

where the a_i are constants depending only on X. This is another fact we will refer to in the proof of the Global Torelli theorem.

Next we turn attention to some standard results about the deformation theory of irreducible symplectic manifolds.

4.3 Deformation Theory

A deformation of a compact manifold X is a smooth proper map $\mathscr{Y} \to S$, where S is an analytic space and such that the fibre of the distinguished point $0 \in S$ is isomorphic to X. It is well known that for any compact Kaehler manifold X there is a semi-universal deformation $\mathscr{Y} \to Def(X)$ where Def(X) refers to a germ of an analytic space and the fibre $\mathscr{Y}_0$ over $0 \in Def(X)$ is isomorphic to X. Furthermore, the Zariski tangent space to Def(X) is naturally isomorphic to $H^1(X, T_X)$. If moreover $H^0(X, T_X) = 0$, i.e if X does not admit any infinitesimal automorphisms, then the semi-universal family is even universal. This is to say that for any deformation $\mathscr{X}_S \to S$ of X there is a unique holomorphic map $S \to Def(X)$ such that $\mathscr{X}_S \cong \mathscr{Y} \times_{Def(X)} S$. Lastly, it can be shown [[4]and references therein] that Def(X) is is smooth if $K_X \simeq O_X$. From the above general facts we can deduce that in the case of irreducible symplectic manifolds X there is a universal family because $H^0(X, T_X)$ is trivial as the following short proof illustrates:

Proof.

$$H^0(X, T_X) = H^0(X, \Omega_X) = H^{10}(X) \simeq H^{01}(X) = H^1(X, O_X) = 0$$

where we used the canonical isomorphism between T_X and Ω_X induced by the non-degenerate two-form σ in the first, the Dolbeault isomorphism in the second equality and refer to [7] for the proof of the fact that the assumed simple connectedness of X implies $H^1(X, O_X) = 0$. □

Moreover, drawing again on the results above, we see that Def(X) is smooth of dimension $h^1(X, T_X) = h^1(X, \Omega_X) = h^{11}(X)$.
Finally, it is worth mentioning that every small deformation of an irreducible symplectic manifold is known to be irreducible symplectic and the universal deformation $\mathscr{Y} \to Def(X)$ is universal for all its fibres $\mathscr{Y}_t$ for t close enough to 0. In other words, the class of hyperkaehler manifolds is stable under small deformations.

4.4 The Period map

In knowledge of the above we can now define the so called period map which lies at the heart of the (Local) Torelli theorem.

$$\begin{aligned} p : &Def(X) \to \mathbb{P}(H^2(X, \mathbb{Z})) \\ &t \mapsto [H^{20}(\mathscr{Y}_t)] \end{aligned}$$

where $\mathscr{Y}_t$ denotes the fibre of the universal family $\mathscr{Y} \to Def(X)$. That this map is well defined follows from the observation that in the smooth setting we have been considering so far, there is a canonical identification of $H^2(X, \mathbb{Z}) \cong H^2(\mathscr{Y}_t, \mathbb{Z})$ given by parallel transport which respects the two BB forms q_X and $q_{\mathscr{Y}_t}$. Furthermore, as will be shown in the sequel, the period map takes values in the period domain

$$\Omega(X) := \left\{ [\sigma] \in \mathbb{P}(H^2(X, \mathbb{C})) | q_X(\sigma) = 0; q_X(\sigma, \bar{\sigma}) > 0 \right\}$$

We are now in the position to state and proof the local Torelli theorem:

Theorem 5. *Let X be an irreducible symplectic manifold with associated BB-form q_X. We denote by*

$$\Omega(X) := \left\{ [\sigma] \in \mathbb{P}(H^2(X, \mathbb{C})) | q_X(\sigma) = 0; q_X(\sigma, \bar{\sigma}) > 0 \right\}$$

the period domain for X inside $\mathbb{P}(H^2(X,\mathbb{C}))$.
If $f: Y \to Def(X)$ *denotes the universal deformation of X and* $X_t = f^{-1}(t)$ *the fibre over a point t, then the local period map*

$$Def(X) \to \Omega(X), t \mapsto H^{2,0}(X_t)$$

is an isomorphism

Proof. We first notice that by a result cited in $[[4], Prop 1.12]$ the universal family $\mathscr{Y} \to Def(X)$ exists and Def(X) is smooth of dimension $h^1(X, T_X)$. Moreover, by op. cit., all the nearby fibres $\mathscr{Y}_t$ are as well irreducible symplectic manifolds.
From Grauert's theorem we furthermore learn that $f_*\Omega^2_{\mathscr{Y}/S}$ is locally free and that there is an isomorphism

