Discrete complex analysis

Convergence results

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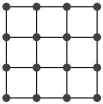
Research Seminar
on Discrete Geometry and Geometry of Numbers
21.11.2023

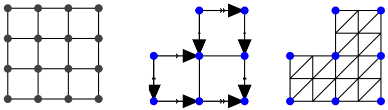
Numerical analysis



Courant...

Applications of discrete complex analysis: an overview

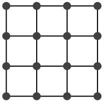
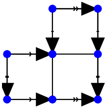
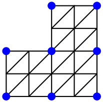
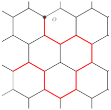
Numerical analysis



Courant. . .

Bobenko–S.

Applications of discrete complex analysis: an overview

Numerical analysis			Statistical physics
			
Courant. . .	Bobenko–S.		Duminil-Copin–Smirnov

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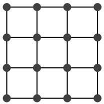 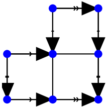 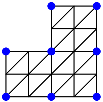	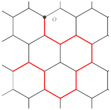 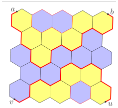
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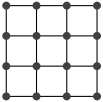 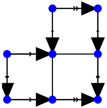 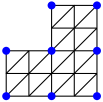	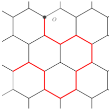 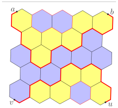 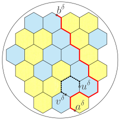
Courant. . . Bobenko–S.	Duminil-Copin–Smirnov Khristoforov–S.–Smirnov

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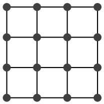

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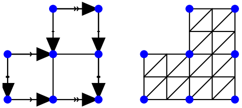
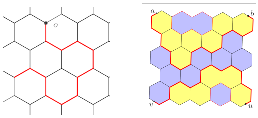
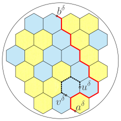


Bobenko–S.

Statistical physics

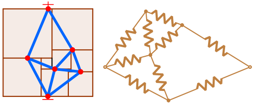


Duminil-Copin–Smirnov



Khristoforov–S.–Smirnov

Combinatorial geometry



Brooks–Smith–Stone–Tutte

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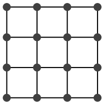
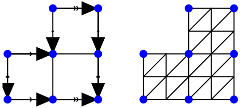


Courant. . .



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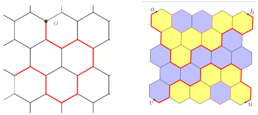

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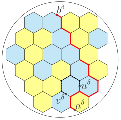

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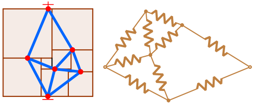

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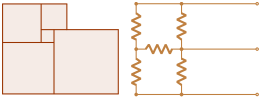

Kenyon

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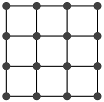

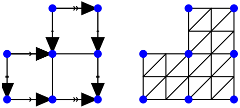

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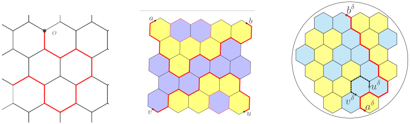
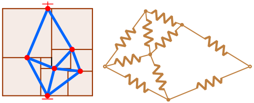
Statistical physics



Duminil-Copin–Smirnov

Khristoforov–S.–Smirnov

Combinatorial geometry



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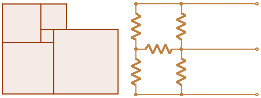

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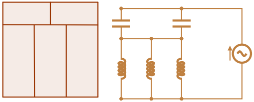
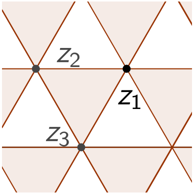


Prasolov–S.

Different discretizations of complex analysis

Triangular lattice



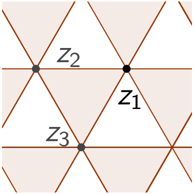
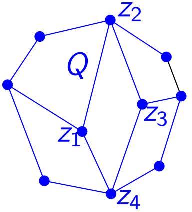
$$f(z_1) + f(z_2) + f(z_3) = 0$$

Dynnikov–Novikov

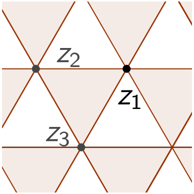
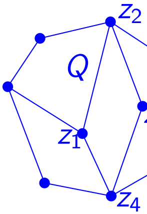
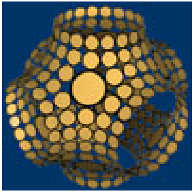
$\downarrow$

integrable systems

Different discretizations of complex analysis

Triangular lattice	Quadrilateral lattice
	
$f(z_1)+f(z_2)+f(z_3)=0$	$\frac{f(z_1)-f(z_3)}{z_1-z_3}=\frac{f(z_2)-f(z_4)}{z_2-z_4}$
Dynnikov–Novikov	Isaacs, Ferrand, ...
$\downarrow$	$\downarrow$
integrable systems	numerical analysis statistical physics combinatorial geometry

Different discretizations of complex analysis

Triangular lattice	Quadrilateral lattice	Circle packing
		
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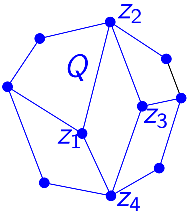
Dynnikov–Novikov	Isaacs, Ferrand, ...	Thurston
↓	↓	↓
integrable systems	numerical analysis statistical physics combinatorial geometry	conformal geometry

Different discretizations of complex analysis

Quadrilateral lattice



$$\frac{f(z_1)-f(z_3)}{z_1-z_3}=\frac{f(z_2)-f(z_4)}{z_2-z_4}$$

Isaacs, Ferrand, ...

