Response to editor's and referees' remarks on "Discrete field theory: symmetries and conservation laws"

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Contents

The author is VERY GRATEFUL to the referees and the editor for reading the manuscript and writing the reviews. The referees' remarks have been incorporated into the manuscript, which has increased the quality of the latter substantially. For convenience, main changes are quoted in this file.

1 Response to the editorial comments

While the two reviewers differ in their overall view, they are agreed on the key points that the claims of the paper are given in an overly sensational way and that the mathematics is not presented at the level of generality claimed, nor with the level of rigor claimed. They are also agreed that the results of the paper are (or would be should the mathematical correctness be resolved) quite interesting. This paper will need substantial revisions to be acceptable.

Reply. The paper has been revised according to the referees' suggestions. In particular, all overly sensational claims from the introduction have been removed; more details have been added to Definition 2.9 and Remarks 3.1, 2.17, to established the claimed levels of rigor and generality.

2 Response to review 1

The preprint concerns a discussion of well-known field theories in terms of discretisations using mostly cubical constructions and notions from algebraic topology. The author claims his ideas are far simpler than that of other authors. This claim cannot be verified.

Reply. The following attempt to put the claim into a falsifiable form has been performed in "Conclusions" section:

```
Compared to known results, the new Noether theorem is simple enough
to write the resulting conservation laws explicitly as one-line for-
mulae (using only standard topological notation) in numerous examples.
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There are claims of a discrete energy conservation theorem not based on a symmetry and if true, this would be extremely exciting.

Reply. The author would be happy to do all the best to make those results clear; see the particular changes listed below.

However, the exposition is so riddled with imprecisions, mis-defined terms, undefined terms and undefined calculations, that I did not get very far into the paper.

Reply. The particular issues pointed out in the review so far are addressed below. The author would be grateful for more comments.

1.1 Quick start. It would help if the author provided Maxwell's equations in 2+1 space.

Reply. Surely. The following text has been added:

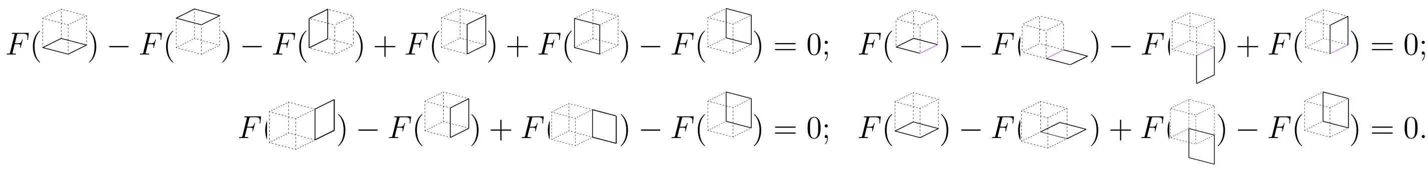
`The Poynting theorem asserts that under` *`Maxwell's equations`* `(where` $\vec{\mathrm{E}} =: (0, \mathrm{E}_x, \mathrm{E}_y)$ `and` $\vec{\mathrm{B}} =: (\mathrm{B}_t, 0, 0)$`)`

$$\frac{\partial \mathrm{B}_t}{\partial t} + \frac{\partial \mathrm{E}_y}{\partial x} - \frac{\partial \mathrm{E}_x}{\partial y} = 0; \qquad \frac{\partial \mathrm{E}_x}{\partial x} + \frac{\partial \mathrm{E}_y}{\partial y} = 0; \qquad \frac{\partial \mathrm{B}_t}{\partial x} + \frac{\partial \mathrm{E}_y}{\partial t} = 0; \qquad \frac{\partial \mathrm{B}_t}{\partial y} - \frac{\partial \mathrm{E}_x}{\partial t} = 0;$$

`the following identity holds ...`

I think there should be 4 equations but there are only 2 discrete ones given

Reply. Exactly. The remaining 2 discrete equations have been written explicitly in equation (1):

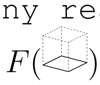
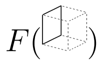
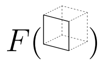


(This has made the comment "(defined in §2.3 and *different* from the ones in Figure 6)" unnecessary, and it has been moved to the caption of Figure 6.)

