Ruled Laguerre minimal surfaces

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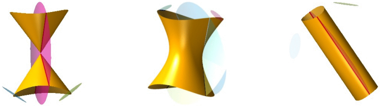
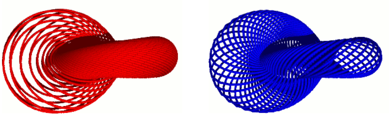
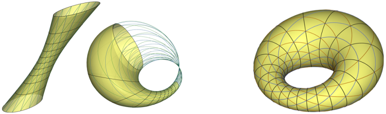
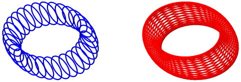
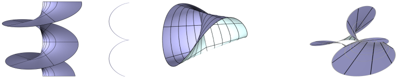
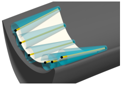
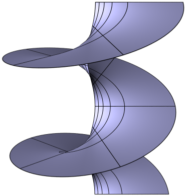
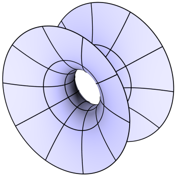
Overview of our research on circles on surfaces

Surfaces ruled by lines & circles  Nilov–S.	Surfaces doubly ruled by circles  S.–Krasauskas
Web from circles on surfaces  Pottmann–Shi–S.	Surfaces doubly ruled by circles  Pakharev–S.
Ruled Laguerre minimal surfaces  S.–Pottmann–Grohs	CNC machining  S.–Bo–Barton–Pottmann

- **Definition.** A *minimal* surface Φ is a local minimizer of the area functional A.
- **Proposition.** *A surface is minimal* $\Leftrightarrow$ *mean curvature* $H \equiv 0$.
- **Examples.**



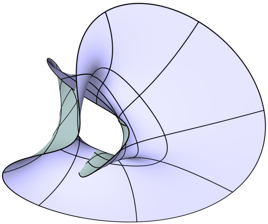


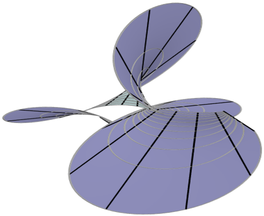
helicoid $x = y \tan z$ catenoid $x^2 + y^2 = \cosh^2 z$

- **Theorem (Catalan, 1842).** *The only ruled minimal surfaces are the plane and the helicoid.*

Laguerre minimal surfaces

- **Definition (Blaschke, 1924).** An *L-minimal* surface Φ is a local minimizer of the functional $\int_\Phi (H^2 - K)/K\, dA$.
- **Examples.** Minimal surfaces; their offsets; spheres.



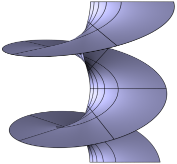


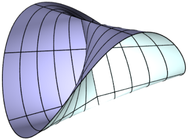
- **Theorem (P.Grohs–H.Pottmann–M.S., 2012).** *All* ruled *L-minimal surfaces up to isometry are the surfaces*

$$\mathbf{R}(\varphi, \lambda) = (A\varphi,\ B\varphi,\ C\varphi + D\cos 2\varphi\,) + \lambda\,(\sin\varphi,\ \cos\varphi,\ 0\,)\,,$$

where $A, B, C, D \in \mathbb{R}$ *are fixed.*

Ruled L-minimal surfaces





helicoid $\mathbf{r}_1$	cycloid $\mathbf{r}_2$	Plücker's conoid $\mathbf{r}_3$
$A = B = D = 0$	$C, D \to 0$	$A = B = C = 0$