↓

numerical analysis
statistical physics
combinatorial geometry

Main notions of discrete complex analysis

A graph $Q \subset \mathbb{C}$ is a *quadrilateral lattice* $\Leftrightarrow$ each bounded face is a quadrilateral.

A function $f : Q \to \mathbb{C}$ is *discrete analytic* $\Leftrightarrow$

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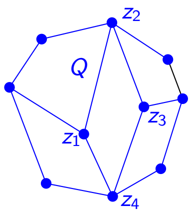
for each face $z_1 z_2 z_3 z_4$ with the vertices listed clockwise. $\mathrm{Re}\, f$ is called *discrete harmonic.*



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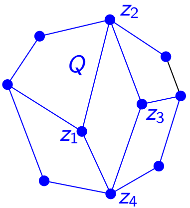
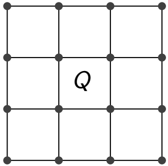
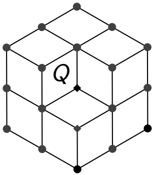
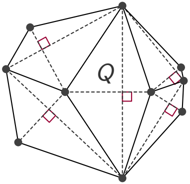




square lattice
Isaacs,Ferrand (1940s)



rhombic lattice
Duffin (1960s)



orthogonal lattice
Mercat (2000s)

The convergence problem in discrete complex analysis

Problem. Prove convergence of discrete harmonic functions to their continuous counterparts as maximal edge length $\to 0$.

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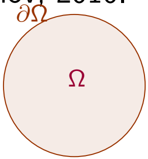
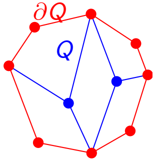


The *Dirichlet problem* in a domain Ω is to find a continuous function $u_{\Omega,g}\colon \mathrm{Cl}\Omega \to \mathbb{R}$ having given boundary values $g\colon \partial\Omega \to \mathbb{R}$ and such that $\Delta u_{\Omega,g} = 0$ in Ω.



The *Dirichlet problem* on Q is to find a discrete harmonic function $u_{Q,g}\colon Q \to \mathbb{R}$ having given boundary values $g\colon \partial Q \to \mathbb{R}$. $(\exists!)$

Convergence theorem for the Dirichlet problem

A sequence $\{Q_n\}$ is *nondegenerate uniform* $\Leftrightarrow \exists\,\mathrm{const} > 0$:

- the angle between the diagonals and the ratio of the diagonals in each quadrilateral face are $> \mathrm{const}$,

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Convergence Theorem for BVP (S. 2013). *Let $\Omega \subset \mathbb{C}$ be a bounded simply-connected domain. Let $g : \mathbb{C} \to \mathbb{R}$ be a smooth function. Take a nondegenerate uniform sequence of finite orthogonal lattices $\{Q_n\}$ such that $\mathrm{Step}(Q_n)$, $\mathrm{Dist}(\partial Q_n, \partial\Omega) \to 0$.* **Then the solution $u_{Q_n,g} : Q_n \to \mathbb{R}$ of the Dirichlet problem on Q_n uniformly converges to the solution $u_{\Omega,g} : \Omega \to \mathbb{R}$ of the Dirichlet problem in Ω.**

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$$\nabla_Q u(z_1z_2z_3z_4) \cdot \overrightarrow{z_1z_3} = u(z_1) - u(z_3),$$
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$$E_Q(u) := \sum_{z_1z_2z_3z_4 \subset Q} |\nabla_Q u(z_1z_2z_3z_4)|^2 \cdot \mathrm{Area}(z_1z_2z_3z_4).$$

Method of the proof: energy estimates

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Convexity Principle. *The energy $E_Q(u)$ is a strictly convex functional on the affine space $\mathbb{R}^{Q^0-\partial Q}$ of functions $u\colon Q^0 \to \mathbb{R}$ having fixed values at the boundary ∂Q.*

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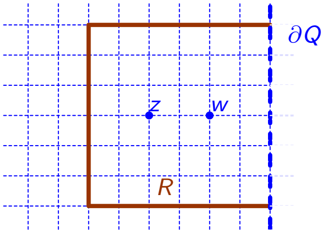
Variational principle. *A function $u\colon Q^0 \to \mathbb{R}$ has minimal energy $E_Q(u)$ among all the functions with the same boundary values if and only if it is discrete harmonic.*

Equicontinuity Lemma. *Let Q be an orthogonal lattice. Let $u\colon Q^0 \to \mathbb{R}$ be a discrete harmonic function. Let $z, w \in B^0$ be two vertices with $|z-w| \geq \mathrm{Step}(Q)$. Let R be a square of side length $r > 3|z-w|$ with the center at $\frac{z+w}{2}$ and the sides parallel and orthogonal to zw. Then $\exists \mathrm{Const}$:*

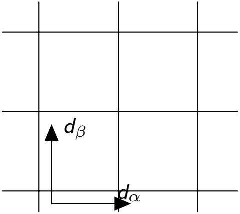
$$|u(z)-u(w)| \leq \mathrm{Const} \cdot E_Q(u)^{1/2} \cdot \log^{-1/2} \frac{r}{3|z-w|} + \\ + \max_{z',w' \in R \cap \partial Q \cap B^0} |u(z')-u(w')|.$$