These simple suggestions have greatly improved readability, thanks!

and the reader is left to assume that F is the 2-form in 2+1 space often written in shorthand as E+B.

Reply. This is exactly the intuition beyond the notation "F". However the reader is not assumed to know that: §1.1 "Quick start" is written for nonspecialists. At this point, the reader is just assumed to understand the definition as written; this could also prevent any further confusion:

`A discrete` *`electromagnetic field`* F `is any real-valued function on the set of faces. Informally, its values` `,` `,`  `discretize` $-\mathrm{B}_t$`,` E_y`,` E_x `respectively, depending on face direction.`

I did not see the point of the T function.

Reply. The author would be happy to do all the best to clarify the point. Hopefully, the following additional phrase could explain it:

`Properties (2)-(3) are exactly what one requests from a discretization of energy density and flux according to the above discretization principles; it is nontrivial to satisfy both properties simultaneously.`

In the case when this short explanation is not convincing enough, the author kindly asks the referee not to give up and to provide a hint what exactly requires further clarification:

- the definition of the function T; or
- the statement of discrete Poynting theorem (3); or
- the analogy to the continuum Poynting theorem stated in bottom of page 2; or
- equation (2) and/or its precise statement in Proposition 2.9; or
- explanation why achieving both (2) and (3) simultaneously is nontrivial.

1.3 Main idea. It is pointless trying to motivate and illustrate when none of the terms referred to in the Figures are defined.

Reply. The author agrees. Figures 2–3 have been moved closer to where the relevant notation is introduced, the former figure being split into two ones for that purpose. References to the particular definitions of the terms used in the figures have been added in the captions.

To illustrate my frustration at trying to guess the quantities in Figure 3: I can guess that f is a face and v a vertex, but what is j? In one formula we have ∂j, indicating j is a chain, and in another we have δj, indicating j is a co-chain. This is bizarre

Reply. Actually the spaces of chains and cochains can be (and are) identified. This is completely similar to identification of both vectors and linear functions in the plane with pairs of real numbers, once we are working in a fixed coordinate system. The following remarks (referred right after the definitions of a cochain and the coboundary in Definitions 1.1 and 2.2) are supposed to remove any confusion:

Remark 3.1. To make the definitions of spacetime and fields more accessible to nonspecialists, we took the liberty to use equivalent definitions of some commonly used notions and to identify spaces connected by the unique fixed isomorphism. Caring for the convenience of specialists as well, now we compare Definition 1.1 with the other ones in literature.

Often *simplicial* (or *cubical*) *k-chains* are defined in a more abstract way, as the elements of the linear space $C_k(M;\mathbb{R})$ generated by the k-dimensional faces of M (with somehow fixed orientation); and *k-cochains* are defined as elements of the dual space $C^k(M;\mathbb{R})$. But space $C_k(M;\mathbb{R})$ comes with the obvious unique distinguished basis: the basis consists of all the k-dimensional faces; the orientation of the faces is determined by the order of their vertices in spacetime M as specified in Definition 2.9; the faces are listed in the dictionary order with respect to the ordered lists of their vertices. The distinguished basis identifies both $C_k(M;\mathbb{R})$ and $C^k(M;\mathbb{R})$ with the set of real-valued functions defined on the set of k-dimensional faces, that is, *k-cochains* in the sense of Definition 1.1. Notice that this identification is not related to spacetime metric.

Thus we do *not* distinguish between chains and cochains. Inserting the obvious isomorphism between their spaces in our formulae would give no advantage but would only complicate notation. However, to make notation compatible with the commonly used one, we sometimes switch between different notation $C^k(M;\mathbb{R})$ and $C_k(M;\mathbb{R})$ for the same object (in our setup).

Remark 2.3. It could be more conceptual to write the Kirchhoff voltage law in the form $\delta Rj = 0$, where R is a map between 1-chains and 1-cochains depending on the resistances. In our setup, chains and cochains are identified and the resistances equal to 1, hence R is the identity map and is omitted.