$$f_*\Omega^2_{\mathscr{Y}/S} \otimes_{O_S} k(s) \to H^0(X, \Omega_{\mathscr{Y}/S})_s = H^{2,0}(X)$$

for all s $\in Def(X)$. Where in the last equality we also used the Dolbeault theorem.
Summing up, this shows that $f_*\Omega^2_{\mathscr{Y}/S}$ is a locally free and invertible sheaf which by flatness of $f: \mathscr{Y} \to Def(X)$ is also compatible with arbitrary base change.
Next, from (4.4) we can deduce that $f_*\Omega^2_{\mathscr{Y}/S}$ defines the period map $p: Def(X) \to \mathbb{P}(H^2(X,\mathbb{C}))$. This also shows the holomorphicity of this map (as defined by a subbundle of $H^2(X,\mathbb{C}) \otimes O_{Def(X)}$). In order to argue that the period map takes values in $\Omega(X)$ we refer to the statements in the previous section (4.2) and the mentioned reference [4].
Lastly, to see that the period map is indeed a local isomorphism, we consider its differential.
The differential of the period map p at zero can be described as the map

$$H^1(X, T_X) \to Hom(H^0(X, \Omega^2_X), H^1(X, \Omega_X)))$$

This last statement follows from general deformation theoretic arguments:
The tangent space to Def(X) is, as remarked earlier, given by $H^1(X, T_X)$, which explains the domain of the claimed map, whereas the codomain is explained by the fact that $\mathbb{P}(H^2(X,\mathbb{C}))$ is parametrizing the 1-dimensional subspaces given by $H^{2,0}(X_t)$ (X and nearby fibres are irreducible symplectic manifolds). Thus by standard deformation theory for Grassmannians we deduce that the tangent space to $\mathbb{P}(H^2(X,\mathbb{C}))$ is given by

$$Hom(H^{20}(X), H^2(X,\mathbb{C})/H^{20}(X))) = Hom(H^{20}(X), H^{11}(X) \oplus H^{02}(X)).$$

It then follows from [[5],Prop 1.20] that this tangent map is given as the composition

$$dp: H^1(X, T_X) \to Hom(H^{20}(X), H^{11}(X)) \subset Hom(H^{20}(X), H^{11}(X) + H^{02}(X)) = Hom(H^0(X, \Omega^2_X), H^1(X, \Omega_X)))$$

as claimed.
Finally, we can use the isomorphism $\Omega_X \cong T_X$ resulting from the irreducible symplecticity of X, to see that the differential dp of the period map is an isomorphism. Hence p is a local isomorphism thanks to the inverse function theorem. This concludes the proof. □

4.5 Moduli Space of marked hyperkaehler manifolds

As opposed to the case of K3 surfaces who all share the so called K3-lattice of given signature, in higher dimensions, the lattice structure of the second cohomology is no longer the same for all hyperkaehler manifolds. We thus have to restrict to a given deformation type. In other words, we fix a lattice Λ of signature $(3, b_2 - 3)$ (We choose the signature this way to be able to use the BB form as polarization) and define the moduli space of Λ-marked hyperkaehler manifolds as

$$\mathscr{M}_\Lambda = \{(X, \phi)\}/\cong \tag{4}$$

where ϕ is an isomorphism $\Lambda \cong H^2(X,\mathbb{Z})$ between the second cohomology lattice endowed with the BB form and the fixed lattice Λ. We call two marked manifolds (X,ϕ) and $(X^{'},\phi^{'})$ isomorphic if there exists a biholomorphic map $f: X \cong X'$ such that $\phi \circ f = \phi'$. It is maybe worth mentioning that for many choices of Λ, the associate moduli space $\mathcal{M}_\Lambda = \emptyset$ which doesn't come as a surprise given the arbitrariness of our choice of signature for the fixed lattice Λ. Next, we would like to show that $\mathcal{M}_\Lambda$, the moduli space of Λ marked hyperkaehler manifolds, has itself the structure of a complex locally Euclidean space of dimension $b_2 - 2$. This is the content of the following proposition for which we provide a proof strongly inspired by [2]

Proposition 1. *The moduli space of* Λ*-marked hyperkaehler manifolds,* $\mathcal{M}_\Lambda$*, has the structure of a complex manifold of dimension* $b_2(X)-2$ *. Furthermore, for any parametrized* $(X,\phi) \in \mathcal{M}_\Lambda$*, Def(X) can be identified with an open neighbourhood of* (X,ϕ) *in* $\mathcal{M}_\Lambda$

