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Discrete field theory: symmetries and conservation laws

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Discrete field theory: symmetries and conservation laws

M. Skopenkov

Abstract

We present a general algorithm constructing a discretization of a classical field theory from a Lagrangian. We prove a new discrete Noether theorem relating symmetries to conservation laws and an energy conservation theorem not based on any symmetry. This gives exact conservation laws for several discrete field theories: electrodynamics, gauge theory, Klein–Gordon and Dirac ones. In particular, we construct a conserved discrete energy-momentum tensor, approximating the continuum one at least for free fields. The theory is stated in topological terms, such as coboundary and products of cochains.

Keywords: discrete field theory, discrete differential geometry, conservation law, Noether's theorem
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Contents

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1 Introduction

Dedicated to the last real scientists, which unlike merchants show both advantages and limitations of their theory.

This work is a try to build a general *discrete field theory*. This has the following motivation:

- getting effective numeric algorithms for field theory;
- putting field theory to a mathematically rigorous basis;
- learning the fundamental laws of nature (which we think is discrete rather than continuous).

Numerous discretizations of particular field theories are known [1, 8, 9, 10, 12, 11, 13, 16]. Our aim is *not* to invent new discretizations, but to extract and study the best among the known ones. Discretizations exhibiting exact (not just approximate) conservation laws have been proved to be most successful for computational purposes [11]. This leads us to the following *principles of discretization*:

- keep approximation of continuum theory;
- keep conservation laws exact;
- drop spatial symmetries easily (which we think are approximate rather than fundamental).

These principles have a built-in difficulty: we have to drop most continuous symmetries, but usually conservation laws are obtained just from such symmetries using the Noether theorem. We develop a new general method to get discrete conservation laws (which we think are reasons of symmetries of the continuum limit rather than consequences). The method is simpler than those of [11, 14, 15, 20].

The following basic warm-up results of discrete field theory are obtained in the present paper:

- discretization of several field theories in a similar fashion keeping conservation laws exact (§2);
- a new discrete Noether theorem relating symmetries to conservation laws (Theorems 1.2 and 3.3);
- a new discrete energy conservation theorem not based on a symmetry (Theorem 1.3 and 2.2).

1.1 Quick start

We start with an elementary and informal description of one result (Theorem 2.2), in the simplest unknown particular case. It is an energy conservation theorem for lattice electrodynamics; more precisely, for electrodynamics in 2 spatial and 1 time dimensions. For these small dimensions we just *draw* everything. The more realistic case of 3 spatial and 1 time dimensions is analogous; see §2.3.

Recall briefly the energy conservation theorem in *continuum* electrodynamics (the Poynting theorem). Let x, y, t be the Cartesian coordinates in space; see Figure 1. *Electric* and *magnetic fields* are arbitrary vector-valued functions $\vec{\mathrm{E}}(x, y, t)$ and $\vec{\mathrm{B}}(x, y, t)$ respectively such that $\vec{\mathrm{E}} \perp Ot$ and $\vec{\mathrm{B}} \parallel Ot$. The *energy density* and the *energy flux* (*the Poynting vector*) are the functions $\frac{1}{2}(\vec{\mathrm{E}}^2 + \vec{\mathrm{B}}^2)$ and $\vec{\mathrm{E}} \times \vec{\mathrm{B}}$ respectively. The Poynting theorem asserts that under Maxwell's equations (which we do not need to write down), for each cube with the edges parallel to the coordinate axes we have

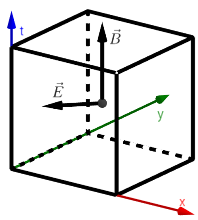
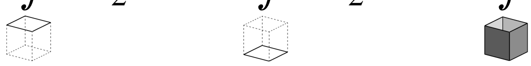
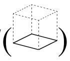
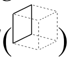
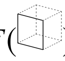


Figure 1: Cube

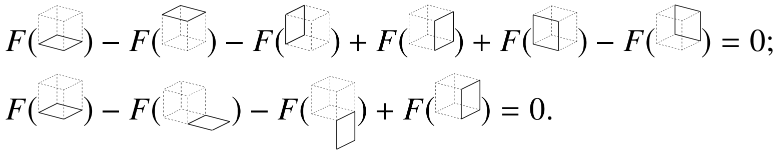
$$\int \frac{\vec{\mathrm{E}}^2 + \vec{\mathrm{B}}^2}{2}\,\mathrm{dA} - \int \frac{\vec{\mathrm{E}}^2 + \vec{\mathrm{B}}^2}{2}\,\mathrm{dA} = \int \vec{\mathrm{E}} \times \vec{\mathrm{B}}\,\mathrm{d}\vec{\mathrm{n}}.$$



Here the cube is shown by dotted lines, and the faces over which a particular integral is taken are in bold. The first two integrals mean the total energy contained in the same square in the Oxy plane at two different moments of time t. The third integral means the total energy flux through the boundary between these two moments. Thus the equation means energy conservation.

Let us discretize. Dissect the unit cube into $N \times N \times N$ equal cubes. Throughout this subsection by *cubes* we mean the latter equal cubes, by *faces* and *edges* — their faces and edges. A discrete *electromagnetic field* F is an arbitrary real-valued function on the set of faces. Informally, its values $F($$)$, $F($$)$, and $F($$)$ discretize $-\mathrm{B}_t$, E_y, and E_x respectively, depending on the face direction.

The well-known discrete homogeneous *Maxwell's equations* are

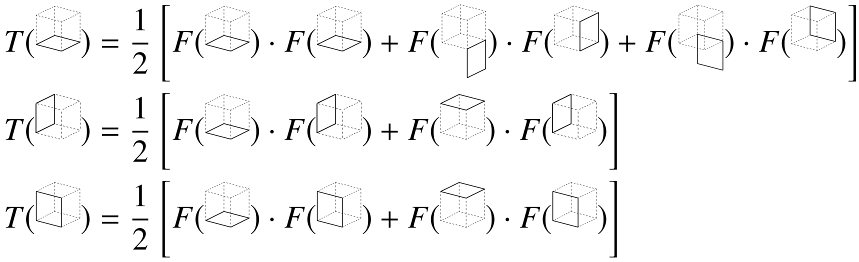


(1)

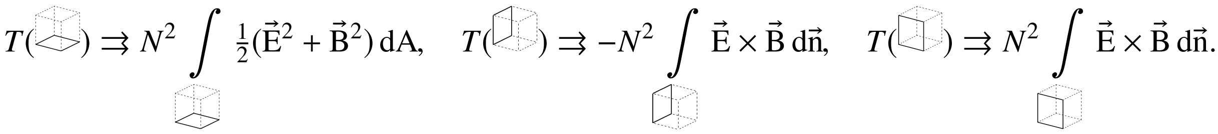
(2)

Here we sum the values of F at the faces of a particular cube (in (1)) and at the faces containing a particular edge (in (2)), with appropriate signs (defined in §2.3 and *different* from the ones in Figure 5). We write one equation per cube and one per nonboundary edge, and impose no boundary conditions.

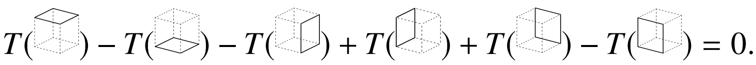
It's time for our new definition. Let T be the function on the set of nonboundary faces given by



The value at a horizontal (respectively, vertical) face discretizes energy density (respectively, flux). Proposition 2.9 asserts that under a natural choice of F we have uniform convergence as $N \to \infty$:



The desired discrete Poynting theorem (particular case of Theorem 2.2) asserts that assuming only Maxwell's equations (1)–(2), for each nonboundary cube we have the identity



(3)

A proof *in pictures* is in §4.1. And we proceed to a systematic discussion of discrete field theory.

1.2 Background

Discrete field theory is actually at least as old as the continuum one. In 1847 G. Kirchhoff stated the laws of an electrical network, which is in fact the simplest model of the theory; see §2.2. In the continuum limit, the laws approximate the Laplace equation; thus the model perfectly serves for numerical solution of the latter. Remarkable approximation theorems were proved by L. Lusternik [17], R. Courant–K. Friedrichs–H. Lewy [8] in 1920s and later generalized, e.g., in [6, 5, 2, 24]. Planar networks lead to the discretization of complex analysis having applications in statistical physics (e.g., obtained in 2010s by S.Smirnov et al. [5]) and even computer graphics [12].