- **Notation:**
  - $\mathbf{r}_1(u,v) = \left(u - \frac{u}{u^2+v^2},\ \frac{v}{u^2+v^2} - v,\ 2\operatorname{Arctan}\frac{u}{v} \right)$
  - $\mathbf{r}_2(u,v) = \left(\operatorname{Arctan}\frac{u}{v} - \frac{uv}{u^2+v^2},\ \frac{u^2}{u^2+v^2},\ 0 \right)$
  - $\mathbf{r}_3(u,v) = \frac{uv}{u^2+v^2} \left(\frac{v}{u^2+v^2} - v,\ u - \frac{u}{u^2+v^2},\ \frac{u}{v} \right)$
  - R^θ = rotation through the angle θ around the z-axis.
- **Theorem.** *All ruled L-minimal surfaces up to isometry are*

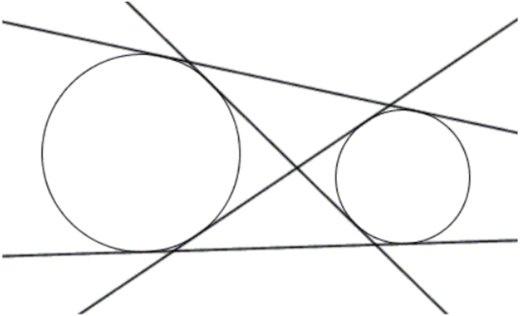
$$\mathbf{r}(u,v) = a_1\mathbf{r}_1(u,v) + a_2\mathbf{r}_2(u,v) + a_3R^\theta\mathbf{r}_3(u,v) \quad \textit{for some } a_1, a_2, a_3, \theta.$$

Main ideas of the proof briefly

- *Isotropic model of Laguerre geometry:* ruled L-minimal surface $\rightsquigarrow$ graph of a biharmonic function carrying a family of isotropic circles.
- *The Pencil theorem:* the top view of the family is a pencil. *Equivalently:* all the rulings of an L-minimal surface are parallel to one plane.
- *Explicit solution* of the biharmonic equation in convenient coordinates associated with the pencil.

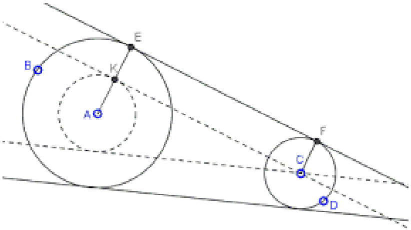
Antique geometry problem.

Construct a common tangent to 2 given circles using a compass and a straightedge.



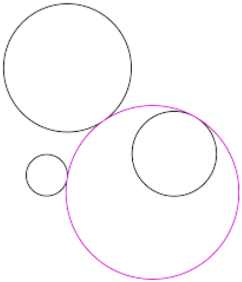
from cut-the-knot.org

Antique geometry problem.
Construct a common tangent to 2 given circles using a compass and a straightedge.
Solution: transform one circle to a point by an *offset*



from cut-the-knot.org

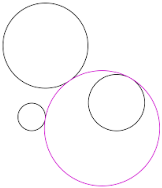
Apollonius problem. Construct a common tangent circle to 3 given circles using a compass and a straightedge.



from wikipedia.org

Apollonius problem. Construct a common tangent circle to 3 given circles using a compass and a straightedge.

Solution: transform one circle to a point by an *offset*, move the point to infinity by an *inversion*, apply the previous problem.



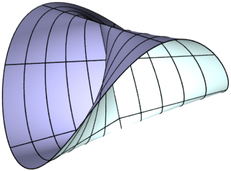
from wikipedia.org

Definition. A *Laguerre transformation* is a transformation of the set of oriented hyperplanes in $\mathbb{R}^n$ taking oriented tangent hyperplanes to an oriented sphere (possibly of radius 0) to oriented tangent hyperplanes to an oriented sphere (possibly of radius 0).

Examples. Offsets, similarities.