Proof. First, by our observations in the subsection considering deformation theoretical aspects, we see that the dimension of Def(X) for a given hyperkaehler manifold X parametrized by $\mathcal{M}_\Lambda$, is given by $h^1(X,T_X) = h^1(X,\Omega_X) = h^{11}(X) = b_2(X)-2$. Moreover, by a previous remark, it is smooth.We can then consider Def(X) as a smooth disk of dimension $b_2(X)-2$. The marking ϕ on X induces a marking on all the other fibres $\mathcal{X}_t$ of the universal family $\mathcal{X} \to Def(X)$. Now the Local Torelli theorem tells us that the period map $p: Def(X) \to \mathcal{M}_\Lambda$ is a local isomorphism and hence for $t \neq \tilde{t} \in Def(X)$, the respective fibres $\mathcal{X}_t$ and $\mathcal{X}_{\tilde{t}}$ with corresponding markings ϕ_t and $\phi_{\tilde{t}}$ are non-isomorphic as marked manifolds. Consequently, we can think of Def(X) as open subset of $\mathcal{M}_\Lambda$ containing (X,ϕ).
Next, we show that the locally induced complex structures indeed glue to give $\mathcal{M}_\Lambda$ the claimed structure of a complex manifold. For (X,ϕ) and $(\tilde{X},\tilde{\phi}) \in \mathcal{M}_\Lambda$ consider the intersection $Def(X) \cap Def(\tilde{X})$ which can of course be empty. Since the two families $\mathcal{X} \to Def(X)$ and $\tilde{\mathcal{X}} \to Def(\tilde{X})$ are universal for all their fibres, the complex structures induced by the distinct families on the fibres over $Def(X) \cap Def(\tilde{X})$ necessarily coincide. Hence the local complex structures, seen as endomorphisms on the respective tangent bundles, glue to give a complex structure on $\mathcal{M}_\Lambda$. Lastly, since all the Def(X) for varying X are smooth, so is $\mathcal{M}_\Lambda$. □

Remark 2.
- *In the context of this Proposition we do not include a Hausdorffness assumption in our definition of manifold. Indeed,* $\mathcal{M}_\Lambda$ *is non-Hausdorff as we will discuss in the sequel.*
- *It is worth noticing that although the complex structures on the fibres do glue, the families themselves, seen as objects over the respective* $Def(X)$*, need not glue to a joint family. This is due to the fact that there may be non-trivial automorphisms making the necessary cocycle condition impossible.*

From the above construction, we can glue the local period maps to a global one

$$p: \mathcal{M}_\Lambda \to \Omega_X$$

which by construction is a locally biholomorphic map.

4.6 The Moduli space made Hausdorff

As pointed out in the previous remark, the moduli space of Λ-marked hyperkaehler manifolds a priori fails to be Hausdorff. This is, there exist tuples of marked hyperkaehler manifolds (X,ϕ) and $(\tilde{X},\tilde{\phi})$ which cannot be separated by disjoint open neighbourhoods. We will refer to those as non-separable points in the sequel. In view of our ultimate goal to establish a Global Torelli theorem, we aim at a generic injectivity of the period map and hence have to identify non-separable points

in $\mathcal{M}_\Lambda$. In other words, we need to turn $\mathcal{M}_\Lambda$ into a Hausdorff space $\bar{\mathcal{M}}_\Lambda$ by identifying those points of the moduli space of marked hyperkaehler manifolds which map to the same point in the period domain.
A way to achieve this is carried out in great detail in [2]. We content ourselves to give a brief outline of the argumentation in loc. cit. and refer the reader to this publications for the details.
The key step in this process is to detect non-separable points, i.e to find a characterisation for the latter. This is basically the content of thms 4.3 and 4.6 in [4]. We here only paraphrase these results.