The author agrees that identification of chains and cochains may look bizarre for specialists. But the whole §2 "Examples" is obviously addressed to nonspecialists, and tracking the difference between chains and cochains would cost too much: this would double the number of objects and complicate the notation, giving nothing in return. Notice that a nonspecialist, who learns the definition of the operators ∂ and δ from Definition 2.2, is not going to be confused by applying both to the same function j on edges. On the other hand, specialists could enjoy §3 "Generalizations", where all $C^k(M;\mathbb{R})$ and $C_k(M;\mathbb{R})$ are thoroughly put in agreement with the common notation (except Table 3, again concerning examples).

Hopefully, Remark 3.1 resolves most of the remaining questions by the referee, but short replies are provided anyway below.

*as is the cap-product formula in Figure 3's caption. The cap product is a map taking a chain of degree p and a cochain of degree q and giving you a chain of degree p-q. It is **not** really like the interior product of the exterior calculus, as claimed in Table 1, as the interior product has as its end result a form and not the input to a form.*

Reply. This is essentially the same question as above: identification of chains and cochains allows to view the output of the cap-product as a cochain, pretty similar to the interior product.

The analogy between the (co)boundary, cup/cap-product and (co)differential, exterior/interior product is even often used to explain the latter to students, because the definitions of the former on a grid are very simple and visual: compare discrete and continuum Maxwell's equations in the replies above.

In any event, it is not the case that a cap product is defined on pairs of co-chains and neither does it give you a function defined on a vertex, as such a function would be a 0-co-chain.

Reply. Again, functions defined on the set of vertices are identified with 0-chains, which allows to view the output of the cap-product as such function.

Unfortunately, the author's own def'n for the cap product, Def'n 2.4 makes no sense as it is written, relying on it does on a lexicographic (undefined here, but looks to be user-defined) ordering of vertices.

Reply. Thank you very much for pointing out this inaccuracy. Definition 2.4 has been rephrased without appealing to the lexicographic ordering:

`Denote by` $\max f$ `the vertex of a face` f `or an edge` f `having the maximal sum of the coordinates. Set` $\max f := f$, `if` f `is a vertex. ...`

For this purpose, the choice of the coordinate system has been made more explicit earlier in Definition 2.2:

`Assume that the coordinate axes are parallel to the edges, and orient edges in the directions of the axes.`

Similarly, mentioning of ordering has been removed from Definitions 2.7–2.8: $u < v$ has been replaced by $\max uv = v$.

The definition of the lexicographic ordering itself, renamed to *dictionary order* for clarity, has been recalled in Definition 2.9:

`Fix the` *`dictionary order`* `of the grid vertices: set` $(x_0, x_1 \dots, x_{d-1}) < (y_0, y_1 \dots, y_{d-1})$ `if and only if` $x_0 = y_0, \dots, x_{k-1} = y_{k-1}$, `and` $x_k < y_k$ `for some` $0 \le k \le d-1$. `Denote by` $\max f$ $(\min f)$ `the maximal (minimal) vertex of a face` f `(on the grid, it is the vertex with the maximal (minimal) sum of the coordinates).`

The ordering of vertices in Figure 4 has been removed because it did not agree with such definition.

The referee's useful remark has allowed to identify and fix a mistake in Definition 2.9: the order of vectors in the positive basis must be of course *opposite* to the dictionary order of the endpoints:

`A` *`positively oriented`* `basis in a face is formed by the` k `vectors starting at the minimal vertex of the face, going along the edges of the face, and listed in the order opposite to the order of the endpoints. E.g., on the grid, the positively oriented basis in a` d`-dimensional face is` $(1/N, 0, \dots, 0), (0, 1/N, \dots, 0), \dots, (0, 0, \dots, 1/N)$, `as` $(1/N, 0, \dots, 0) > (0, 1/N, \dots, 0) > \dots > (0, 0, \dots, 1/N)$.

This is how the definition was actually understood in the paper; hence this change does not affect the rest of the paper.