Discrete field theory was closely related to topology from the youth of both subjects. The Kirchhoff laws are naturally stated in terms of the *boundary* and the *coboundary* operators; see §2.2 for an elementary introduction. Such formulation is usually attributed to H. Weyl; see [13, §1F, p. 31] for an elaborate historical survey. In 1930s G. de Rham established correspondence between these operators and the exterior derivative and its dual; see [1] for a survey and [23] for general philosophy. This lead to the above discrete Maxwell equations (1)–(2); see also §2.3 and [3, 13, 16, 22].

The next major step was done by A. Kolmogorov and J. Alexander in 1930s, who invented a product discretizing the exterior product in a sense. Kolmogorov commented that such discretization was his original motivation. The construction was soon modified by H. Whitney and others to give the now-famous *cup-product* [25]. The original product was anticommutative, whereas the cup-product was associative. One cannot get both properties simultaneously (this fact is crucial for rational homotopy theory). This reflects a general phenomenon that not all properties survive under discretization. We choose the associative cup-product as a discretization of the exterior product, in contrast to [12].

Later there appeared discrete models for other classical fields: e.g., *Feynman checkerboard* from 1940s and *Regge calculus* from 1960s for the Dirac and the gravitational field respectively.

In 1970s F. Wegner and K. Wilson introduced *lattice gauge theory* as a computational tool for gauge theory; see [18] or §2.4 for an elementary introduction and [9] for details. Using it, Wilson established confinement of quarks in large-coupling limit. The general-coupling case remains a famous open problem. The theory culminated in determining the proton mass with an error < 2% in a sense.

In 1980s A. Connes developed a formalism, dealing (to some extent) uniformly with continuous and discrete geometries [7]. Using it, A. Dimakis et al. discretized the Yang-Mills equations [10, Eq. (4.15)]. Corollary 2.3 extends their result by adding sources and the crucial unitarity constraint.

In 1990s J. Marsden et al. discretized basic general theorems of field theory: the Euler–Lagrange equations and the Noether theorem on a 2-dimensional grid; see [20, Eq. (5.2) and (5.7)], cf. [15, Eq. (60) and (69)], [14, Theorem 5.2.37]. These results extend the ones obtained earlier for 1-dimensional difference equations; see [14] for references. Discrete Euler–Lagrange equations in §1.4 are straightforward generalizations of the known ones; but Discrete Noether Theorem 1.2 is different. M. Kraus et al. have stepped beyond the Lagrangian formulation [15]. A general discretization approach to hydrodynamics was introduced by E. Gawlik et al. in 2010s [11, §4]. They derived general Euler–Poincare equations and Kelvin–Noether theorem [11, §3]. Their approach was based on discretization of the diffeomorphism group, thus was applicable to rather specific class of models. In 2017 E. Mansfield et al. discussed conservation laws for finite-element approximations [19].

There was a folklore belief that no conserved discrete energy-momentum tensor exists in this framework. E.g., in 2016 D. Chelkak, A. Glazman, and S. Smirnov introduced a "halfway" conserved tensor [4, Corollary 2.12(1)], cf. [21]. Even the notion of a rank 2 symmetric tensor itself is hard to discretize [1, §7]. But in §1.1 and §2.3 we construct an exactly conserved discrete energy-momentum tensor, approximating the continuum one at least for free fields.

Great success of discrete models forces to search for a general discretization method and even to reconsider the old idea that the Universe is discrete rather than continuous.

1.3 Main idea

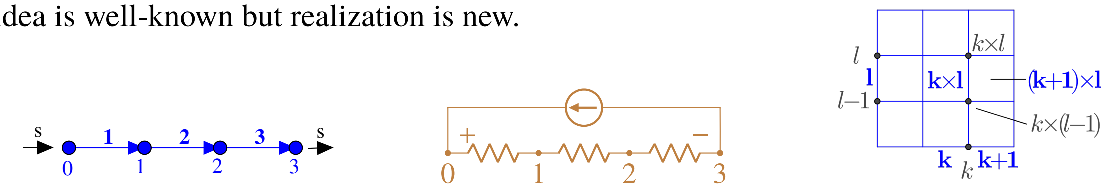
We propose the following discretization algorithm for field theories:

1) take a continuum Lagrangian written in terms of exterior calculus operations from Table 1;

2) replace the exterior calculus operations by cochain operations using Table 1 *literally*;

3) get equations of motions/conservation laws from discrete Euler–Lagrange/Noether theorems.

This idea is well-known but realization is new.



$[\partial j](k) = j(\mathbf{k}) - j(\mathbf{k}+\mathbf{1})$; $[\psi \frown \Delta](\mathbf{k}) = \psi(\mathbf{k})\Delta(k-1)$; $[\psi \times \phi](k \times l) = \psi(k)\phi(l)$

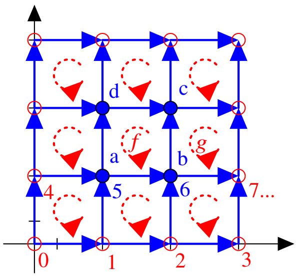
$[\delta\phi](\mathbf{k}) = \phi(k) - \phi(k-1)$; $[\psi \smile \Delta](\mathbf{k}) = \psi(\mathbf{k})\Delta(k)$; $[\psi \times \phi](\mathbf{k} \times \mathbf{l}) = \psi(\mathbf{k})\phi(\mathbf{l})$.

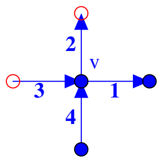
Figure 2: A pipeline, a network, the Cartesian square, boundary, coboundary, cap-, cup-, cross-product.

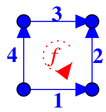
Results of applying the algorithm to basic field theories are discussed in §2. The output discrete theories are usually simpler than the input continuum ones; knowledge of the latter is not required for understanding the former. All the output theories of §2 are known, but some obtained conservation laws are new. As a tool, we use discrete covariant differentiation (see §2.4 and [10]) and build a new discretization of tensor calculus involving non-antisymmetric tensors (see §2.3). This is done in terms of cochain operations from Table 1. These operations appear naturally and are defined easily; see some examples in Figures 2–3, where symbols in bold denote edges.

Table 1: Correspondence between continuum and discrete notions

Continuum		Discrete		Definition
Algorithmic part I. Replacement in Lagrangian and action:				
differentiable manifold (spacetime)	M	simplicial or cubical complex with fixed vertices ordering	M	1.1
k-form, $\mathbb{R}$- or $\mathbb{C}^{m\times n}$-valued	ϕ	k-cochain, $\mathbb{R}$- or $\mathbb{C}^{m\times n}$-valued	ϕ	1.1
exterior derivative	d	coboundary	δ	2.9
exterior product	$\wedge$	cup-product	$\smile$	2.18
interior product	$\lrcorner$	cap-product	$\frown$	2.18
connection 1-form,Lie-algebra-valued	A	connection,not Lie-algebra-valued	A	2.17
curvature 2-form, Lie-algebra-valued	F	curvature, not Lie-algebra-valued	F	2.17
covariant exterior derivative	$\mathrm{D_A}$	covariant coboundary	D_A	2.20, 2.18
raising all indices	$\sharp$	sharp-operator (new notion)	$\#$	2.9 for $M=I_N^d$
vector of the Dirac γ-matrices	γ	the Dirac 1-chain (new notion)	γ	2.21 for $M=I_N^4$
function on $\mathbb{R}$ or $\mathbb{C}^{m\times n}$ (e.g., ln or Tr)	f	the same function on $\mathbb{R}$ or $\mathbb{C}^{m\times n}$	f	—
spacetime integration of a 0-form	$\int_{\mathrm{M}} d\mathrm{V}\cdot$	sum of the values of a 0-chain	ϵ	2.2
Informal part II. Correspondence in equations of motion and conservation laws:				
codifferential, $\sharp$-conjugated	$\sharp\delta\flat$	boundary	∂	2.9
covariant codifferential,$\sharp$-conjugated	$\sharp \mathrm{D_A^*}\flat$	covariant boundary	D_A^*	2.20, 2.18
interior product	$\llcorner$	cop-product (new notion)	$\stackrel{*}{\frown}$	2.18
tensor product over $C^\infty(M)$	$\otimes$	chain-cochain cross-product	$\times$	2.10
type $(1,1)$ tensor	T	type $(1,1)$ tensor (new notion)	T	2.10
integration of its k-th component	$\int_\pi \mathrm{T}_k$	flux (new notion)	$\langle T,\pi\rangle_k$	2.12 for $M=I_N^d$
integration of a k-form	$\int_\pi \phi$	pairing	$\langle\phi,\pi\rangle$	4.2