Theorem 6.
- *If (X,ϕ) and $(\tilde{X},\tilde{\phi})$ are inseparable in $\mathcal{M}_\Lambda$, then X and $\tilde{X}$ are birational*
- *If X and $\tilde{X}$ are birational projective irreducible symplectic manifolds, then there exist two different markings $\phi : H^2(X,\mathbb{Z}) \cong \Lambda$ and $\tilde{\phi} : H^2(\tilde{X},\mathbb{Z}) \cong \Lambda$ such that (X,ϕ) and $(\tilde{X},\tilde{\phi}) \in \mathcal{M}_\Lambda$ are inseparable points.*

Having detected the algebraic notion behind the topological property of non-separateness, the Hausdorff-reduction of the moduli space of marked hyperkaehler manifolds $\mathcal{M}_\Lambda$ can be shown to exist with the following properties (see [2]; Corollary 4.10):

Theorem 7. *The period map $p : \mathcal{M}_\Lambda \to \Omega \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ factors over the Hausdorff-reduction $\bar{\mathcal{M}}_\Lambda$. This is, there are a complex Hausdorff manifold $\bar{\mathcal{M}}_\Lambda$ and locally biholomorphic maps factoring the period map $p : \mathcal{M}_\Lambda \to \bar{\mathcal{M}}_\Lambda \to \Omega \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ such that (X,ϕ) and $(\tilde{X},\tilde{\phi})$ in $\mathcal{M}_\Lambda$ map to the same point in $\bar{\mathcal{M}}_\Lambda$ iff they are inseparable points in $\mathcal{M}_\Lambda$*

Huybrechts in [2] then uses the above construction of $\mathcal{M}_\Lambda$ to reproof a Global Torelli theorem for hyperkaehler manifolds in the framework of moduli spaces of marked manifolds, which was originally proved by Verbitsky using Teichmueller theory. We would like to give account of this remarkable result by highlighting some of the relevant steps in the course of the proof but leaving out on most of the detailed calculations presented in [2].
This said, we first state the Global Torelli theorem proved by Verbitsky in [3].

Theorem 8. *Let Λ be a lattice of signature $(3, b_2 - 3)$ and $\mathcal{M}^0$ a connected component of the moduli space of hyperkaehler manifolds $\mathcal{M}_\Lambda$. Then the period map*

$$\begin{aligned} p : \mathcal{M}^0 &\to \Omega_\Lambda \\ (X,\phi) &\mapsto [\phi(H^{20}(X))] \end{aligned}$$

is generically injective

The key idea behind the proof of this theorem in [2] is to show that the (extended) period map $\bar{\mathcal{M}}_\Lambda \to \Omega_\Lambda$ is a covering of the simply connected period domain Ω_Λ by a connected component $\bar{\mathcal{M}}_\Lambda^0$ of the Hausdorff reduction of the moduli space of marked hyperkaehler manifolds $\bar{\mathcal{M}}_\Lambda$. Due to the simple connectedness of the base space this immediately shows that the map is indeed an isomorphism. In order to establish the claimed generic injectivity in Verbitsky's theorem it then only remains to see that the map $\mathcal{M}_\Lambda \to \bar{\mathcal{M}}_\Lambda$ is generically $1:1$ which follows from the fact that the locus of points being separable in $\mathcal{M}_\Lambda$ is the complement of a countable union of closed subsets.
We next wish to outline some of these ideas in slightly more detail, especially shall we explicitly state Huybrechts proof of the surjectivity of the period map as well as the covering space results as we shall include the latter in the proof of a smooth version of the Global Torelli theorem in [1] in the subsequent section.
In order to be able to present Huybrechts surjectivity proof, we first have to introduce some notions and preliminary results from [2] which we state without proof.

Definition 2. *We call a three dimensional subspace $W \subset \Lambda \otimes \mathbb{R}$ positive if the restriction of the BB-form to W is strictly positive*

Definition 3. *To such a positive 3 space we can now associate a twistor line $T_W = \Omega_\Lambda \cap \mathbb{P}(W \otimes \mathbb{C})$.*

This twistor line T_W is obviously given by a smooth quadric inside $\mathbb{P}(W \otimes \mathbb{C}) \cong \mathbb{P}^2$ and hence simply a projective line, $T_W \cong \mathbb{P}^1$. Furthermore, we call a twistor line generic if $W^\perp \cap \Lambda = 0$ and two points $x, y \in \Omega_\Lambda$ equivalent if there are twistor lines $T_{W_1}, ..., T_{W_n}$ and points $x = x_1, ..., x_{n+1} = y$ with $x_i, x_{i+1} \in T_{W_i}$.
It can then be shown (see [2], Prop.3.7) that any two points $x, y \in \Omega_\Lambda$ are equivalent.
As a last ingredient for the proof of the global surjectivity of the period map we need the following technical proposition [[2], Prop. 5.4]

Proposition 2. *Let $(X, \phi) \in \bar{\mathcal{M}}_\Lambda$ such that the period $P(X, \phi)$ is contained in a generic twistor line $T_W \subset D$. Then there is a unique lift of T_W to a curve in $\bar{\mathcal{M}}_\Lambda$ through (X, ϕ), i.e there is a commutative diagram with (X, ϕ) in the image of $\tilde{i}$.*

$$\begin{array}{ccc} T_W & \xrightarrow{\tilde{i}} & \bar{\mathcal{M}}_\Lambda \\ & \searrow^{i} & \downarrow P \\ & & \Omega_\Lambda \end{array}$$

Herewith we now state and proof global surjectivity of the period map, closely following [[2], $Thm 5.5$].