Laguerre geometry

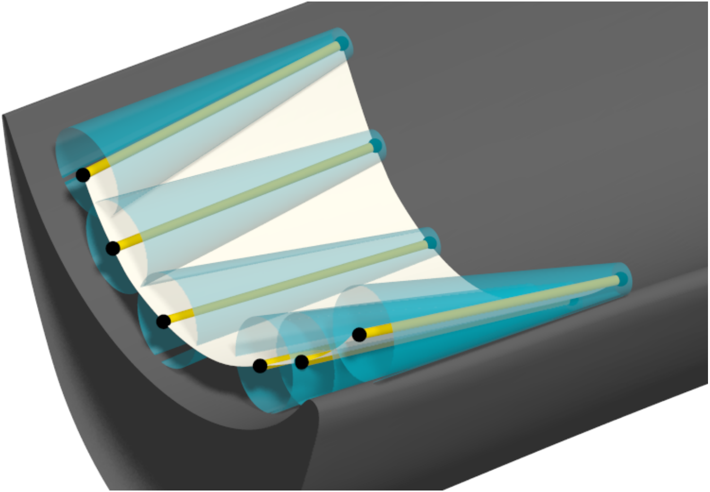
- **Example** (of an L-transformation): offset operation.
- **Definition.** $ST\mathbb{R}^3 = \{(r, P) : P \ni r$ is an or. plane in $\mathbb{R}^3\}$.
- **Definition.** An *L-transformation* is a map $ST\mathbb{R}^3 \to ST\mathbb{R}^3$ taking planes to planes and spheres to spheres.
- **Definition.** A *Legendre surface* is an immersed surface $(\mathbf{r}; \mathbf{P}) : \mathbb{R}^2 \to ST\mathbb{R}^3$ such that $d\mathbf{r}(u; v) \perp \mathbf{P}(u; v)$.
- **Example.** An oriented surface or curve $\rightsquigarrow$ Legendre surface.



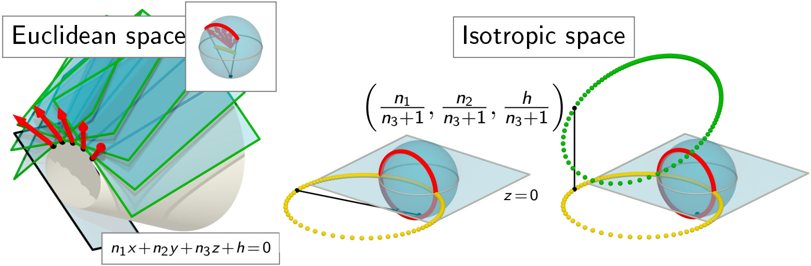
Plucker's conoid
$z = x^2/(x^2 + y^2)$

cycloid
$\mathbf{r}(t) = (t \sin t; 1 - \cos t; 0)$

Surfaces enveloped by a family of cones



Isotropic model of Laguerre geometry



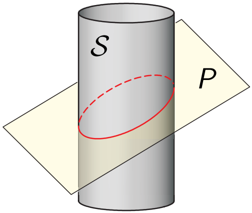
Isotropic model of Laguerre geometry

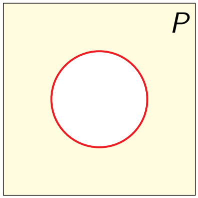
- oriented plane $n_1x + n_2y + n_3z + h = 0,\ n_3 \neq -1$
  $\rightsquigarrow$ point $\frac{1}{n_3+1}(n_1, n_2, h)$ in isotropic space
- L-transformation $\rightsquigarrow$ i-M-transformation

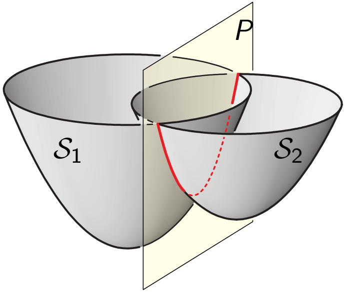
Surface in Laguerre geometry	Corresponding object of isotropic geometry
oriented plane	point
oriented sphere	non-isotropic plane
cone	non-isotropic line
cone	i-circle of elliptic type
cone	i-circle of parabolic type
oriented sphere	i-sphere of parabolic type
parabolic cyclide	i-paraboloid
L-minimal surface	i-Willmore surface

- **Definition.** The *isotropic plane* is $\mathbb{R}^2$ with $\|(x,t)\| = t$.
- **Definition.** The *isotropic space* is $\mathbb{R}^3$ with $\|(x,y,z)\|^2 = x^2 + y^2$.