The author is especially grateful to the referee for the help to fix this issue.

The author defines a k-dim'l field or k-cochain as a real valued function on k-faces. Actually, co-chains are linear maps on linear sums of k-faces, However, the author then writes $C^k(M; R) =$

$C_k(M; R)$. *But the left hand side of this equation is the standard notation for co-chains, which are functions on chains, while the right hand side is the standard notation for chains, ie the linear sums of the faces themselves. These two spaces are totally different!!*

Reply. Fixation of basis gives the unique isomorphism from a linear space to its dual space, and thus allows to identify the spaces $C^k(M; R)$ and $C_k(M; R)$. This has been clarified in Remark 3.1.

Further, co-chains in $C^k(M; R)$ *assign constants to each one-element chain, and not, as is implied, a variable function.*

Reply. A constant, i.e., a real number, assigned to each simplex is exactly a function on the set of simplices. Thus the referee actually gives the same definition as in the text under review, which is not surprising: the definition is quite natural.

It does not seem to me that the author has the standard definitions of what he is talking about clear in his/her own mind. It is simply not the case, as claimed in 1.6, that the paper is written with mathematical rigour. Rigour begins with the standard definitions and with precise statements. I am not averse to the use of diagrams of cubical elements as seen on page 3 as the input to functions, to illustrate a point. But the absolute basics of mathematical rigour involves defining your terms: this includes saying what space every object is in, where the maps are going to and from, and why each calculation is well-defined. It is not up to the reader to guess. It is up to the author to write clearly.

Reply. The author agrees that this work can only make sense if written in mathematical level of rigor as specified in Section 1.6. However different definitions are considered as "standard" in different communities, thus one can hardly require "standard definitions", but rather definitions *equivalent to the commonly used ones*, with the equivalence either obvious or proved.

The author would be grateful for pointing out any particular imprecise statements, not well-defined calculations, or definitions, where the equivalence to any other commonly used ones requires explanation. This could provide a possibility to fix those issues. So far the questions raised above reduce to identification of chains and cochains, and to the definition of the lexicographic ordering, hopefully resolved in Remark 3.1 and Definition 2.9.

3 Response to review 3

(1) *The introduction (page 2, before subsection 1.1) seems some-what detached from the rest of the paper. For instance, the principles of discretization (starting around line 15) are not discussed anywhere else in the paper, and it is not clear to me how the discussion in the rest of the paper is influenced by or supports these principles. The introduction should be slightly updated to connect better to the main body of the paper. See point 4 below.*

Reply. References to the discretization principles have been added throughout the paper. For instance, the following phrase has been added right after (2)-(3):

`Properties (2)-(3) are exactly what one requests from a discretization of energy density and flux according to the above discretization principles; it is nontrivial to satisfy both properties simultaneously.`

And after "One expects the following properties of the momentum flux":

`(the latter property being required by the discretization principles from §1)`

See also the sentence right before Remark 2.9 and the new "Conclusions" subsection.

(2) *Also in section 1, there are several fundamental claims for which the papers offers no motivation at all. E.g. page 2, line 8: we think [nature] is discrete rather than fundamental" or slightly below: we think [spatial symmetries] are approximate rather than fundamental". The paper offers no support for these claims. Nor are these claims supported enough to motivate the investigation in the paper. Such claims should either be dropped or motivated better.*

Reply. Both claims have been dropped, with the phrase containing the former updated as follows:

`creating an alternative candidate for a fundamental field theory`

(3) *Similarly on page 4, line 32: reconsider the old idea that the Universe is discrete rather than continuous". This might be personal taste, but I think such statements are sensational, rather than scientific. Generally, I think scientists should be more careful in distinguishing between reality and mathematical models for it.*

Reply. The phrase has been replaced by the following one:

`build the whole field theory starting from discrete rather than continuous space and time.`

(4) *I really think the paper is lacking a "discussion" section somewhere where the obtained results are put into context with the literature and the principles discussed in the introduction. Partly the material is there in the "limitations", "background" and "open problems" sections. Such a section should be added, either at the end of the paper or in the introduction.*

Reply. New "Conclusions" section has been added as suggested.