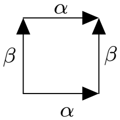


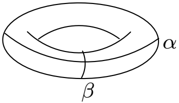
2
Discrete analytic functions in Riemann surfaces

Riemann surfaces

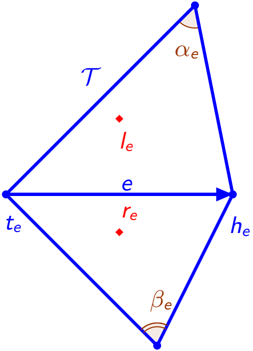
Riemann surface	Analytic functions
planar domain	functions $u(x,y)+iv(x,y)$ s.t. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$
quotient $\mathbb{C}$ by a lattice	doubly periodic analytic functions
complex algebraic curve $a_{nm}z^n w^m+\dots+a_{00}=0$	analytic functions in both w and z
polyhedral surface	continuous functions which are analytic on each face







$\mathcal{R}$	a polyhedral surface
$\mathcal{T}$	its triangulation
$\mathcal{T}^0$	the set of vertices
$\vec{\mathcal{T}}^1$	the set of oriented edges
$\mathcal{T}^2$	the set faces



A *discrete analytic function* is a pair $(u\colon \mathcal{T}^0 \to \mathbb{R}, v\colon \mathcal{T}^2 \to \mathbb{R})$ such that $\forall e \in \vec{\mathcal{T}}^1$

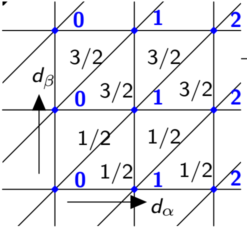
$$v(l_e) - v(r_e) = \frac{\cot\alpha_e + \cot\beta_e}{2}(u(h_e) - u(t_e)).$$

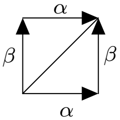
(Duffin, Pinkall–Polthier, Desbrun–Meyer–Schröder, Mercat)

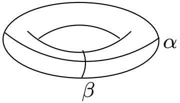
Remark. $\mathcal{T}$ is a *Delauney* triangulation of $\mathbb{R}^2 \Rightarrow u \sqcup iv$ is discrete analytic on Q (in the sense of **Part 1** of the slides).

Discrete Abelian integrals of the 1st kind

$p\colon \widetilde{\mathcal{R}} \to \mathcal{R}$	the universal covering
$\{\alpha, \beta\}$	the basis of $\pi_1(\mathcal{R})$
$\{d_\alpha, d_\beta\}$	the automorphisms of p





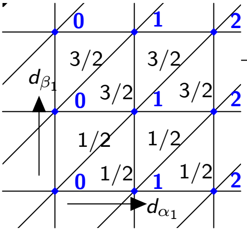


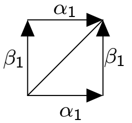
$$\widetilde{\mathcal{T}} \quad \overset{p}{\to} \quad \mathcal{T} \quad \approx \quad S^1 \times S^1$$

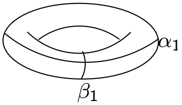
A *discrete Abelian integral of the 1st kind* with periods $A, B \in \mathbb{C}$ is a discrete analytic function $(\mathrm{Re} f : \widetilde{\mathcal{T}}^0 \to \mathbb{R}, \mathrm{Im} f : \widetilde{\mathcal{T}}^2 \to \mathbb{R})$ such that $\forall z \in \widetilde{\mathcal{T}}^0, \forall w \in \widetilde{\mathcal{T}}^2$

$$[\mathrm{Re} f](d_\alpha z) - [\mathrm{Re} f](z) = \mathrm{Re}\, A; \quad [\mathrm{Re} f](d_\beta z) - [\mathrm{Re} f](z) = \mathrm{Re}\, B;$$
$$[\mathrm{Im} f](d_\alpha w) - [\mathrm{Im} f](w) = \mathrm{Im}\, A; \quad [\mathrm{Im} f](d_\beta w) - [\mathrm{Im} f](w) = \mathrm{Im}\, B.$$

$p\colon \widetilde{\mathcal{R}} \to \mathcal{R}$	the universal covering
$\{\alpha_k, \beta_k\}_{k=1}^{g}$	the basis of $\pi_1(\mathcal{R})$
$\{d_{\alpha_k}, d_{\beta_k}\}_{k=1}^{g}$	the automorphisms of p







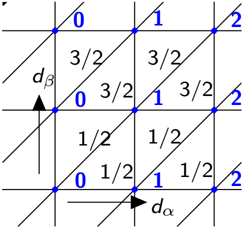
$$\widetilde{\mathcal{T}} \quad \xrightarrow{p} \quad \mathcal{T} \quad \approx \quad S^1 \times S^1$$

A *discrete Abelian integral of the 1st kind* with periods $A_1, \dots, A_g, B_1, \dots, B_g \in \mathbb{C}$ is a discrete analytic function $(\mathrm{Re} f\colon \widetilde{\mathcal{T}}^0 \to \mathbb{R}, \mathrm{Im} f\colon \widetilde{\mathcal{T}}^2 \to \mathbb{R})$ such that $\forall z \in \widetilde{\mathcal{T}}^0, \forall w \in \widetilde{\mathcal{T}}^2$

$$\mathrm{Re} f(d_{\alpha_k} z) - \mathrm{Re} f(z) = \mathrm{Re}\, A_k; \quad \mathrm{Re} f(d_{\beta_k} z) - \mathrm{Re} f(z) = \mathrm{Re}\, B_k;$$
$$\mathrm{Im} f(d_{\alpha_k} w) - \mathrm{Im} f(w) = \mathrm{Im}\, A_k; \quad \mathrm{Im} f(d_{\beta_k} w) - \mathrm{Im} f(w) = \mathrm{Im}\, B_k.$$

Existence & Uniqueness Theorem (Bobenko–S. 2012)
$\forall A \in \mathbb{C}$ there is a discrete Abelian integral of the 1st kind with the A-period A. It is unique up to constant.