Object	Definition
point	point in isotropic space
non-isotropic line	line non-parallel to the z-axis
non-isotropic plane	plane non-parallel to the z-axis
i-circle of elliptic type	ellipse whose top view is a circle
i-circle of parabolic type	parabola with z-parallel axis
i-sphere of parabolic type	paraboloid of revolution with z-parallel axis
i-paraboloid	graph of a quadratic function $z = F(x,y)$
i-Willmore surface	graph of a (multi-valued) biharmonic function $z = F(x,y)$







i-circle of elliptic type
(top view is a circle)

i-circle of parabolic type

- **The Pencil theorem** Let $F(x,y)$ be a biharmonic function in a region $U \subset \mathbb{R}^2$. Let S_t, $t \in I$, be an analytic family of circles in the plane. Suppose that for each $t \in I$ we have $S_t \cap U \neq \emptyset$ and the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then either S_t, $t \in I$, is a pencil of circles or

$$F(x,y) = A((x-a)^2 + (y-b)^2) + \\ + \frac{B(x-c)^2 + C(x-c)(y-d) + D(y-d)^2}{(x-c)^2 + (y-d)^2}$$

for some $a, b, c, d, A, B, C, D \in \mathbb{R}$.

- **Lemma on crossing circles.** Let S_t, $t \in I$, be a family of pairwise crossing circles in the plane distinct from a pencil of circles. Let F be an arbitrary function defined in the set $U = \bigcup_{t\in I} S_t$. Suppose that for each $t \in I$ the restriction $F|_{S_t}$ is a restriction of a linear function. Then

$$F = A((x-a)^2 + (y-b)^2) + B$$

for some $a, b, A, B \in \mathbb{R}$.

- **Lemma on circles with a common point.** Let S_t, $t \in I$, be a family of pairwise crossing circles in the plane passing through the origin O. Assume that no three circles of the family belong to one pencil. Let F be an arbitrary function defined in the set $U = \bigcup_{t \in I} S_t - \{O\}$. Suppose that for each $t \in I$ the restriction $F\big|_{S_t - \{O\}}$ is a restriction of a linear function. Then

$$F(x, y) = A((x-a)^2 + (y-b)^2) + \frac{Bx^2 + Cxy + Dy^2}{x^2 + y^2}$$

for some $a, b, A, B, C, D \in \mathbb{R}$.

Lemma on nested circles

- **Lemma on nested circles.** Let S_1 and S_2 be the pair of circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Let F be a function biharmonic in the whole plane $\mathbb{R}^2$. Suppose that for each $t = 1, 2$ the restriction $F\,|_{S_t}$ is a restriction of a linear function. Then

$$F(x, y) = (x^2 + y^2)(Ax + By + C) + ax + by + c$$

  for some $a, b, c, A, B, C \in \mathbb{R}$.

- **Proposition.** Let

$$F(x, y) = (x^2 + y^2)(Ax + By + C) + ax + by + c,$$

  where $A^2 + B^2 \neq 0$. Suppose that the restriction of the function F to a circle $S \subset \mathbb{R}^2$ is linear. Then the center of the circle S is the origin.