(5) *On a mathematical level: Many objects are defined only for* $M = I_d^N$ *only. I think this is completely fine for the objective of the paper. But at several points the author claims that the results are easily extended to general simplicial complexes, which I find somewhat nonchalant. I would either give some more details about the generalization or drop these remarks (specifically, Remark 2.16 on page 22, Remark 3.1 on page 25)*

Reply. The part of Remark 3.1 concerning such generalization has been dropped. In the other remark, a detailed explanation has been given:

Remark 2.17. The results of §2.4 remain true for arbitrary spacetime, if one drops all #-operators. The section is intentionally written so that all the definitions, propositions, and corollaries (but not necessarily the particular examples outside those environments) remain true, if I_N^d is replaced by an arbitrary cubical complex M, the dictionary order on I_N^d is replaced by the ordering on M fixed in Definition 1.1, and all #-operators are dropped. For a simplicial complex M, the definitions should be modified according to Remarks 2.15–2.16. The proofs of the resulting generalizations are analogous, only for a simplicial complex each instance of the fourth vertex ``d'' of a face $abcd$ is just removed, and a direct checking is used instead of Lemma 4.5.

Following are some smaller remarks and typos: Some figures (2,3) appear before the relevant notation is introduced and are slightly hard to understand without them.

Reply. Figures 2–3 have been moved closer to where the relevant notation is introduced, the former figure being split into two ones for that purpose. References to the particular definitions of the terms used in the figures have been added in the captions.

page 13, line 33: Amper should be spelled Ampère.

Reply. Fixed (2 instances)

page 16, paragraph in lines 33-36: I think this point is important - it is very confusing at first why a tensor should be a function on M x M - as opposed to just M. I think emphasizing this point would facilitate understanding of this concept for the reader.

Reply. The author agrees completely. The following phrase has been added:

We emphasize that they are functions on faces of the *Cartesian square* $M\times M$ rather than of spacetime M itself. We shall see that such functions appear naturally in examples in §2.

Definition 2.14, page 20: I think it would be helpful for the reader to recall the relation between A and U here.

Reply. In fact the relation cannot be just "recalled" because it is introduced much later. Neither this relation is used at this stage. To make the path to the statement of the main result of the subsection (Corollary 2.3) as short as possible, the (unnecessary at this stage) definitions of A, F, D_A^* are postponed. To clarify this point, the following phrase has been added in Definition 2.14:

`So far the notations` $D_A^* j$ `and` $D_A^* \# F$ `should be viewed as indivisible. Separate conceptual definitions of` A, F, D_A^* `are postponed until the end of §2.4, where (10)-(11) become easy propositions.`

Definition 2.15, page 20: A slight pedantry: Probably the author means by $T_u G$ the linear subspace parallel to the tangent space to G at u? The latter is not a subspace, only an affine space.

Reply. The difference is indeed important. Fixed as suggested.

4 Author's own changes

Essential change I. There was a serious mess with the signature of the Minkowski metric in some examples: the energy-density entry $T^{00} = g^{00}T_0^0$ was negative rather than positive. Now metric-signature convention has been fixed to $(+,-,...,-)$ everywhere (giving $T^{00} \geq 0$) and the convention has been made explicit. This has NOT affected Sections 4.2-4.4 (containing proofs with involved computations) at all; only the following points in §§2.2,2.3,2.5,4.1,4.5 were affected:

- Definition 2.9: in definition of $\#F$, a factor of $(-1)^k$ added, the phrase '$k > 0$' added
- Remark 2.7: '$(-,+,...,+)$' replaced by '$(+,-,...,-)$'
- Subsubsections 'Approximation' of §2.3 and §2.4, the first paragraphs: in the definition of the energy-momentum tensor, it has been added

  `(for the metric of signature` $(+,-,...,-)$`)`
- Definition 2.18-1.19, Corollary 2.5-2.7, Proposition 2.14: the sign before each '$\#$' operator changed.