$[\partial j](v) = -j(\mathbf{1}) - j(\mathbf{2}) + j(\mathbf{3}) + j(\mathbf{4})$; $[\delta j](f) = j(\mathbf{1}) + j(\mathbf{2}) - j(\mathbf{3}) - j(\mathbf{4})$;
$[\phi \frown \psi](v) = \phi(\mathbf{3})\psi(\mathbf{3}) + \phi(\mathbf{4})\psi(\mathbf{4})$; $[\phi \smile \psi](f) = \phi(\mathbf{1})\psi(\mathbf{2}) - \phi(\mathbf{4})\psi(\mathbf{3})$;

Figure 3: A 3×3 grid, boundary, coboundary, cap-, and cup-product

The algorithm provides conservation laws only for symmetries which are preserved by the discretization. Thus we usually guarantee charge conservation (based on the automatically preserved gauge symmetry) and energy-momentum conservation (not based on any symmetry in our setup).

We stress that Part I of Table 1 gives an *algorithm*, not just an analogy (as Part II). However putting a continuum Lagrangian to the required input form is not always possible and can be ambiguous:

Example 1.1. The simplest Lagrangian of continuum electrodynamics can be written as $\mathcal{L}[\phi] = \sharp d\phi \lrcorner d\phi$, where ϕ is a real-valued 1-form on $\mathbb{R}^{3,1}$ (*vector-potential*). The resulting discretization $\mathcal{L}[\phi] = \#\delta\phi \frown \delta\phi$ gives the known *discrete Maxwell equations* briefly recalled in §2.3.

Example 1.2. The same continuum Lagrangian can be written as $\mathcal{L}[\mathrm{A}] = \sharp \mathrm{F}[\mathrm{A}]^* \lrcorner \mathrm{F}[\mathrm{A}]$, where $\mathrm{A} = i\phi$ is a u(1)-connection 1-form and $\mathrm{F}[\mathrm{A}] = d\mathrm{A} + \mathrm{A}\wedge\mathrm{A} = d\mathrm{A}$ is the curvature 2-form on $\mathbb{R}^{3,1}$. Here $\mathrm{A}\wedge\mathrm{A} = 0$ identically because A assumes values in an Abelian Lie algebra.

The resulting discretization is $\mathcal{L}[A] = \#F[A]^* \frown F[A]$, where $F[A] = \delta A + A \smile A$. The discretization turns out to be different from Example 1.1 because $A \smile A \neq 0$ and $F[A] \neq \delta A$ anymore. It is equivalent to famous *lattice gauge theory* recalled in §2.4.

So, depending on the choice of the input form of the Lagrangian, in Examples 1.1 and 1.2 we get two unequivalent discretizations of one continuum theory, both very useful in their own contexts.

Remark 1.1. In Table 1 we intentionally include no discretization for the Hodge star or products other than exterior, interior, tensor products. In all the examples, we have succeeded to avoid them.

Continuum and discrete notations fit not that well. But both are commonly used in their own contexts (except a few new discrete objects, for which we keep the continuum notation in a different font).

1.4 Statements

Let us formally state the main new results in their simplest form. Formal definitions of some used notions and generalizations of the results to nontrivial connections are postponed until further sections.

Definition 1.1. A *finite simplicial* (respectively, *cubical*) *complex* is a finite set of simplices (respectively, hypercubes) in a Euclidean space of some dimension satisfying the following properties:

1) the intersection of any two simplices (respectively, hypercubes) from the set is either empty or their common face;

2) all the faces of a simplex (respectively, a hypercube) from the set belong to the set as well.

Spacetime M is an arbitrary finite simplicial or cubical complex with fixed vertices ordering. For a cubical complex, we require that the minimal and the maximal vertex of each 2-dimensional face are opposite. (Typical examples of spacetimes are a path with N edges or an $N \times N$ grid with the lexicographic vertices ordering; see Figures 2–3.)

A k-dimensional *field* or k-*cochain* is a real-valued function defined on the set of k-dimensional faces of M. Denote by $C^k(M;\mathbb{R}) = C_k(M;\mathbb{R})$ the set of all k-dimensional fields; cf. Remark 3.1.

A *Lagrangian* is a function $\mathcal{L}\colon C^k(M;\mathbb{R}) \to C_0(M;\mathbb{R})$. The *action functional* $C^k(M;\mathbb{R}) \to \mathbb{R}$ is the sum of the values of the Lagrangian over all the vertices. A field is *on shell* (i.e., lying on the shell given by the equations of classical physics), if it is a stationary function for the action functional.

References to definitions of *(co)boundary*, chain-cochain *cap*- and *cross-products* are in Table 1.

Informally, a Lagrangian is *local*, if its value at a vertex depends only on the values of the field ϕ and the coboundary $\delta\phi$ at the faces for which the vertex is maximal. Informally, *partial derivatives* with respect to ϕ and $\delta\phi$ are fields of dimension k and $k+1$ respectively, obtained by differentiating the Lagrangian as if ϕ and $\delta\phi$ were independent variables. Formal definitions are in Definition 3.1.

The following theorem is a straightforward generalizaion of known ones; cf. [20, Eq. (5.2)].

Theorem 1.1 (Discrete Euler–Lagrange equations)**.** *Let* $\mathcal{L}\colon C^k(M;\mathbb{R}) \to C_0(M;\mathbb{R})$ *be a local Lagrangian. Then a field* $\phi \in C^k(M;\mathbb{R})$ *is on shell, if and only if the following equation holds:*

$$\partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} + \frac{\partial \mathcal{L}[\phi]}{\partial \phi} = 0. \tag{4}$$

(Here a plus sign stands because the boundary operator ∂ for $k = 0$ discretizes *minus* divergence.)

A *current* is a 1-dimensional field $j \in C_1(M;\mathbb{R})$. A current is *conserved*, if $\partial j = 0$.

The Noether theorem gives a conserved current for each continuous symmetry of the Lagrangian.

Theorem 1.2 (Discrete Noether theorem)**.** *Let* $\mathcal{L}\colon C^k(M;\mathbb{R}) \to C_0(M;\mathbb{R})$ *be a local Lagrangian and* $\phi \in C^k(M;\mathbb{R})$ *be a field on shell. The Lagrangian is* invariant under an infinitesimal transformation $\Delta \in C^k(M;\mathbb{R})$*, i.e.,*

$$\frac{\partial}{\partial t}\mathcal{L}[\phi + t\Delta]\bigg|_{t=0} = 0, \tag{5}$$

if and only if the following current is conserved:

$$j[\phi] = \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \frown \Delta. \tag{6}$$

This theorem is different from known discretizations of the Noether theorem in [14, 15, 20].

Discrete spacetime has no continuous symmetries, but there is still a corresponding conserved tensor. Conserved tensors are defined in Definition 2.10; they are functions on faces of $M \times M$.

Theorem 1.3 (Energy-momentum conservation)**.** *For each local Lagrangian* $\mathcal{L}\colon C^k(M;\mathbb{R}) \to C_0(M;\mathbb{R})$ *and each field* $\phi \in C^k(M;\mathbb{R})$ *on shell we have the following conserved* energy-momentum *tensor:*

$$T[\phi] = \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \times \delta\phi + \frac{\partial \mathcal{L}[\phi]}{\partial \phi} \times \phi. \tag{7}$$

This theorem is completely new. An integral form of this conservation law on a grid is stated in §2.3 (see Theorem 2.2 sketched already in §1.1). In particular, to tensor (7) defined on $M \times M$ we assign a conserved quantity defined on the grid M itself. In many examples, (6)–(7) approximate their continuum analogues; see Theorem 2.1, Propositions 2.9, 2.13, 2.15, and Remark 2.14.