Biharmonic continuation

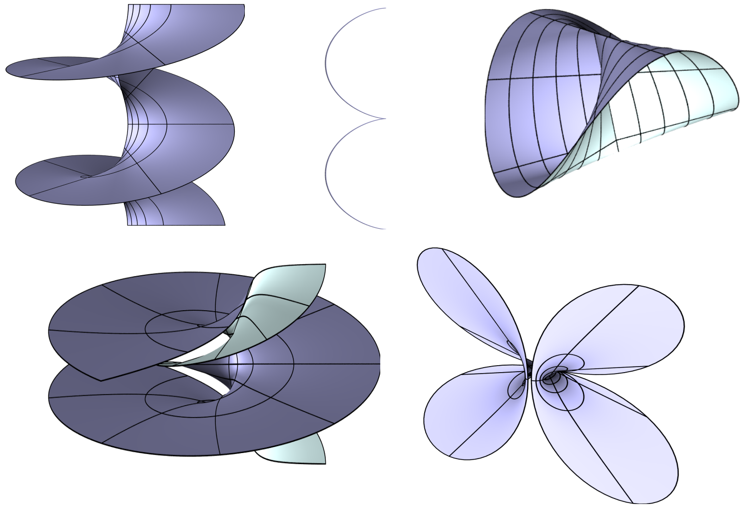
- **Notation.** S_1 and S_2 — a pair of circles in $\mathbb{R}^2$; r_1 and r_2 — reflections w.r.t. S_1 and S_2; $\Sigma_{12} = \{\, x \in \mathbb{R}^2 \ : \ r_1(x) = r_2(x) \,\}$.
- **Double symmetry principle.** Let F be a function biharmonic in a simply-connected region $U \subset \mathbb{R}^2$ nicely arranged with respect to a pair of circles $S_1 \neq S_2$. Suppose that for each $t = 1, 2$ the restriction $F\,|_{S_t \cap U}$ is a restriction of a linear function. Then F extends to a function biharmonic in the open set $r_1(U) \cap r_2(U) - \Sigma_{12}$.

- **Lemma on continuation.** Let S_t, $t \in I$, be a family of nested circles in the plane distinct from a pencil of circles. Let $F : U \to \mathbb{R}$ be a function biharmonic in a region $U \subset \mathbb{R}^2$ such that $U \cap S_t \neq \emptyset$ for each $t \in I$. Suppose that for each $t \in I$ the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then the function F extends to a function biharmonic in the whole plane $\mathbb{R}^2$.

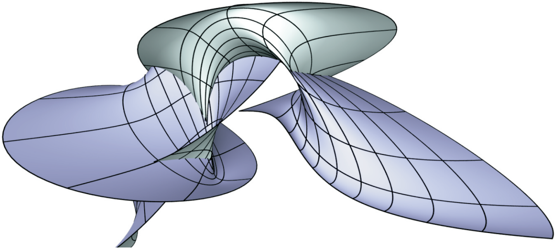
Corollaries of the Pencil theorem

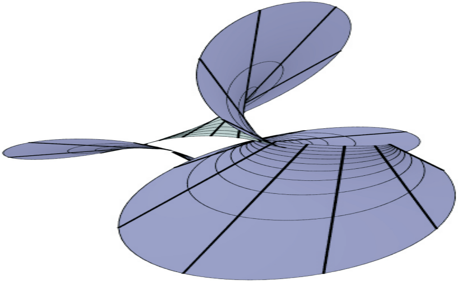
- **Corollary (on envelopes of cones).** Let Φ be an L-minimal surface enveloped by an analytic family $\mathcal{F}$ of cones. Then either the surface Φ is a parabolic cyclide or a sphere, or the Gaussian spherical image of the family $\mathcal{F}$ is a pencil of circles in the unit sphere.
- **Corollary (on ruled surfaces).** A ruled L-minimal surface is a Catalan surface, i. e., contains a family of line segments parallel to one plane.