Theorem 9. *Let $\mathcal{M}^0_\Lambda$ be a connected component of the moduli space $\mathcal{M}_\Lambda$ of marked hyperkaehler manifolds. Then the restriction of the period map $P : \mathcal{M}^0_\Lambda \twoheadrightarrow \Omega_\Lambda \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ is surjective.*

Proof. As we remarked above, any two points $x, y \in \Omega_\Lambda$ are equivalent. It is thus sufficient to prove that whenever 2 points x and y are contained in a generic twistor line T_W, then $x \in P(\mathcal{M}^0_\Lambda)$ iff $y \in P(\mathcal{M}^0_\Lambda)$. But this is easily seen to be true by invoking the previous technical proposition. Indeed, by the latter any generic twistor line T_W through x can be lifted to a curve through any given preimage (X, ϕ) of x. Moreover, thanks to the commutativity of the diagram in the technical proposition, y will be contained in the image of the lift of T_W, which concludes the proof. □

As stated above in the outline of the strategy, the next step in Huybrecht's proof of Verbitskys Global Torelli theorem consists in establishing the (restriction of) the period map as a covering space map between $\bar{\mathcal{M}}^0_\Lambda$ and the period domain. As is shown in [2] a 'local version' of the proposition stating that any two points in the period domain be equivalent (see [2] for the precise definition of local in this context) and some general topological results on the characterisation of covering maps as local homeomorphisms satisfying certain properties [see [2] ,Prop.5.6] lead to the following two results in [2] we will need in our proof of the Global Torelli Theorem à la Bakker et al. in the upcoming section

Lemma 1 (Cor. 5.8 in [2])**.** *If $\bar{\mathcal{M}}^0_\Lambda$ is a connected component of $\bar{\mathcal{M}}_\Lambda$, then the period map $p : \bar{\mathcal{M}}^0_\Lambda \to \Omega_\Lambda$ is a covering space.*

Due to the simple connectedness of Ω_Λ we here from immediately infer

Corollary 1. *If $\bar{\mathcal{M}}^0_\Lambda$ is a connected component of $\bar{\mathcal{M}}_\Lambda$, then the period map $p : \bar{\mathcal{M}}^0_\Lambda \to \Omega_\Lambda$ is an isomorphism.*

After this summary of a Global Torelli theorem for irreducible symplectic manifolds originally established by Verbitsky, we would like to turn to the main part of the present text dealing with the proof of a Global Torelli theorem partly inspired by the authors of [1] which, in its original form, even applies to the more general context of so called primitive symplectic varieties.

5 Generalizations to the singular setting - Primitive symplectic varieties

The at first sight amazing fact that most of the above outlined results carry over almost verbatim to primitive symplectic varieties which by definition are non-smooth, is foremost due to two properties of this class of varieties. Firstly, they can be shown to be of Fujiki class $\mathscr{C}$ i.e they admit a resolution of singularities by a compact Kaehler manifold and secondly that (see Thm 3.4 in [1]) any symplectic variety and hence in particular any primitive symplectic variety has at worst rational singularities. Most proofs of results analogous to those seen in the previous sections rely on these two cardinal properties. In this sense primitive symplectic manifolds almost by definition build the most general setting in which Torelli theorems for hyperkaehler manifolds can be shown to hold using a similar reasoning. We mean to be rather concise in this section and content ourselves with a short overview of the most important results that mutatis mutandis do carry over to this generalized context and refer the interested reader to the detailed exposition in [1] for proofs and details.

For our final aim to give a proof of a strengthened version of the Global Torelli theorem of Verbitsky stated in the previous section for hyperkaehler manifolds it is sufficient to state generalizations of some of the results mentioned in the previous sections in order to motivate and justify the proof strategy in [1].

We proceed along the line in (4.1) and start with the BB form defined there.

5.1 Beauville-Bogomolov-Fujiki Form

The first important fact in this context is that compact symplectic varieties admit a pure Hodge structure on the torsion free part of their second cohomology $H^2(X,\mathbb{Z})_{tf}$ [[1], Cor. 3.5].