The *discrete period matrix* $\Pi_{\mathcal{T}}$ (*period matrix* $\Pi_{\mathcal{T}}$) is the B-period of the discrete Abelian integral (Abelian integral) of the 1st kind with the A-period 1.
It is a 1×1 matrix for a surface of genus 1.



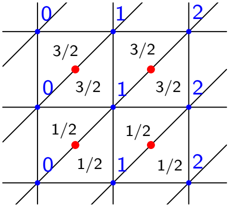
$\Pi_{\mathcal{T}} = i = \Pi_{\mathcal{R}}$

Notation.

$\gamma_z := 2\pi(\text{the sum of angles meeting at } z)^{-1}$
$\gamma_z > 1 \Leftrightarrow$ "curvature" > 0
$\gamma_{\mathcal{R}} := \min_{z \in \mathcal{T}^0}\{1, \gamma_z\}$

Existence & Uniqueness Theorem (Bobenko–S. 2012)
For any numbers $A_1, \dots, A_g \in \mathbb{C}$ there exist a discrete Abelian integral of the 1st kind with A-periods $A_1, \dots, A_g$. It is unique up to constant.

Let $\phi^l_\mathcal{T} = (\mathrm{Re}\,\phi^l_\mathcal{T} \colon \widetilde{\mathcal{T}}^0 \to \mathbb{R}, \mathrm{Im}\,\phi^l_\mathcal{T} \colon \widetilde{\mathcal{T}}^2 \to \mathbb{R})$ be the unique (up to constant) discrete Abelian integral of the 1st kind with A-periods $A_k = \delta_{kl}$.
The *discrete period matrix* $\Pi_\mathcal{T}$ is the $g \times g$ matrix whose columns are the B-periods of $\phi^1_\mathcal{T}, \dots, \phi^g_\mathcal{T}$.



Example. For $\mathcal{R} = \mathbb{C}/(\mathbb{Z} + \eta\mathbb{Z})$:

$\mathrm{Re}\,\phi^1_\mathcal{T}(z) = \mathrm{Re}\,z$,
$\mathrm{Im}\,\phi^1_\mathcal{T}(w) = \mathrm{Im}\,w^*$,
where w^* is the circumcenter of a face w.

The complex structure on polyhedral surfaces

Polyhedral metric $\rightsquigarrow$ complex structure

Identify each face $w \in \widetilde{T}^2$ with a triangle in $\mathbb{C}$ by an orientation-preserving isometry.

A function $f\colon \widetilde{\mathcal{R}} \to \mathbb{C}$ is *analytic*, if it is continuous and its restriction to the interior of each face is analytic.

Let $\phi^l_{\mathcal{R}}\colon \widetilde{\mathcal{R}} \to \mathbb{C}$ be the unique (up to constant) Abelian integral of the 1st kind with A-periods $A_k = \delta_{kl}$.

The *period matrix* $\Pi_{\mathcal{R}}$ is the $g \times g$ matrix whose columns are the B-periods of $\phi^1_{\mathcal{R}}, \dots, \phi^g_{\mathcal{R}}$.

$\gamma_z := 2\pi(\text{the sum of angles meeting at } z)^{-1}$

$\gamma_z > 1 \Leftrightarrow$ "curvature" > 0

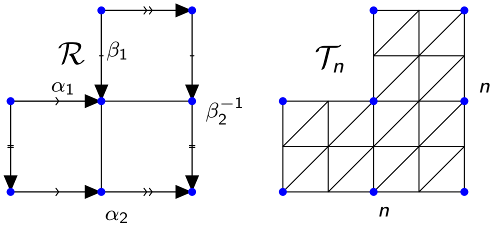
$\gamma_{\mathcal{R}} := \min_{z \in T^0}\{1, \gamma_z\}$

Convergence Theorem for Period Matrices (Bobenko–S. 2013) $\forall\delta > 0\ \exists \mathrm{Const}_{\delta,\mathcal{R}}, \mathrm{const}_{\delta,\mathcal{R}} > 0$ *such that for any triangulation* $\mathcal{T}$ *of* $\mathcal{R}$ *with the maximal edge length* $h < \mathrm{const}_{\delta,\mathcal{R}}$ *and with the minimal face angle* $> \delta$ *we have*

$$\|\Pi_{\mathcal{T}} - \Pi_{\mathcal{R}}\| \le \mathrm{Const}_{\delta,\mathcal{R}} \cdot \begin{cases} h, & \text{if } \gamma_{\mathcal{R}} > 1/2; \\ h|\log h|, & \text{if } \gamma_{\mathcal{R}} = 1/2; \\ h^{2\gamma_{\mathcal{R}}}, & \text{if } \gamma_{\mathcal{R}} < 1/2. \end{cases}$$

Corollary. **The discrete period matrices** *of a sequence of triangulations of the surface with the maximal edge length tending to zero and with face angles bounded from zero* **converge to the period matrix** *of the surface.*

Model surface:



Computations using a software by S. Tikhomirov:

n	$\|\Pi_{\mathcal{T}_n} - \Pi_{\mathcal{R}}\|$	$\|\Pi_{\mathcal{T}_n} - \Pi_{\mathcal{R}}\| \cdot h^{-2\gamma_{\mathcal{R}}}$
8	0.611	1.22
16	0.363	1.15
32	0.220	1.11
64	0.136	1.08
128	0.084	1.07
256	0.053	1.06

A sequence $\{\mathcal{T}_n\}$ is *nondegenerate uniform* $\Leftrightarrow \exists \mathrm{const} > 0$:

- the minimal face angle is $> \mathrm{const}$;
- $\forall e \in \vec{\mathcal{T}}_n^1$ we have $\alpha_e + \beta_e < \pi - \mathrm{const}$;
- the number of vertices in an arbitrary disk of radius equal to the maximal edge length $(=: \mathrm{Size}(\mathcal{T}_n))$ is $< \mathrm{const}^{-1}$.

Convergence Theorem for Abelian integrals (Bobenko–S. 2013) *Let $\{\mathcal{T}_n\}$ be a nondegenerate uniform sequence of triangulations of $\mathcal{R}$ with $\mathrm{Size}(\mathcal{T}_n) \to 0$. Let $z_n \in \widetilde{\mathcal{T}}_n^0$ converge to $z_0 \in \widetilde{\mathcal{R}}$ and $w_n \in \widetilde{\mathcal{T}}_n^2$ contain z_n.* **Then the discrete Abelian integrals of the 1st kind** $\phi^l_{\mathcal{T}_n} = (\mathrm{Re}\,\phi^l_{\mathcal{T}_n} : \widetilde{\mathcal{T}}_n^0 \to \mathbb{R}, \mathrm{Im}\,\phi^l_{\mathcal{T}_n} : \widetilde{\mathcal{T}}_n^2 \to \mathbb{R})$ **normalized by** $\mathrm{Re}\,\phi^l_{\mathcal{T}}(z_n) = \mathrm{Im}\,\phi^l_{\mathcal{T}}(w_n) = 0$ **converge to the Abelian integral of the 1st kind** $\phi^l_{\mathcal{R}} : \widetilde{\mathcal{R}} \to \mathbb{C}$ **normalized by** $\phi^l_{\mathcal{R}}(z_0) = 0$ *uniformly on compact subsets.*

Discrete Riemann–Roch theorem

A *discrete meromorphic function* is an arbitrary pair $(\mathrm{Re} f : \mathcal{T}^0 \to \mathbb{R}, \mathrm{Im} f : \mathcal{T}^2 \to \mathbb{R})$.

$\mathrm{res}_e\, f := \mathrm{Im} f(r_e) - \mathrm{Im} f(l_e) + \nu(e)\mathrm{Re} f(h_e) - \nu(e)\mathrm{Re} f(t_e)$

A *divisor* is a map $D\colon \mathcal{T}^0 \sqcup \mathcal{T}^1 \sqcup \mathcal{T}^2 \to \{0, \pm 1\}$.

$(f) := I_{\mathrm{Re} f=0} - I_{\mathrm{res}_e f \neq 0} + I_{\mathrm{Im} f=0}$; $\quad l(D) := \dim\{f : (f) \geq D\}$

A *discrete Abelian differential* is an odd map $\omega\colon \vec{\mathcal{T}}^1 \to \mathbb{R}$.

$\mathrm{res}_w\, \omega := \sum_{e \in \vec{\mathcal{T}}^1 : l_e = w} \omega(e)$; $\quad \mathrm{res}_z\, \omega := i \sum_{e \in \vec{\mathcal{T}}^1 : h_e = z} \nu(e)\omega(e)$.

$(\omega) := -I_{\mathrm{res}_z \omega \neq 0} + I_{\omega=0} - I_{\mathrm{res}_w \omega \neq 0}$; $\quad i(D) := \dim\{\omega : (\omega) \geq D\}$

D is *admissible* $\Leftrightarrow (-1)^k D(\mathcal{T}^k) \leq 0$; $\quad \deg D := \sum_z D(z)$.

Discrete Riemann–Roch Theorem (Bobenko–S. 2012)

For admissible divisors D on a triangulated surface of genus g

$$l(-D) = \deg D - 2g + 2 + i(D).$$

3
Convergence via energy estimates

The *energy* of a function $u\colon \Omega \to \mathbb{R}$ is $E_\Omega(u) := \int_\Omega |\nabla u|^2 dA$.
The *gradient* of a function $u\colon Q^0 \to \mathbb{R}$ at a face $z_1z_2z_3z_4$ is the unique vector $\nabla_Q u(z_1z_2z_3z_4) \in \mathbb{R}^2$ such that

$$\nabla_Q u(z_1z_2z_3z_4) \cdot \overrightarrow{z_1z_3} = u(z_1) - u(z_3),$$
$$\nabla_Q u(z_1z_2z_3z_4) \cdot \overrightarrow{z_2z_4} = u(z_2) - u(z_4).$$

The *energy* of the function $u\colon Q^0 \to \mathbb{R}$ is

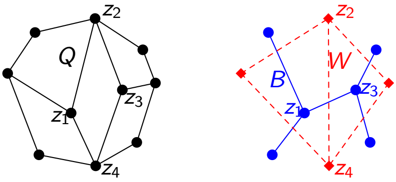
$$E_Q(u) := \sum_{z_1z_2z_3z_4 \subset Q} |\nabla_Q u(z_1z_2z_3z_4)|^2 \cdot \mathrm{Area}(z_1z_2z_3z_4).$$

Convexity Principle. *The energy $E_Q(u)$ is a strictly convex functional on the affine space $\mathbb{R}^{Q^0-\partial Q}$ of functions $u\colon Q^0 \to \mathbb{R}$ having fixed values at the boundary ∂Q.*

Variational principle. *A function $u\colon Q^0 \to \mathbb{R}$ has minimal energy $E_Q(u)$ among all the functions with the same boundary values if and only if it is discrete harmonic.*

Physical interpretation

A *direct-current network*/*alternating-current network* is a connected graph with a marked subset of vertices (*boundary*) and a positive number/complex number with positive real part (*conductance*/*admittance*) assigned to each edge.