After straightforward modification, these main results generalize to:

- complex- or vector-valued fields: the real part of the rhs of (6) and (7) is conserved;
- several interacting fields: one equation (4) per field; the sum of all currents (6) is conserved;
- nonfree boundary conditions: equation (4) and conservation laws hold apart the boundary.

1.5 Limitations

So far the proposed general discrete field theory has no applications (as a mathematical theory) and is not falsifiable (as a candidate for a fundamental physical theory).

Most of the technical issues concern the discretization of energy conservation and tensor calculus:

On one hand, the new notion of energy-momentum tensor (7) seems to be too abstract and too general. It discretizes not the continuum energy-momentum tensor precisely but a related object mapped to the latter; see Remark 2.8. Depending on a particular Lagrangian, (7) approximates either the nonsymmetric canonical energy-momentum tensor, or the symmetric Belinfante–Rosenfeld one, or even a nonconserved tensor; see Remark 2.10.

On the other hand, discrete non-antisymmetric tensor calculus from §2.3 seems to be too restrictive: it includes only type (1, 1) tensors and only the trivial connection; integration is defined only on a grid. The way of further generalization is unclear: e.g., for lattice gauge theory from §2.4, a naive way to define a real gauge invariant energy-momentum tensor leads to a nonconserved tensor; cf. Remark 2.14.

Approximation of continuum theories by discrete ones is not discussed at all, with the following two exceptions. First, for electrical networks the known approximation result is recalled in §2.2. Second, for the completely new discrete energy-momentum tensor the continuum limit is found in §2.

Some other limitations are stated as open problems in §5.

1.6 Overview

In §2 we give basic examples of discrete field theories. It contains an exposition of known results for nonspecialists and also a few new ones; §2 is independent from §1. In §3 we state the main results in their full generality. The only prerequisites for §3 are the definitions cited in Part I of Table 1 and Definitions 2.13, 2.15. In §4 we prove the results of §§1–3. In §5 we state open problems.

The paper is written in a mathematical level of rigor, i.e., all the definitions, conventions, and theorems (including corollaries, propositions, lemmas) should be understood literally. Theorems remain true, even if cut out from the text. The proofs of theorems use the statements but not the proofs of the other ones. Most statements are much less technical than the proofs; that is why the proofs are kept in a separate section. Remarks are informal and are not used elsewhere (hence skippable) unless the opposite is explicitly indicated.

We tried our best to make the results accessible to nonspecialists and to minimize the background assumed from the reader. The required notions are introduced little by little in examples in §2.

2 Examples

2.1 One-dimensional field theory

Toy model

First we illustrate our main results in the trivial particular case of dimension 1.

Consider a pipeline of N identical pipes in series with sources at the two endpoints pumping incompressible fluid in and out; see Figure 2 to the left. Let s be the intensity of each source (measured in litres/second). The *current* $j(\mathbf{k})$ through $\mathbf{k}$-th pipe (measured in litres/second) satisfies

- *Mass conservation law*: $j(\mathbf{1}) = j(\mathbf{N}) = s$ and $j(\mathbf{k}+\mathbf{1}) = j(\mathbf{k})$ for each $\mathbf{k} = \mathbf{1}, \dots, \mathbf{N}-\mathbf{1}$.

This just means that $j(\mathbf{k}) = s$ for $\mathbf{k} = \mathbf{1}, \dots, \mathbf{N}$. Throughout §2.1 we use bold font for edge numbers.

Formally, we define $s \in \mathbb{R}$ to be a fixed number and the *current* to be a function $j\colon \{\mathbf{1}, \dots, \mathbf{N}\} \to \mathbb{R}$ satisfying the mass conservation. (There is no formal difference between symbols in different fonts.)

Let us state a least action principle for the toy model. A *potential* ϕ of the flow is a function $\phi\colon \{0, \dots, N\} \to \mathbb{R}$ such that $\phi(k-1) - \phi(k) = j(\mathbf{k})$ for each $k = 1, \dots, N$. Clearly, it satisfies

- *the Laplace equation*: $\phi(k+1) - 2\phi(k) + \phi(k-1) = 0$ for each $k = 1, \dots, N-1$;

- *the least action principle*: among all functions on $\{0, \dots, N\}$, ϕ minimizes the functional

$$\frac{1}{2}\sum_{k=1}^{N}(\phi(k) - \phi(k-1))^2 - s\phi(0) + s\phi(N)$$

The first term is the total fluid kinetic energy. The functional is the sum of the values of the function

$$\mathcal{L}[\phi](k) = \frac{1}{2}(\underbrace{\phi(k) - \phi(k-1)}_{[\delta\phi](\mathbf{k})})^2 - s(k)\phi(k), \qquad \text{where} \qquad s(k) := \begin{cases} +s, & \text{if } k = 0 \\ 0, & \text{if } 1 \le k \le N-1 \\ -s, & \text{if } k = N. \end{cases}$$

Generalization

Such a "least action" formulation of the model has a straightforward generalization. The following definition is a particular case of Definition 3.1 below.

A *local Lagrangian* $\mathcal{L}$ is a self-map of the set of all real-valued functions on $\{0, \dots, N\}$ such that

$$\mathcal{L}[\phi](k) = L_k(\phi(k), \phi(k) - \phi(k-1))$$

for some differentiable function $L_k\colon \mathbb{R}^2 \to \mathbb{R}$. The 2 arguments of L_k are denoted by ϕ and $\delta\phi$. Set

$$\frac{\partial\mathcal{L}[\phi]}{\partial\phi}\colon \{0, \dots, N\} \to \mathbb{R}, \qquad \frac{\partial\mathcal{L}[\phi]}{\partial\phi}(k) := \left.\frac{\partial L_k(\phi, \delta\phi)}{\partial\phi}\right|_{\phi=\phi(k),\, \delta\phi=\phi(k)-\phi(k-1)};$$

$$\frac{\partial\mathcal{L}[\phi]}{\partial(\delta\phi)}\colon \{\mathbf{1}, \dots, \mathbf{N}\} \to \mathbb{R}, \qquad \frac{\partial\mathcal{L}[\phi]}{\partial(\delta\phi)}(\mathbf{k}) := \left.\frac{\partial L_k(\phi, \delta\phi)}{\partial(\delta\phi)}\right|_{\phi=\phi(k),\, \delta\phi=\phi(k)-\phi(k-1)}.$$

We also set $\frac{\partial\mathcal{L}}{\partial(\delta\phi)}(\mathbf{0}) = \frac{\partial\mathcal{L}}{\partial(\delta\phi)}(\mathbf{N}+\mathbf{1}) = 0$. E.g., in the toy model: $\frac{\partial\mathcal{L}[\phi]}{\partial\phi}(k) = -s(k)$, $\frac{\partial\mathcal{L}[\phi]}{\partial(\delta\phi)}(\mathbf{k}) = \delta\phi(\mathbf{k})$.

The following obvious proposition is a particular case of Theorem 1.1 above:

Proposition 2.1 (the Euler–Lagrange equation)**.** *Let $\mathcal{L}[\phi]$ be a local Lagrangian. A function ϕ is stationary for the functional $\sum_{k=0}^{N}\mathcal{L}[\phi](k)$, if and only if for each $k = 0, \dots, N$ we have*

$$\frac{\partial\mathcal{L}}{\partial(\delta\phi)}(\mathbf{k}) - \frac{\partial\mathcal{L}}{\partial(\delta\phi)}(\mathbf{k}+\mathbf{1}) + \frac{\partial\mathcal{L}}{\partial\phi}(k) = 0.$$

E.g., in the toy model above, the Euler–Lagrange equation is the Laplace equation. That model had a built-in conservation law, hidden after the least-action formulation. The following obvious proposition reveals conservation laws hidden in the Lagrangian; it is a particular case of Theorem 1.2.