Furthermore, there is a non-degenerate quadratic form $q_X : H^2(X,\mathbb{R}) \to \mathbb{R}$ whose associated bilinear form has signature $(3, b_2(X)-3)$. Due to its analogous properties to (4.2) its again refered to as BBF form.

5.2 Deformation Theory

The guiding principle concerning the deformation theory of primitive symplectic varieties is that most results from the smooth setting carry over if restricted to locally trivial deformations. Notice here that any deformation of a hyperkaehler manifold is locally trivial, whereas in the singular setting this is no longer true.

More precisely, the general guiding principle manifests itself as follows: By [1] section 4.3 and references therein, there is a miniversal deformation $\mathscr{X} \to S$ of a given compact complex space Z which is versal in any point of S.

This miniversal family is referred to as Kuranishi family and the base space $Def(Z) := S$ is called Kuranishi space.

If Z is a complex space satisfying $H^0(Z, T_Z) = 0$ then any miniversal deformations is actually universal.

Moreover, the restriction of the Kuranishi family to the closed complex subspace $Def^{lt}(Z) \subset Def(Z)$ parametrizing locally trivial deformations of Z is a locally trivial deformation of Z and again miniversal for locally trivial deformations.

It can furthermore be shown (see [1]; Lemma 4.6) that for primitive symplectic varieties X, $H^0(X, T_X) = 0$ and hence the (restricted) Kuranishi family is universal over the smooth space $Def^{lt}(X)$ of locally trivial deformations. Moreover, as in the smooth case, the dimension of $Def^{lt}(X)$ is given by $h^{11}(X)$.

Lastly, it is important to mention that every small locally trivial deformation of a primitive sym-

plectic variety X is again primitive symplectic. In particular is the locally trivial Kuranishi family universal for all its fibres.
Concerning the period map we do not intend to add any further comment, the relevant statements and definitions, in particular the Local Torelli theorem, carry over almost verbatim. The proof of the latter is slightly more involved as one is now dealing with complex spaces instead of regular schemes and thus in particular to proof the existence of the period map one has to pass to the smooth locus and replace the argument involving the no longer appplicable Grauert's theorem we used in (4.4). Finally, in order to be able to state the Global Torelli theorem of [1] we need to address the construction of the moduli space of marked primitive symplectic varieties. In great parts it is again similar to the one of marked hyperkaehler manifolds but there are some changes to be imposed

5.3 Moduli space of marked primitive symplectic varieties

Given a lattice Λ we will again denote by $\mathscr{M}_\Lambda$ the now analytic coarse moduli space of Λ-marked (same definition as for hyperkaehler manifolds) primitive symplectic varieties. As a set, $\mathscr{M}_\Lambda$ consist of isomorphism classes of Λ-marked primitive symplectic varieties (X,μ). It can be given the structure of a non-Hausdorff complex manifold by glueing locally points in the bases of the locally trivial Kuranishi family over which the fibres are isomorphic in the same way we showed in the case of hyperkaehler manifolds. Furthermore, one can, using slightly different arguments, show that a Hausdorff reduction $\bar{\mathscr{M}}_\Lambda$ of $\mathscr{M}_\Lambda$ exists and results from identifying exactly the non-separable points of $\mathscr{M}_\Lambda$.
In order to proceed we need the following definition:

Definition 4. *We call a Hodge structure on* Λ *semi-polarized if* $q:\Lambda\otimes\Lambda\to\mathbb{R}(-2)$ *is a morphism of Hodge structures. Moreover it is called hyperkaehler it is pure of weight 2 with* $h^{20}=h^{02}=1$*, the signature of q is* $(3,b_2-3)$ *and q is positive definite on the real space underlying* $H^{20}\oplus H^{02}$.

Hyperkaehler Hodge structures on Λ are parametrized by the period domain

$$\Omega_\Lambda=\{[\sigma]\in\mathbb{P}(\Lambda_{\mathbb{C}})|q(\sigma)m=0;q(\sigma,\bar{\sigma})>0\}$$