- The graph B is naturally an *alternating-current network*
- *Admittance* $c(z_1z_3) := i\frac{z_2-z_4}{z_1-z_3} \Rightarrow \mathrm{Re}\, c(z_1z_3) > 0$
- *Voltage* $V(z_1z_3) := f(z_1) - f(z_3)$
- *Current* $I(z_1z_3) := if(z_2) - if(z_4)$
- *Energy* $E(f) := \mathrm{Re} \sum_{z_1z_3} V(z_1z_3)\bar{I}(z_1z_3)$.

Gradient Convergence Lemma.
$|\nabla g - \nabla_Q(g\,|_{Q^0})| \leq \mathrm{Const} \cdot eh \max_{z \in \mathrm{Conv}(\partial Q)} |D^2 g(z)|$.
Proof. Consider a face $z_1 z_2 z_3 z_4$ of the lattice Q. By the Rolle theorem there is a point $z \in z_1 z_3$ (possibly outside $z_1 z_2 z_3 z_4$ but inside the convex hull $\mathrm{Conv}(\partial Q)$) such that $(\nabla g(z) - [\nabla_Q g](z_1 z_2 z_3 z_4)) \cdot \overrightarrow{z_1 z_3}/|\overrightarrow{z_1 z_3}| = 0$. Thus $(\nabla g - \nabla_Q g) \cdot \overrightarrow{z_1 z_3}/|\overrightarrow{z_1 z_3}| \leq \mathrm{Const} \cdot h \max_{z \in \mathrm{Conv}(z_1 z_2 z_3 z_4)} |D^2 g(z)|$ in $z_1 z_2 z_3 z_4$. The same inequality holds with $z_1 z_3$ replaced by $z_2 z_4$. By projection the lemma follows.

Energy Convergence Lemma. *Let $\partial\Omega$ be smooth and $\{Q_n\} \subset \Omega$ be a nondegenerate uniform sequence of quadrilateral lattices such that $\mathrm{Size}(Q_n)$, $\mathrm{Dist}(\partial Q_n, \partial\Omega) \to 0$. Let $g\colon \mathbb{C} \to \mathbb{R}$ be a C^2 function. Then $E_{Q_n}(g\big|_{Q_n^0}) \to E_\Omega(g)$.*

Proof idea. *Discontinuous* piecewise-linear “interpolation”: $I_Q g\colon z_1z_2z_3z_4 \to \mathbb{R}$ is the linear function s.t.

$$I_Q g(z_1) = g(z_1),$$
$$I_Q g(z_3) = g(z_3),$$
$$I_Q g(z_2) - I_Q g(z_4) = g(z_2) - g(z_4).$$

Thus $\nabla_Q g = \nabla I_Q g, E_Q(g) = E_{\Omega\cap Q}(I_Q g) \Rightarrow$ convergence.

Remark. Discontinuity $\Rightarrow$ usual finite element method helpless!

Proof of Energy Convergence Lemma. Denote by $\widehat{Q}_n$ the domain enclosed by the curve ∂Q_n. Since Q_n approximates Ω and $\partial\Omega$ is smooth it follows that some neighborhood Ω' of Ω contains all the lattices Q_n and $\mathrm{Area}(\Omega - \widehat{Q}_n), \mathrm{Area}(\widehat{Q}_n - \Omega) \to 0$ as $n \to \infty$. Since the domain Ω is bounded and the function $g\colon \mathbb{C} \to \mathbb{R}$ is smooth it follows that ∇g is bounded in $\mathrm{Conv}(\Omega')$. Thus the integrals $E_\Omega(g)$, $E_{\widehat{Q}_n}(g)$ exist and $E_\Omega(g) - E_{\widehat{Q}_n}(g) = E_{\Omega - \widehat{Q}_n}(g) - E_{\widehat{Q}_n - \Omega}(g) \to 0$ as $n \to \infty$. By Gradient Approximation Lemma we get $E_{\widehat{Q}_n}(g) - E_{Q_n}(g\left|_{Q_n^0}\right.) \to 0$ as $n \to \infty$, and the lemma follows.

Hölderness

$u\colon B^0 \to \mathbb{R}$ is *Hölder* $\Leftrightarrow |u(z) - u(w)| \leq \mathrm{const} \cdot |z - w|^p$.

Discrete harmonic functions are Hölder:

- with $p = 1/2$ on *square* lattices (Courant et al 1928);
- with $p = 1$ on *rhombic* lattices (Chelkak–Smirnov, Kenyon 2008 Integrability!);
- with some p on *orthogonal* lattices (Saloff-Coste 1997).

Remark. (Informal meaning of integrability)
For any discrete analytic function $f\colon Q^0 \to \mathbb{C}$ its *primitive* $F(z_m) := \sum_{k=1}^{m-1} \frac{f(z_k)+f(z_{k+1})}{2}(z_{k+1} - z_k)$ is discrete analytic $\Leftrightarrow$ Q is *parallelogrammic*.