Proposition 2.2 (the Noether theorem)**.** *If a local Lagrangian $\mathcal{L}[\phi]$ is* invariant under an infinitesimal transformation $\Delta(k)$*, i.e.,*

$$\left.\frac{\partial}{\partial t}\mathcal{L}[\phi + t\Delta]\right|_{t=0} = 0,$$

then for each stationary function ϕ for $\sum_{k=0}^{N}\mathcal{L}[\phi](k)$ the following function is conserved*, i.e. constant:*

$$j(\mathbf{k}) = \frac{\partial\mathcal{L}[\phi]}{\partial(\delta\phi)}(\mathbf{k})\Delta(k-1).$$

E.g., in the above toy model, apart the endpoints, the Lagrangian is invariant under the transformation $\phi \mapsto \phi - t$, where $t \in \mathbb{R}$. The resulting Noether conserved function is exactly $j(\mathbf{k}) = \phi(k-1) - \phi(k)$.

Momentum conservation

Let us state a less intuitive momentum conservation. The introduced discrete momentum tensor is a completely new object. First we give a heuristic motivation (cf. §2.2), then a formal definition.

In the toy model above, momentum circulation is physically clear. The momentum of the fluid in the pipe $\mathbf{k}$ is proportional to $j(\mathbf{k})$. During time Δt, the volume proportional to $j(\mathbf{k})\Delta t$ moves to the next pipe. Thus the momentum flux through the vertex k per unit time is proportional to $j(\mathbf{k})^2$. (We ignore pressure and do not care of the proportionality constant because this is just a heuristic anyway.)

Now consider a *free field*, i.e., $\mathcal{L}[\phi](k) = [\delta\phi](\mathbf{k})^2 + m^2\phi(k)^2$, where $m \geq 0$. Let ϕ be a stationary function, i.e. just a function satisfying the equation $\phi(k-1) - (2+m^2)\phi(k) + \phi(k+1) = 0$ for each $0 < k < N$. One expects the following properties of the momentum flux $\sigma(k)$ through a vertex k:

- $\sigma(k) = j(\mathbf{k})^2$ for $m = 0$, i.e., for a linear potential ϕ;
- $\sigma(k)$ depends only on $\phi(k)$, $\delta\phi(\mathbf{k})$, $\delta\phi(\mathbf{k}+\mathbf{1})$, and is homogeneous quadratic in these values;
- $\sigma(k) = \text{const}$ apart the endpoints, i.e., the momentum is conserved.

The simplest function $\sigma(k)$ satisfying these properties is (we skip a direct checking)

$$\sigma(k) = \delta\phi(\mathbf{k}+\mathbf{1})\delta\phi(\mathbf{k}) - m^2\phi(k)^2 = \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\delta\phi)}(\mathbf{k}+\mathbf{1})\delta\phi(\mathbf{k}) - \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\phi}(k)\phi(k).$$

Remark 2.1. A naive way to discretize the momentum flux would be to take the continuum momentum flux of a piecewise-linear extension of ϕ. But the resulting quantity is not conserved in a reasonable sense. Our function $\sigma(k)$ is very different from such naive "finite-element" discretization.

For an arbitrary Lagrangian, the formula for $\sigma(k)$ is not applicable literally but still suggestive. Since the formula involves the product of the values of $\delta\phi$ at distinct edges, it is reasonable to view it as a "projection" of a more fundamental quantity defined on the Cartesian square of the pipeline.

Definition 2.1. (This is a particular case of Definition 2.10.) The *Cartesian square* of a path with N edges is the grid $N \times N$; see Figure 2 to the right. The vertices of the grid have form $k \times l$, where k and l are vertices of the path. The 1×1 squares have form $\mathbf{k} \times \mathbf{l}$, where $\mathbf{k}$ and $\mathbf{l}$ are edges.

For functions ψ, ϕ on the set of vertices (respectively, edges) of the path denote by $\psi \times \phi$ the function on the vertices (respectively, 1×1 squares) of the grid given by $[\psi \times \phi](k \times l) = \psi(k)\phi(l)$ (respectively, by $[\psi \times \phi](\mathbf{k} \times \mathbf{l}) = \psi(\mathbf{k})\phi(\mathbf{l})$). A real-valued function on the disjoint union of the sets of vertices and 1×1 squares of the grid is a *type* $(1,1)$ *tensor*. (E.g., for the toy model, equation (7) gives the tensor equal s^2 on each 1×1 square and vanishing on each nonboundary vertex.)

A tensor T is *conserved*, if for each $0 < k < N$ and $0 < l \leq N$ the following equation holds:

$$T(k \times l) - T(k \times (l-1)) + T(\mathbf{k} \times \mathbf{l}) - T((\mathbf{k}+\mathbf{1}) \times \mathbf{l}) = 0.$$

I.e., we have one equation per *vertical* nonboundary edge; see Figure 2 to the right.

Remark 2.2. This is a well-known discretization of the Cauchy–Riemann equations [2, Eq. (2.2)], up to orientation. Thus tensor conservation means one half of the Cauchy–Riemann equations (for *vertical* edges only), like in [4, Corollary 2.12(1)], although our setup is very different from theirs.

The following obvious corollary of Proposition 2.1 is a particular case of Theorem 1.3.

Proposition 2.3 (Momentum conservation)**.** *Let $\mathcal{L}[\phi]$ be a local Lagrangian and ϕ be a stationary function for the functional $\sum_{k=0}^{N} \mathcal{L}[\phi](k)$. Then the tensor given by* (7) *is conserved.*

Define the *flux* of a tensor T through a vertex k by the formula $\frac{1}{2}T((\mathbf{k}+\mathbf{1})\times\mathbf{k}) - \frac{1}{2}T(k \times k)$. E.g., for the free field, the flux of tensor (7) equals exactly $\sigma(k)$. A tensor T is *symmetric*, if $T(k \times l) = T(l \times k)$ for all vertices or edges k, l. E.g., tensor (7) is symmetric essentially *only* for the free field (in spite of being a tensor on 1-dimensional spacetime). A conserved symmetric tensor has constant flux (this is a version of Proposition 2.7 below). E.g., for the toy model, the flux of tensor (7) is $j(\mathbf{k})^2/2$.

The same toy model describes the electrical network of N unit resistors in series (as well as many other systems); see Figure 2 to the middle. Now we switch entirely to the language of networks.

2.2 Electrical networks

Basic model

Consider an $N \times N$ grid of unit resistors; see Figure 3. A standard problem is to find currents in the grid, given the current sources at the boundary. It is solved using the following mathematical model.

Definition 2.2. Each of the N^2 unit squares of the grid is called a *face*. Orient the boundary ∂f of each face f counterclockwise. Orient edges in the directions of the coordinate axes. A *function on vertices/edges/faces* is a real-valued function defined on the set of vertices/edges/faces of the grid.

A *source* s is a function on vertices vanishing at all the nonboundary vertices. The *current generated* by the source s, or the *current on shell*, is the function on edges satisfying the two equations:

- *the Kirchhoff current law or charge conservation law*: $\partial j = -s$;
- *the Kirchhoff voltage law in the case of unit resistances*: $\delta j = 0$.

Here the *boundary* ∂j and the *coboundary* δj of a function j on edges are the functions on vertices and faces respectively given by the following formulae (see Figure 3 to the middle and to the right):

$$[\partial j](v) = \sum_{e \text{ ending at } v} j(e) - \sum_{e \text{ starting at } v} j(e),$$
$$[\delta j](f) = \sum_{e \text{ oriented along } \partial f} j(e) - \sum_{e \text{ oriented opposite to } \partial f} j(e),$$

for each vertex v and face f, where the sums are over edges e containing v and contained in ∂f respectively. Denote by $\epsilon s := \sum_v s(v)$ the sum over all vertices v (ϵ acts only on functions on vertices).

The following existence and uniqueness result is well-known.

Proposition 2.4. *A current generated by a source s exists, if and only if $\epsilon s = 0$. If a current generated by the source s exists, then it is unique.*

Electrical potential

Let us state a least-action principle for electrical networks. Throughout §2.2 j is a current on shell.

Definition 2.3. An *electrical potential* ϕ is a function on vertices satisfying

- *the Ohm law in the case of unit resistances*: $j = -\delta\phi$.

Here the *coboundary* $\delta\phi$ is the function on edges given by the formula

$$[\delta\phi](uv) = \phi(v) - \phi(u),$$

where uv is an edge starting at u and ending at v.