For a primitive symplectic variety X with $(H^2(X,\mathbb{Z}),q_X)\cong(\Lambda,q)$ we denote by $\mathscr{M}^+$ the moduli space of Λ-marked locally trivial deformations of X. With this definition, $\mathscr{M}^+$ is the union of connected components of the total moduli space $\mathscr{M}_\Lambda$. In this setting, we have the period map $p:\mathscr{M}^+\to\Omega_\Lambda$ which is a local isomorphism by the local Torelli theorem fitting into the following diagram

$$\begin{array}{ccc} \mathscr{M}^+ & \xrightarrow{H} & \bar{\mathscr{M}}^+ \\ & \underset{P}{\searrow} & \downarrow{\scriptstyle \bar{P}} \\ & & \Omega_\Lambda \end{array}$$

where H denotes the Hausdorff reduction of $\mathscr{M}^+$ and $\bar{P}$ is a local homeomorphism.
Lastly notice that, as in the smooth setting above, we have the automorphism group of the lattice Λ, $O(\Lambda)$, acting on the three spaces $\mathscr{M}^+,\bar{\mathscr{M}}^+,\Omega_\Lambda$ by changing the marking. The maps H, P and $\bar{P}$ respect this action. Lastly, for any connected component $\mathscr{M}\subset\mathscr{M}^+$, we denote by $Mon(\mathscr{M})\subset O(\Lambda)$ the image of the monodromy representation on the second cohomology which is defined up to conjugation. Given these preparatory notions we are now in the position to state the Global Torelli theorem for primitive symplectic varieties[[1]; Thm 8.2]

6 Global Torelli Theorem

Theorem 10 (Global Torelli)**.** *Assume that $rk(\Lambda) \geq 5$ and let $\mathscr{M}$ be a connected component of $\mathscr{M}^+$. Then the following holds*

- *The monodromy group $Mon(\mathscr{M}) \subset O(\Lambda)$ is of finite index*
- *$\bar{P}$ is an isomorphism of $\bar{\mathscr{M}}$ onto the complement in Ω_Λ of countably many maximal Picard rank periods*
- *If X is $\mathbb{Q}$-factorial and terminal, then the same is true for every point $(X,\mu) \subset \mathscr{M}$ and $\bar{P}$ is an isomorphism of $\bar{\mathscr{M}}$ onto Ω.*

A few comments on the above theorem are in order. Firstly, the assumption on the Betti numbers $b_2 \geq 5$ Bakker et al impose is due to the fact that their proof relies on Ratner theory, more precisely they invoke a theorem from Verbitsky [8] on the characterisation of orbit closures which only holds under this restriction. In the smooth setup however, the use of Ratner theory can be bypassed if one is willed to use Huybrechts global surjectivity theorem instead and hence the restriction on the Betti numbers need not be imposed. Secondly, as for hyperkaehler manifolds every deformation is locally trivial, there is no need to introduce the notion of $\mathscr{M}^+$ in this setting.
This said, we are in the position to rephrase and reprove the above theorem in the restrictive case where the more general primitive symplectic varieties are replaced by hyperkaehler manifolds.

Theorem 11 (Global Torelli for hyperkaehler manifolds)**.** *Let $\mathscr{M}$ be a connected component of the moduli space of Λ-marked hyperkaehler manifolds $\mathscr{M}_\Lambda$. Then the following holds*

- *The monodromy group $Mon(\mathscr{M}) \subset O(\Lambda)$ is of finite index*
- *$\bar{P}$ is an isomorphism of $\bar{\mathscr{M}}$ onto the period domain Ω_Λ*

Remark 3. *Obviously the second statement of the theorem is exactly the content of (1) proven in [2] and hence is only listed for completeness. The only statement that remains to be proven is the one concerning the monodromy group.*

Proof. The first step in our argumentation closely follows [[1], Thm 8.5] whereas the remaining ones directly use results proven in [2].
First Step:

Proposition 3. *Let $p \in \Omega_\Lambda$ be a very general period with Picard group generated by a positive vector. Then $\bar{P}^{-1}(p)$ is finite.*

Remark 4. *In the sequel by a very general point in a space we mean a point located in the complement of a countable union of closed subspaces.*

Proof. We first notice that by Lemma 4.7 in [2] the fibre of a very general marked hyperkaehler manifold in $\mathscr{M}$ under the Hausdorff reduction map $\mathscr{M} \to \bar{\mathscr{M}}$ is just one point and it is hence sufficient to show that $P^{-1}(p)$ is finite.
We say that an ample line bundle has BBF square d if $q_X(c_1(L)) = d$.