Problem (Chelkak, 2011). Are discrete harmonic functions Hölder with $p = 1$ on orthogonal lattices?

The main energy estimate

Equicontinuity Lemma. *Let Q be an orthogonal lattice. Let $u\colon Q^0 \to \mathbb{R}$ be a discrete harmonic function. Let $z, w \in B^0$ be two vertices with $|z-w| \ge \mathrm{Size}(Q)$. Let R be a square of side length $r > 3|z-w|$ with the center at $\frac{z+w}{2}$ and the sides parallel and orthogonal to zw. Then $\exists \mathrm{Const}$: $|u(z)-u(w)| \le$*

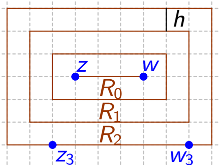
$$\mathrm{Const}\cdot E_Q(u)^{1/2}\cdot\log^{-1/2}\frac{r}{3|z-w|} + \max_{z',w' \in R\cap\partial Q\cap B^0} |u(z')-u(w')|.$$

Proof for a square lattice (cf. Lusternik 1926).

Assume $R\cap\partial Q=\emptyset$, $u(z)\ge u(w)$.

$R_m :=$ rectangle $2mh \times (2mh+|z-w|)$.

$m \le \frac{r-|z-w|}{2h} \Rightarrow R_m \subset R \Rightarrow \exists z_m, w_m \in \partial R_m : u(z_m) \ge u(z), u(w_m) \le u(w)$ Thus



$$E_Q(u) \ge \sum_{m=0}^{[(r-|z-w|)/2h]} \frac{|u(z_m)-u(w_m)|^2}{8m+2|z-w|/h} \ge \frac{|u(z)-u(w)|^2}{8}\log\frac{r}{3|z-w|}.$$

The *laplacian* of a function $u\colon Q^0 \to \mathbb{R}$: $[\Delta_Q u](z) := -\frac{\partial E_Q(u)}{\partial u(z)}$.

Remark. For a *parallelogrammic lattice* Q and a quadratic function g we have $\Delta_Q g = \Delta g$.

Laplacian Approximation Lemma *Let Q be a quadrilateral lattice, R be a square of side length $r > \mathrm{Size}(Q)$ inside ∂Q, and $g\colon \mathbb{C} \to \mathbb{R}$ be a smooth function. Then $\exists \mathrm{Const}$ such that*

$$\left| \sum_{z \in R \cap B^0} [\Delta_Q(g\,|_{Q^0})]\,(z) - \int_R \Delta g\, dA \right| \le$$

$$\mathrm{Const} \cdot \left(r \cdot \mathrm{Size}(Q) \max_{z \in R} |D^2 g(z)| + r^3 \max_{z \in R} |D^3 g(z)| \right).$$

The *energy* of a function $u\colon \widetilde{\mathcal{R}} \to \mathbb{R}$ is $E_{\mathcal{R}}(u) := \int_{\mathcal{R}} |\nabla u|^2 dA$.

The *energy* of a function $u\colon \widetilde{\mathcal{T}}^0 \to \mathbb{R}$ is

$$E_{\mathcal{T}}(u) := \sum_{e \in \mathcal{T}^1} \frac{\cot \alpha_e + \cot \beta_e}{2} (u(h_e) - u(t_e))^2 = E_{\mathcal{R}}(I_{\mathcal{T}} u),$$

where $I_{\mathcal{T}} u$ is the piecewise-linear interpolation of u.

Energy Convergence Lemma for Abelian Integrals. $\forall \delta > 0$ *and* $\forall u\colon \widetilde{\mathcal{R}} \to \mathbb{R}$ — *smooth multi-valued function* $\exists \mathrm{Const}_{u,\delta,\mathcal{R}}, \mathrm{const}_{u,\delta,\mathcal{R}} > 0$ *such that for any triangulation* $\mathcal{T}$ *of* $\mathcal{R}$ *with the maximal edge length* $h < \mathrm{const}_{u,\delta,\mathcal{R}}$ *and with the minimal face angle* $> \delta$ *we have*

$$|E_{\mathcal{T}}(u\,|_{\widetilde{\mathcal{T}}^0}) - E_{\mathcal{R}}(u)| \le \mathrm{Const}_{u,\delta,\mathcal{R}} \cdot \begin{cases} h, & \text{if } \gamma_{\mathcal{R}} > 1/2; \\ h|\log h|, & \text{if } \gamma_{\mathcal{R}} = 1/2; \\ h^{2\gamma_{\mathcal{R}}}, & \text{if } \gamma_{\mathcal{R}} < 1/2. \end{cases}$$

Energy Conservation Principle. *Let f be a discrete Abelian integral of the 1st kind with periods $A_1, \dots, A_g, B_1, \dots, B_g$. Then* $E_{\mathcal{T}}(\mathrm{Re} f) = -\mathrm{Im} \sum_{k=1}^{g} A_k \bar{B}_k$.

Corollary. $\exists$ *discrete harmonic* $u_{\mathcal{T},A_1,\dots,A_g,B_1,\dots,B_g} : \widetilde{\mathcal{T}}^0 \to \mathbb{R}$ *with arbitrary periods* $A_1, \dots, A_g, B_1, \dots, B_g \in \mathbb{R}$.