The following well-known existence and uniqueness result is straightforward.

Proposition 2.5. *For each current on shell there is a unique up to additive constant electrical potential.*

The following properties of an electrical potential ϕ may serve as equivalent definitions:

- *the Laplace equation with the Neumann boundary condition*: $\partial\delta\phi = s$;
- *the least action principle*: among all the functions on vertices, ϕ minimizes the functional

$$\mathcal{S}[\phi] = \frac{1}{2} \sum_{\text{edges } uv} (\phi(u) - \phi(v))^2 - \sum_{\text{vertices } v} s(v)\phi(v) = \epsilon\mathcal{L}[\phi], \qquad \text{where}$$
$$\mathcal{L}[\phi] = \tfrac{1}{2}\delta\phi \frown \delta\phi - s \frown \phi.$$

Here the cap-product $\frown$ is defined as follows; see Figure 3 to the middle.

Definition 2.4. Order the vertices lexicographically with respect to their coordinates. Denote by $\max f$ ($\min f$) the maximal (minimal) vertex of a face, edge, or vertex f. The *cap-product* $\phi \frown \psi$ of two functions ϕ and ψ on faces (respectively, edges or vertices) is the function on vertices given by

$$[\phi \frown \psi](v) = \sum_{f\,:\,\max f = v} \phi(f)\psi(f),$$

where the sum is over faces (respectively, edges or vertices) f such that $\max f = v$.

Magnetic field

There is one more discrete field in an electrical network: the current j generates a magnetic field.

Definition 2.5. A *magnetic field* F (or *magnetic flux through faces in the* $(0, 0, -1)$*-direction*) *generated* by a current j on shell is a function on faces satisfying the following equation apart the grid boundary:

- *the Ampere law in the case of unit-area faces*: $-\partial F = j$.

Here the *boundary* ∂F is the function on edges given by the formula

$$[\partial F](e) = F(f) - F(g)$$

for each pair of adjacent faces f and g such that ∂f (respectively, ∂g) is oriented along (respectively, opposite to) the common edge e; see Figure 3 to the left. (The definition of $[\partial F](e)$ for boundary edges e is not required for this subsection and is postponed until §2.3.)

The following well-known existence and uniqueness result is straightforward.

Proposition 2.6. *For each current on shell there is a unique up to additive constant magnetic field.*

Throughout §2.2 the functions ϕ and F are an electrical potential and a magnetic field respectively.

Remark 2.3. The pair (ϕ, F) and $-j$ are discretizations of an analytic function and its derivative [5, 2].

Definition 2.6. A *magnetic vector-potential* A of the field F is a function on edges such that $\delta A = F$.

A magnetic vector-potential A has the following properties (proved analogously to §2.3):

- *the source equation*: $-\partial\delta A = j$ apart the grid boundary;
- *gauge invariance*: $A + \delta g$ is a vector-potential of the same field for any function g on vertices;
- *the least action principle*: among all the functions on edges, A minimizes $\mathcal{S}[A] = \epsilon\mathcal{L}[A]$, where

$$\mathcal{L}[A] = \tfrac{1}{2}\delta A \frown \delta A + j \frown A.$$

Energy and momentum

Let us state energy and momentum conservation in an electrical network in a simple heuristic form.

For functions ϕ, ψ on faces (respectively, edges or vertices), denote by $\langle\phi, \psi\rangle = \sum_f \phi(f)\psi(f)$ the sum over all faces (respectively, edges or vertices). The obvious identity $\langle\delta\phi, j\rangle = \langle\phi, \partial j\rangle$ implies

- *the Tellegen theorem or* total *energy conservation*: $\langle\delta\phi, j\rangle + \langle\phi, s\rangle = 0$.

Now we study *local* conservation and the flow of energy. Energy flows in the direction of the Poynting vector, hence *transversely* to (not along) the resistors. This is why we define energy flow in a subdivision of the grid. The cross-product formula for the Poynting vector is then discretized directly.

Definition 2.7. The *doubling* is the $2N \times 2N$ grid with the vertices at vertices, edge midpoints, and face centers of the initial $N \times N$ grid. Orient the edges still in the direction of the coordinate axes.

The *heat power* W is the function on the vertices v of the doubling given by the formula

$$W(v) = \begin{cases} -[\delta\phi](e)j(e), & \text{if } v \text{ is the midpoint of an edge } e; \\ 0, & \text{if } v \text{ is the center of a face or a vertex of the initial grid.} \end{cases}$$

The *Poynting vector* or *energy flux* S is the function on edges uv of the doubling, $u < v$, given by

$$S(uv) = \begin{cases} [\delta\phi](e)F(f), & u \text{ and } v \text{ are the centers of a vertical edge } e \text{ and a face } f \text{ or vice versa;} \\ -[\delta\phi](e)F(f), & u \text{ and } v \text{ are the centers of a horizontal edge } e \text{ and a face } f \text{ or vice versa;} \\ 0, & u \text{ or } v \text{ is a vertex of the initial grid.} \end{cases}$$

The *Lorentz force* L is defined analogously to S, only $\delta\phi$ is replaced by $-j/2$ (so $L = S/2$ in our basic model). The *magnetic pressure* P (or *momentum flux of the magnetic field towards the edges in the normal direction*) is the function on nonboundary vertices v of the doubling given by the formula

$$P(v) = \begin{cases} F(f)F(f)/2, & \text{if } v \text{ is the center of a face } f; \\ F(f)F(g)/2, & \text{if } v \text{ is the midpoint of the common edge of faces } f \text{ and } g; \\ 0, & \text{if } v \text{ is a vertex of the initial grid.} \end{cases}$$

A straightforward consequence of these definitions and the Kirchhoff laws is:

- *Energy conservation*: $\partial S - W = 0$.
- *Momentum conservation for the magnetic field*: $\delta P + L = 0$ on those edges of the doubling which contain the face-centers of the initial grid.

In §2.3 we introduce a more conceptual form of the two laws, explaining the latter restriction.

Now we state a less visual momentum conservation law for the *electric* field. This is a new result. One expects the following properties of the momentum flux $\sigma(e)$ across edges e of the initial grid:

- $\sigma(e)$ equals the momentum flux of a continuum electric field across e, if the potential is linear;
- $\sigma(e)$ depends only on the values of $\delta\phi$ at the edges intersecting e and is bilinear in these values;
- $\delta\sigma = 0$ apart the grid boundary: the momentum flux across the boundary of each face vanishes.

The simplest function σ satisfying these properties is defined as follows; cf. Figure 4 and Remark 2.1.

Definition 2.8. The *momentum flux of the electric field across edges in the negative normal direction*, or *the electric part of the Maxwell stress tensor*, is the pair $\sigma = (\sigma_1, \sigma_2)$ of functions on edges disjoint with the grid boundary given by the following formula for each $k = 1, 2$:

$$\sigma_k(uv) = \frac{(-1)^k}{2} \begin{cases} \delta\phi(uu_+)\delta\phi(uv) + \delta\phi(vv_+)\delta\phi(uv), & \text{if } uv \parallel Ox_k; \\ \delta\phi(uv)\delta\phi(uv) - \delta\phi(vv_+)\delta\phi(v_-v), & \text{if } uv \perp Ox_k, \end{cases}$$

where uu_+, v_-v, vv_+, are the edges orthogonal to uv such that $u < u_+$ and $v_- < v < v_+$; see Figure 4.

Corollary 2.1 (Momentum conservation for the electric field)**.** *For each electric potential ϕ on shell we have $\delta\sigma_1 = \delta\sigma_2 = 0$ on each face not intersecting the grid boundary.*

Remark 2.4. The function σ_k is minus the flux (given by Definition 2.12) of the energy-momentum tensor $T[\phi] = \delta\phi \times \delta\phi$ (given by Theorem 1.3 for the Lagrangian $\mathcal{L}[\phi] = \frac{1}{2}\delta\phi \frown \delta\phi - s \frown \phi$).