Elliptic families of cones

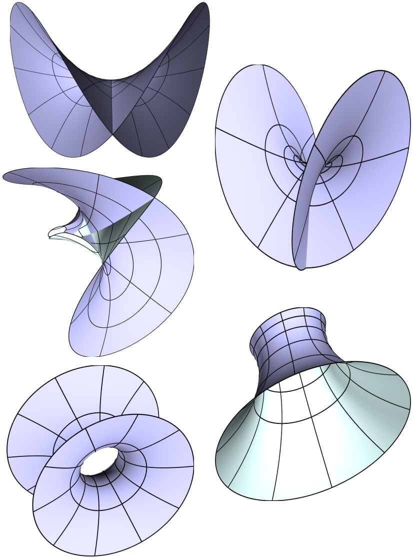


Elliptic families of cones

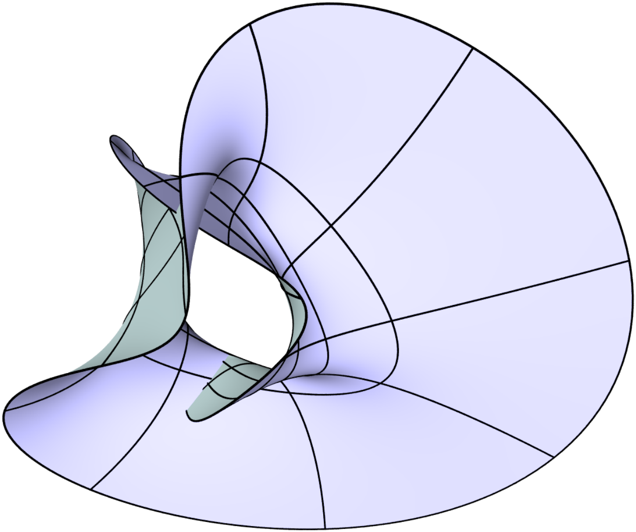




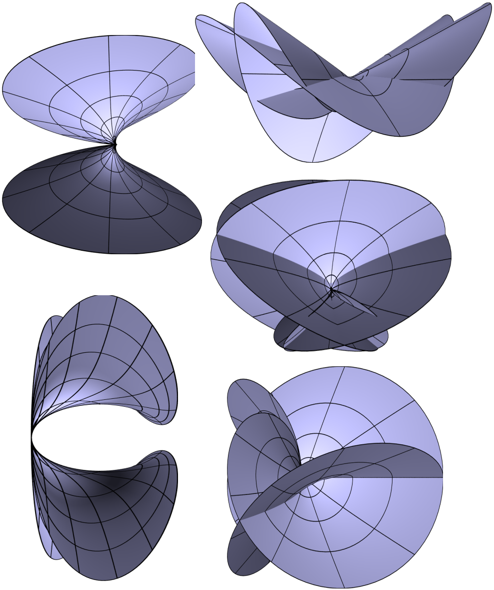
Hyperbolic families of cones



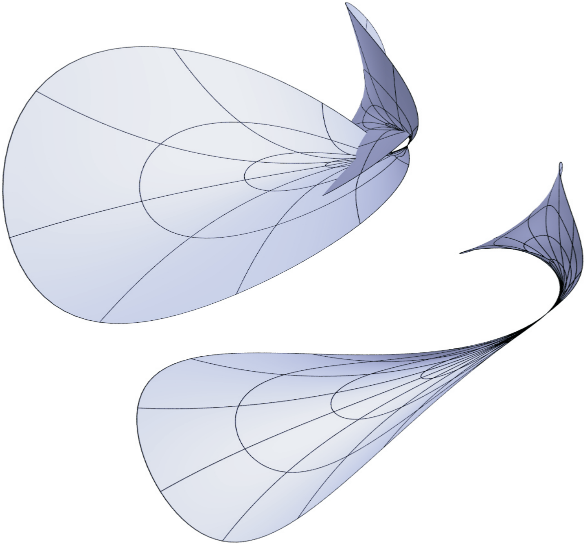
Hyperbolic families of cones



Parabolic families of cones



Parabolic families of cones



References

M. Skopenkov, H. Pottmann, P. Grohs, Ruled Laguerre minimal surfaces, Math. Z. 272 (2012), 645-674.

Acknowledgements

THANKS!