- Before Proposition 2.13 and in the Proof of Proposition 2.13: several signs changed in the defition of j^l, T_k^l and $\partial^n \phi$
- Before Definition 3.1: notation g^{ll} changed to $g(v,k,l)$ (also in Table 3) index j changed to l for sameness (also in (18)-(19 and in the proof of Lemma 4.1)), a factor of $(-1)^{k+1}$ added in the definition of $g(v,k,l)$.
- Examples 1.1-1.2 and Definition 2.21: $R^{3,1}$ replaced by $R^{1,3}$
- Examples 1.1-1.2: a minus sign added before all '$\#$' (4 times), the word 'simplest' removed, the last sentence of Example 1.2 made more accurate:

  `It is famous` *`compact Abelian lattice gauge theory`* `recalled in §2.4.`
- Subsubsection 'Approximation' of §2.2, the first displayed formula: the sign of σ_{kl} was changed to fit the common definition of the Maxwell stress tensor.
- Definition 2.8: in definition of σ_k, the sign was changed to fit the continuum analogue
- Figure 4: the sign after σ_k was changed (2 times)
- Remark 2.5: the word 'minus' removed

- Proof of Theorem 2.1 in §4.5: in the last displayed formula, the signs of two intermediate terms changed

Essential change II. A particular case of one of the main new constructions has been found in literature. The following citations have been added:

- 3rd paragraph of §1: a reference to Dorodnitsyn's book from 2001 added.
- In §1.2"Background" added:

  `But in 2000s A. Dorodnitsyn disretized energy and momentum conservation in some particular cases [12, Example in §8], and in §1.1, §2.3 we finally construct...`
- After Theorem 1.2 and in Corollary 2.1: a reference to Dorodnitsyn's book from 2001 added.
- Before Definition 2.8 it is written:

  `This is essentially [12, Example in §8]`
- After Theorem 2.2 added:

  `Some particular cases of this theorem were established in [12,§8] by a different method.`
- Reference to a remarkable survey by C.M. Bender-L.R. Mead-K.A. Milton added with the following phrase:

  `Compare with the efforts put to achieve the gauge covariance in the remarkable survey [2,§9].`

Minor changes:

- Throughout: replaced "discrete electrodynamics" by more common "lattice electrodynamics", replaced $\cong$ by $\rightrightarrows$, replaced $\ln$ by $\log$, replaced "values at ∂I_N^d" by "values on ∂I_N^d"
- Proof of (3) in §4.1: TWICE the lhs... , wrong signs in the last two square brackets, terms 3,6,f, fixed.
- In Corollary 2.15, 'in particular' part and its proof added.
- §1: Theorem 1.3 and 2.2 $\to$ Theorems 1.3 and 2.2
- §1.1: "smooth" added before "vector-valued functions", "inward" added before "energy flux", paragraph "Let us discretize…" and the sentence before (3) rephrased slightly
- §1.2: The following phrase added after mentioning Feynman checkerboard:

  `see [23] for an elementary introduction and survey of the former model.`
- §1.2: The following phrase removed for brevity: "Using it, Wilson established confinement of quarks in large-coupling limit. The general-coupling case remains a famous open problem."
- §1.3: The following phrase added:

  `In this subsection, in contrast to the rest of the paper, we assume familiarity with the basics of continuum field theory.`
- §1.4: formally state $\to$ precisely state

- Definition 1.1: definitions of a face and a stationary function added:

  The simplices/cubes of M are called *faces* of M.

  The *action* $\mathcal{S}\colon C^k(M;\mathbb{R}) \to \mathbb{R}$ is the sum of the values of the Lagrangian over all the vertices. A field $\phi \in C^k(M;\mathbb{R})$ is *on shell* (i.e., lying on the shell given by the equations of classical physics), if it is *stationary* for the action functional, i.e., $\frac{\partial}{\partial t}\mathcal{S}[\phi + t\Delta]\big|_{t=0} = 0$ for each $\Delta \in C^k(M;\mathbb{R})$.

- §1.6: Remarks 2.15, 2.16 added to prerequisites to §3.

- §2.1: bold font for edge numbers → bold font for pipe numbers.