Claim 1. *Pairs (X,L)consisting of a hyperkaehler manifold X of a fixed deformation type and an ample line bundle L of fixed BBF-square form a bounded family.*

Proof. By Lemma 5.7 in [1] the BBF form is deformation invariant and by Cor.5.21 in loc.cit. the holomorphic Euler characteristic is given by $\mathscr{X}(L) = f_X(q_X(c_1(L)))$ where f_X denotes a polynomial which is furthermore also invariant under small deformations. In order to prove the claim we thus need to show that there is uniform k such that L^k is very ample. But this follows from Prop 2.7 in [10], which says that given a bounded family $\{L\}$ (i.e with fixed coefficients in the Hilbert polynomial) there is $k \in \mathbb{Z}$ such that L_t is k-regular for all members of the bounded family. We shortly recall that k-regular means that $H^i((L(k-i))) = 0$ for all $i > 0$.
This shows that we can choose a common k in the Serre vanishing theorem that works for all members of the family. Reduced to our situation we deduce that there is some fixed N such that for any pair (X, L) the variety X can be embedded in $\mathbb{P}^N$ via some fixed power L^k of L.
Let H be the corresponding Hilbertscheme of subschemes of $\mathbb{P}^N$ of bounded degree and let $f : \mathscr{X} \to H$ denote the universal family. Then by the openness of symplecticity and smoothness, there is an open subset $\tilde{H}$ over which the fibres of f are hyperkaehler manifolds.
Next, by generic smoothness of f, there is an open subset V of $\tilde{H}$ such that $f^{-1}V \to V$ is a smooth proper morphism. I.e. the restriction of f defines a small deformation.
It follows from the above that the set of pairs (X, L) as in the statement of the proposition together with the choice of an embedding into $\mathbb{P}^N$ as above is an open subscheme V of H. Thus the $\mathbb{C}-pts$ of the quotient stack $[PGL_{n+1} \diagdown V]$ then parametrize isomorphism classes of pairs (X, L). Moreover, as $H^0(X, T_X) = 0$ for any hyperkaehler manifold, Aut(X) is o-dimensional as a Lie group and hence finite. By general theory on quotient stacks it then follows that $[PGL_{n+1} \diagdown V]$ is a Deligne-Mumford stack. Moreover, there is an étale atlas $S \twoheadrightarrow [PGL_{n+1} \diagdown V]$. Summing up, there exist, dependent on fixed deformation type and choice of BBF-square, a finite-type scheme S and a smooth family $f : \mathscr{X} \to S$ of hyperkaehler manifolds and a relatively ample line bundle $\mathscr{L}$ on $\mathscr{X}$ with the property that every (X, L) as in the statement appears at least once and at most finitely many times as a fibre.
Next, we construct a period map on every component S_0 of the scheme S from above as follows: For some $v \in \Lambda$ with $q(V) = d$ we define a period map $P_v : S_0 \to O(v^\perp) \diagdown \Omega_{v^\perp}$. P_v is then a local isomorphism and as the fibres are algebraic, by [9] also quasi-finite. If we then fix a period $p \in \Omega_\Lambda$ and v as a generator of the Picard group with BBF square d, it follows from this that there are finitely many isomorphism classes of pairs (X, L) whith X a hyperkaehler manifold deformation equivalent to some given X^+ and L an ample line bundle of BBF square d such that $H^2(X, \mathbb{Z})$ and p are abstractly isomorphic as polarized Hodge structures. Given that the Picard rank of p by assumption is 1 , we see that furthermore, there are only finitely many isomorphism classes of projective X deformation equivalent to X^+ and with second cohomology lattice abstractly isomorphic to p as polarized Hodge structures. We conclude the proof of the claim by noticing that by thm 6.9 in [1], every point in the fibre $P^{-1}(p)$ is projective and uniquely polarized by a class of BBF square d. □

□

Second Step:
By Huybrechts surjectivity of the period map (9), smoothness of $\bar{P} : \bar{\mathscr{M}}_\Lambda \to \Omega_\Lambda$ and pathconnectedness of Ω_Λ we thus see that $\bar{\mathscr{M}}_\Lambda$ has finitely many connected components $\mathscr{M}$. Thus $Mon(\mathscr{M})$, by standard theory of local systems the stabilizer of the component $\mathscr{M}$, is of finite index in $O(\Lambda)$.
This ends the proof of the first statement of the Global Torelli theorem. The second claim follows as remarked above directly from (1). □

To sum up, the main aim of the present text was two fold. On the one hand it should give a brief and by no means exhaustive account on some interesting results and findings on hyperkaehler manifolds and to some extent primitive symplect varieties whereas on the other hand two of the corner stones of the moduli theory of these varieties, the Global Torelli theorems, should be presented in a slightly modified fashion by combining proof ideas of two different authors.

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