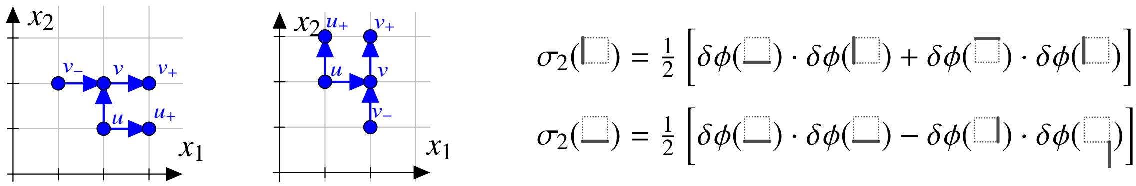


Figure 4: Notation in Definition 2.8 of discrete momentum flux. The square uvv_+u_+ is shown by dotted lines to the right. The edge at which a particular cochain is evaluated is shown in bold.

Approximation

The basic network model indeed converges to a continuum one, as the grid becomes finer and finer.

The continuum model is a *homogeneous conducting plate* defined as follows. Let I^2 be the unit square, $\vec{\mathrm{n}}$ be the unit inner normal vector field on $\partial\mathrm{I}^2$ besides the corners, $*$ be the counterclockwise rotation through $\pi/2$ about the origin (the *Hodge star*), $\delta_{kl} = \delta_k^l := \begin{cases} 1, & \text{if } k = l; \\ 0, & \text{if } k \neq l. \end{cases}$

A *source* s is a continuous function on $\partial\mathrm{I}^2$. The *fields* $\vec{\mathrm{j}},\phi,\mathrm{F},\mathrm{W},\vec{\mathrm{S}},\vec{\mathrm{L}},\mathrm{P},\mathcal{L},\sigma$ *generated* by s are continuous scalar/vector/matrix fields on I^2, being C^1 and satisfying the following conditions apart $\partial\mathrm{I}^2$:

$$-\nabla\phi = \vec{\mathrm{j}}, \qquad \mathrm{W} = -\nabla\phi\cdot\vec{\mathrm{j}}, \qquad \vec{\mathrm{S}} = -*\nabla\phi\cdot\mathrm{F}, \qquad \mathcal{L} = \tfrac{1}{2}(\nabla\phi)^2 \quad \left(= \tfrac{1}{2}\mathrm{d}\phi\lrcorner\mathrm{d}\phi\right),$$

$$*\nabla\mathrm{F} = \vec{\mathrm{j}}, \qquad \vec{\mathrm{L}} = *\vec{\mathrm{j}}\cdot\mathrm{F}, \qquad \mathrm{P} = \tfrac{1}{2}\mathrm{F}\cdot\mathrm{F}, \qquad \sigma_{kl} = -\frac{\partial\phi}{\partial\mathrm{x}^k}\frac{\partial\phi}{\partial\mathrm{x}^l} + \frac{1}{2}\delta_{kl}(\nabla\phi)^2,$$

and the following boundary condition on $\partial\mathrm{I}^2$ besides the corners:

$$\vec{\mathrm{j}}\cdot\vec{\mathrm{n}} = \mathrm{s}.$$

In other words, $\phi + i\mathrm{F}$ is an analytic function such that $\frac{\partial}{\partial\vec{\mathrm{n}}}\phi = -\mathrm{s}$; the other fields are expressions in it.

Let the unit square I^2 be dissected into N^2 equal squares. Given a source s_N, define the fields $j_N, \phi_N, F_N, W_N, S_N, L_N, P_N, \mathcal{L}_N, \sigma_N$ on the resulting grid literally as above on the grid of size $N \times N$.

Remark 2.5. It would be somewhat more conceptual to modify the above Amper law for the resulting grid because the faces are not unit squares anymore. This leads just to normalization of the fields by powers of N. We avoid such modification for simplicity.

Clearly, the continuum model has more symmetries than the discrete one: e.g., $\mathcal{L}$ is rotational-invariant whereas $\mathcal{L}_N$ is not, at least in a naive sense; cf. [14, Definition 5.2.36].

Dissect each side of $\partial\mathrm{I}^2$ into $N + 1$ (not N) equal segments called *auxiliary segments*. Write $a_N(x) \rightrightarrows b_N(x)$ for functions a_N, b_N on a set M_N, if $\max_{x\in M_N}|a_N(x) - b_N(x)| \to 0$ as $N \to \infty$.

Theorem 2.1 (Approximation theorem)**.** *Let* $\mathrm{s}\colon \partial\mathrm{I}^2 \to \mathbb{R}$ *be a continuous source with* $\int_{\partial\mathrm{I}^2}\mathrm{s}\,\mathrm{dl} = 0$. *Dissect* I^2 *into* N^2 *equal squares and define a discrete source* s_N *on the resulting grid by the formula*

$$s_N(v) := \int_{v_-v_+}\mathrm{s}\,\mathrm{dl},$$

where $v_-v_+ \subset \partial I^2$ *is the arc formed by* 1 *or* 2 *auxiliary segments containing a vertex* $v \in \partial I^2$. *Take continuous fields* $\vec{\mathrm{j}}$, ϕ, F, W, $\vec{\mathrm{S}}$, $\vec{\mathrm{L}}$, P, $\mathcal{L}$, σ *and discrete ones* j_N, ϕ_N, F_N, W_N, S_N, L_N, P_N, $\mathcal{L}_N$, $\sigma_N = (\sigma_{N,1}, \sigma_{N,2})$ *generated by the sources. Assume that* ϕ, F *and* ϕ_N, F_N *vanish at the center of* I^2 *and at one of the vertices or faces closest to the center respectively. Take* $r > 0$. *Then on the set of all vertices* v*, edges* e*, faces* f*, edge-midpoints* e'*, and face-centers* f' *at distance* $\geq r$ *from* $\partial\mathrm{I}^2$ *we have:*

$$\phi_N(v) \rightrightarrows \phi(v), \qquad Nj_N(e) \rightrightarrows N\int_e \vec{\mathrm{j}}\cdot\mathrm{d}\vec{\mathrm{l}}, \qquad N^2W_N(e') \rightrightarrows \mathrm{W}(e'), \qquad NS_N(e'f') \rightrightarrows 2N\int_{e'f'}\vec{\mathrm{S}}\cdot\mathrm{d}\vec{\mathrm{l}},$$

$$N^2\mathcal{L}_N(v) \rightrightarrows \mathcal{L}(v), \qquad F_N(f) \rightrightarrows N^2\int_f \mathrm{F}\,\mathrm{dS}, \qquad P_N(e') \rightrightarrows \mathrm{P}(e'), \qquad NL_N(e'f') \rightrightarrows N\int_{e'f'}\vec{\mathrm{L}}\cdot\mathrm{d}\vec{\mathrm{l}},$$

$$N^2\sigma_{N,k}(e) \rightrightarrows N\int_e\left(\sigma_{k2}\,\mathrm{dx}^1 - \sigma_{k1}\,\mathrm{dx}^2\right) \qquad \textit{as } N \to \infty.$$

The theorem is essentially known; it is easily deduced from highly nontrivial known results in §4.

2.3 Lattice electrodynamics

A standard problem in electrodynamics is to find forces between given charges and currents. This is done in two steps: first the field generated by the charges and currents is computed, then — the action of the field upon them. For a discretization, continuum spacetime is replaced by a 4-dimensional grid.

Generation of the field by the current

Definition 2.9. The *d-dimensional grid* I_N^d is the hypercube $0 \le x_0, x_1 \dots, x_{d-1} \le N$ in $\mathbb{R}^d$ dissected into N^d unit hypercubes. Order the grid vertices lexicographically with respect to their coordinates.

Fix the following orientation of k-dimensional faces of I_N^d. A *positively oriented* basis in a face is formed by the k vectors starting at the minimal vertex of the face, going along the edges of the face, and ordered according to the ordering of the endpoints. A k-dimensional face f and a $(k-1)$-dimensional face $e \subset \partial f$ are *cooriented* (repectively, *opposite oriented*), if the ordered set consisting of the outer normal to e in f and a positive basis in e is a positive (respectively, negative) basis in f.