- Remark 2.1: continuum → usual (continuum)

- Definition 2.2: "acts" → "defined for"; convention added:

  Hereafter an empty sum is set to be 0.

- Definition 2.3: the last line rephrased:

  where uv denotes an oriented edge starting at u and ending at v hereafter.

- Before Definition 2.7: *total* energy conservation → *global* energy conservation

- Definition 2.9: added

  Set $\partial F = 0$ and $\delta F = 0$ for $k \leq 0$ and $k \geq d$ respectively, and $C^k(I_N^d;\mathbb{R}) = \{0\}$ for $k < 0$ or $k > d$.

- Remark 2.8: objective reality → realistic model

- Remark 2.9: analogy to Einstein tensor emphasized:

  Here the role of the δ_k^l-term is the same as in the Einstein tensor: it makes the 'contraction' map commute with certain codifferentials...

- Definition 2.11: is supported by → has support on

- Remark 2.11: energy → energy density

- Figure 7: labels e and f added

- Remark 2.14: the last formula rewritten in an equivalent form.

- Definition 2.17: alternative physical terminology (gauge potential and field strength) mentioned

- Before Definition 2.18:

  ...shown in Table 2 AND IN (10)-(11) are sufficient for all our examples.

- Definition 2.18: after "if such face f exists, then it is unique" added "by Definition 1.1"

- Table 2: "undefined" replaced by "$= 0$"

- Definition 2.20: particular cases $k = 0$ and 1 added

- Remark 2.18: fixed misprint - "Then the COVARIANT coboundary"

- Beginning of §2.6: added

  `In this subsection, the ''topological'' notation seems to be less clear than the original ''coordinate'' one [?], but we keep the former for sameness.`

- After Definition 2.21: reflects the general lattice *fermion doubling* phenomenon $\to$ is a manifestation of lattice *fermion doubling* phenomenon.

- Table 3: a new row added to avoid confusion of partial derivatives wrt ϕ and U

- Right before Theorem 3.2: added:

  For fixed $\phi \in C^k(M; \mathbb{C}^{1\times n})$, a local Lagrangian $\mathcal{L}[\phi, U]$ in the sense of Definition 3.1 is a local Lagrangian in the sense of this paragraph (by the second paragraph of Remark 2.19).

- Before Theorem 3.4: definition of gauge invariance recalled

- Before Definition 4.1: We need to $\to$ For that purpose we are going to

- End of the proof of Proposition 4.1: added

  by the assumptions $f \subset h \perp \mathrm{e}_l$ and $f \parallel \mathrm{e}_k$.

- Beginning of the proof of Proposition 4.1: "by Definition 2.10 we have" inserted before "$\partial T(e\times f) = T([e] \times \partial[f]) + T(\delta[e] \times [f])$"

- End of the proof of Proposition 2.7: added

  `because the closed hypersurface can be tiled by` d`-dimensional faces of the doubling.`

- Proof of Proposition 2.8: Reference to Definition 2.12 and the phrase "because $f \not\parallel \mathrm{e}_k$ otherwise" added

- Proof of Lemma 4.8: added

  (see Proposition 2.12), using the formula for $\check{D}^*_A(\phi \frown \psi)$ from Lemma 4.6 instead of (22)

- Proof of Corollary 2.3: Reference to Proposition 2.11 added

- Proof of Corollary 2.4: Reference to Proposition 2.11 added

- Proof of Lemma 4.9: fixed misprint - missing ∂ added (4 times)

- Proof of Proposition 2.12: fixed misprint - missing $*$ added

- Proof of Corollary 2.10: Reference to Definition 16 replaced by reference to Proposition 2.11 and Definition 2.13

- Proof of Proposition 2.14: 8sgn... replaced by an alternative expression equal to 2sgn..., and 4sgn... replaced by an alternative expression equal to sgn...

- Proof of Corollary 2.12, 2.16: minor simplification

- Thanks to A. Rassadin and P. Pylyavskyy added.

Also a few inessential stylistic changes not included in the above list have been made.