Variational Principle. $u_{\mathcal{T},A_1,\dots,A_g,B_1,\dots,B_g}$ *has minimal energy among all the multi-valued functions with the same periods.*

Lemma. $E_{\mathcal{T}}(u_{\mathcal{T},P})$ *and* $E_{\mathcal{R}}(u_{\mathcal{R},P})$ *are quadratic forms in* $P \in \mathbb{R}^{2g}$ *with the block matrices*

$$E_{\mathcal{T}} := \begin{pmatrix} \mathrm{Re}\Pi_{\mathcal{T}^*}(\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1}\mathrm{Re}\Pi_{\mathcal{T}} + \mathrm{Im}\Pi_{\mathcal{T}} & (\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1}\mathrm{Re}\Pi_{\mathcal{T}} \\ \mathrm{Re}\Pi_{\mathcal{T}^*}(\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1} & (\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1} \end{pmatrix},$$

$$E_{\mathcal{R}} := \begin{pmatrix} \mathrm{Re}\Pi_{\mathcal{R}}(\mathrm{Im}\Pi_{\mathcal{R}})^{-1}\mathrm{Re}\Pi_{\mathcal{R}} + \mathrm{Im}\Pi_{\mathcal{R}} & (\mathrm{Im}\Pi_{\mathcal{R}})^{-1}\mathrm{Re}\Pi_{\mathcal{R}} \\ \mathrm{Re}\Pi_{\mathcal{R}}(\mathrm{Im}\Pi_{\mathcal{R}})^{-1} & (\mathrm{Im}\Pi_{\mathcal{R}})^{-1} \end{pmatrix}.$$

Proof of the convergence of period matrices

Convergence Theorem for Period Matrices. $\forall \delta > 0$ $\exists \mathrm{Const}_{\delta,\mathcal{R}}, \mathrm{const}_{\delta,\mathcal{R}} > 0$ *such that for any triangulation* $\mathcal{T}$ *of* $\mathcal{R}$ *with the maximal edge length* $h < \mathrm{const}_{\delta,\mathcal{R}}$ *and with the minimal face angle* $> \delta$ *we have*

$$\|\Pi_\mathcal{T} - \Pi_\mathcal{R}\| \le \lambda(h) := \mathrm{Const}_{\delta,\mathcal{R}} \cdot \begin{cases} h, & \text{if } \gamma_\mathcal{R} > 1/2; \\ h|\log h|, & \text{if } \gamma_\mathcal{R} = 1/2; \\ h^{2\gamma_\mathcal{R}}, & \text{if } \gamma_\mathcal{R} < 1/2. \end{cases}$$

Proof modulo the above lemmas.

$$0 \le E_\mathcal{T}(u_{\mathcal{T},P}) - E_\mathcal{R}(u_{\mathcal{R},P}) \le E_\mathcal{T}(u_{\mathcal{R},P}\left|_{\widetilde{\mathcal{T}}^0}\right.) - E_\mathcal{R}(u_{\mathcal{R},P}) \le \lambda(h)$$

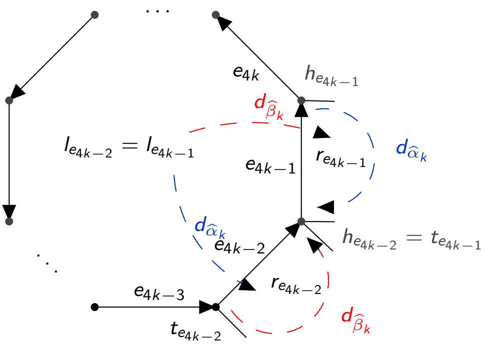
$$\implies \|E_\mathcal{T} - E_\mathcal{R}\| \le \lambda(h) \implies \|\Pi_\mathcal{T} - \Pi_\mathcal{R}\| \le \lambda(h).$$

Riemann bilinear identity

Lemma. *Let $u\colon \widetilde{\mathcal{T}}^0 \to \mathbb{R}$ and $u'\colon \widetilde{\mathcal{T}}^2 \to \mathbb{R}$ be multi-valued functions with periods $A_1, \dots, A_g, B_1, \dots, B_g$ and $A'_1, \dots, A'_g, B'_1, \dots, B'_g$, respectively. Then*

$$\sum_{e\in\mathcal{T}^1} (u'(l_e) - u'(r_e))(u(h_e) - u(t_e)) = \sum_{k=1}^{g} (A_k B'_k - B_k A'_k).$$

Proof plan.
1. Check the identity for the *canonical cell-decomposition*.
2. Perform edge subdivisions.



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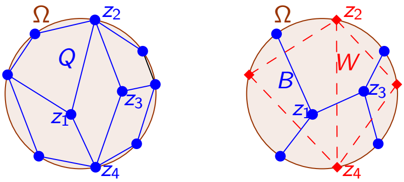
Open problems

Probabilistic interpretation



Let Q be an *orthogonal lattice*. Set $c(z_1z_3) := i\frac{z_2-z_4}{z_1-z_3} > 0$. Consider a *random walk* on the graph B with transition probabilities proportional to $c(z_1z_3)$.

Problem. The trajectories of a loop-erased random walk on B converge to SLE_2 curves in the scaling limit.

Remark. *Rhombic lattices*: Chelkak–Smirnov, 2008.

Open problems

Problem. Generalize Convergence Theorem to:

1. **nonorthogonal** quadrilateral lattices;
2. **discontinuous** boundary values (for convergence of discrete harmonic measure, the Green function, the Cauchy and the Poisson kernels);
3. **mixed** boundary conditions;
4. **infinite** lattices and unbounded domains;
5. **higher** dimensions;
6. **other** elliptic PDE.

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