The *boundary* ∂F and the *coboundary* δF of a function F on k-dimensional faces e are the functions on $(k-1)$- and $(k+1)$-dimensional faces v and f respectively given by (see Figure 5)

$$[\partial F](v) = \sum_{e \text{ cooriented with } v} F(e) - \sum_{e \text{ oriented opposite to } v} F(e),$$
$$[\delta F](f) = \sum_{e \text{ cooriented with } f} F(e) - \sum_{e \text{ oriented opposite to } f} F(e).$$

The *Minkowski sharp operator* # applied to a function F on k-dimensional faces f is

$$[\# F](f) := \begin{cases} -F(f), & \text{if } f \parallel (1, \underbrace{0, \dots, 0}_{d-1 \text{ zeroes}}), \\ F(f), & \text{if } f \perp (1, 0, \dots, 0). \end{cases}$$

An *electromagnetic vector-potential A generated* by a current j is a function on edges satisfying

- *The source equation*: $-\partial \# \delta A = j$.

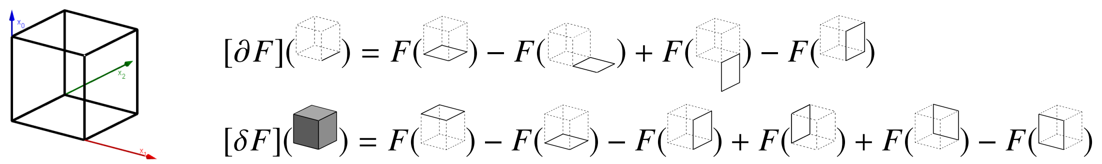


Figure 5: Boundary and coboundary (see Definition 2.9). A nonboundary 3-face (to the left) is shown again by dotted lines (to the right). The face at which a particular cochain is evaluated is in bold.

Remark 2.6. We do not discuss conditions under which the vector-potential exists and is unique.

The operator # is new. It is a discrete analogue of raising indices in the metric of signature $(-, +, \dots, +)$. We use it instead of a discrete Hodge star [22] to avoid working with the dual lattice, which would complicate the theory and its generalization to other spacetimes.

For an arbitrary spacetime the operators ∂ and δ (but not #) are defined analogously except that the lexicographic ordering is replaced by the one fixed in Definition 1.1.

The following 3 properties of an electromagnetic vector-potential A generated by a current j immediately follow from the well-known identities $\delta\delta = 0$ and $\partial\partial = 0$; cf. (1)–(2):

- *the Maxwell equations*: $\delta F = 0$ and $-\partial \# F = j$, where $F := \delta A$ is the *electromagnetic field*;
- *Gauge invariance*: $A + \delta g$ is generated by the same current j for any function g on vertices;
- *Charge conservation*: $\partial j = 0$, if there exists a vector-potential generated by the current j.

Corollary 2.2. *An electromagnetic vector-potential A is generated by a current j, if and only if A is a stationary function for the functional $\mathcal{S}[A] = \epsilon\mathcal{L}[A]$, where*

$$\mathcal{L}[A] = -\tfrac{1}{2}\#\delta A \frown \delta A - j \frown A.$$

Remark 2.7. Electrodynamics in linear nondispersive media is discretized analogously, only the Minkowski sharp operator is replaced by a linear operator depending on the media.

To convince the reader that discrete electrodynamics is an objective reality, let us informally sketch a network model for it [16]. Set $d = 4$. For each edge of the grid I_N^{d-1}, take an oscillatory circuit consisting of one (nonconstant) current source, one unit capacitor, and as many unit-transformer coils as there are faces containing the edge; see Figure 6 to the bottom-left. Join the obtained circuits in the shape of the grid, join the transformer cores in the shape of the 1-dimensional skeleton of the dual grid, join the capacitor dielectric cores in the shape of the 2-dimensional skeleton of the dual grid. We get an electric, a magnetic, and a dielectric network coupled together; a part is shown in Figure 6. We conjecture that the integrals of appropriate currents and voltages over time intervals $[n, n+1]$, where $n \in \mathbb{Z}$, satisfy the discrete Maxwell equations above.

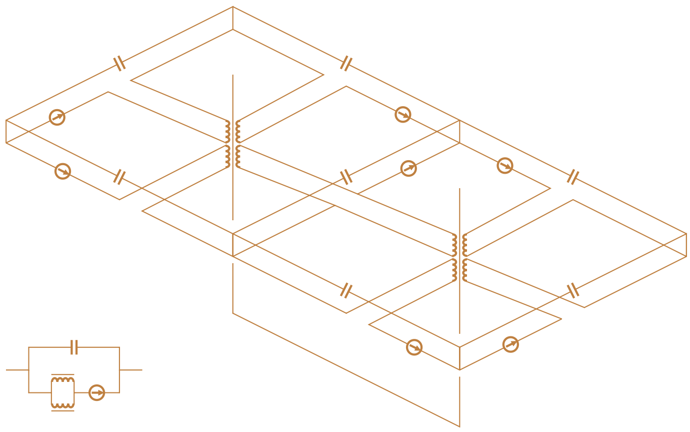


Figure 6: A network model for discrete electrodynamics; cf. [16]

Action of the field on the current

The field acts on the current by the Lorenz force, which we are going to discretize now. The rest of §2.3 contains completely new notions and results (except the cross-product); cf.[3].

Definition 2.10. Let $I_N^d \times I_N^d$ be the Cartesian square of the d-dimensional grid. It is a $2d$-dimensional grid with the faces of the form $e \times f$, where e and f are faces of I_N^d of arbitrary dimension.

A *tensor* of *type* $(q, 1)$, where $q = 1$ or 0, is a function on all faces $e \times f$ of $I_N^d \times I_N^d$ such that $\dim f - \dim e = 1 - q$. The *chain-cochain cross-product* of fields ϕ, ψ with $\dim\phi - \dim\psi = 1 - q$ is the tensor

$$[\psi \times \phi](e \times f) = \begin{cases} \psi(e)\phi(f), & \text{if } \dim e = \dim\psi \text{ and } \dim f = \dim\phi; \\ 0, & \text{if } \dim e \neq \dim\psi \text{ or } \dim f \neq \dim\phi. \end{cases}$$

The *boundary* operator ∂ is the unique linear map between the spaces of type $(1,1)$ and $(0,1)$ tensors such that for each fields ϕ, ψ with $\dim\phi = \dim\psi$ we have

$$\partial(\psi \times \phi) = \partial\psi \times \phi + \psi \times \delta\phi$$

(cf. Definition 2.1 above and equation (22) below). A type $(1,1)$ tensor T is *conserved*, if $\partial T = 0$.

The main motivation for this definition is that it satisfies the principles of discretization from §1, as we see later. Let us clarify the relation to continuum theory (this is not used elsewhere in the paper).

Remark 2.8. Equivalently, the set of type $(q,1)$ tensors is $\bigoplus_{p=0}^{d} C_p(I_N^d;\mathbb{R}) \otimes_{\mathbb{R}} C^{p-q+1}(I_N^d;\mathbb{R})$. Thus it discretizes the space $\bigoplus_{p=0}^{d} \Omega^p(\mathrm{I}^d)^* \otimes \Omega^{p-q+1}(\mathrm{I}^d)$ rather than the space $\mathrm{T}_1^q(\mathrm{I}^d)$ of continuum type $(q,1)$ tensors. (Here $\Omega^p(\mathrm{I}^d)$ denotes the set of C^∞ p-forms on the unit hypercube I^d and $\otimes$ denotes the tensor product over $\Omega^0(\mathrm{I}^d)$). But the former space is mapped to the latter by the 'contraction' map

$$\mathrm{T}^{m_1\dots m_p}_{n_1\dots n_{p-q+1}} \mapsto \begin{cases} \mathrm{T}^{m_1\dots m_p}_{km_1\dots m_p}, & \text{if } q=0;\\ \mathrm{T}^{lm_2\dots m_p}_{km_2\dots m_p} - \frac{1}{2p}\delta^l_k \mathrm{T}^{m_1\dots m_p}_{m_1\dots m_p}, & \text{if } q=1, p>0. \end{cases}$$

(Summation over repeating indices is understood.) Since no discretization of the image is available (at least for $q=1$), the discretization of the domain is proclaimed to be space of type $(q,1)$ tensors. The map is chosen to commute with certain codifferentials when T has certain symmetry properties (i.e., $\sharp\mathrm{T}$ is symmetric wrt interchanging m_p and n_p but antisymmetric wrt interchanging m_p and m_q):

$$\begin{array}{ccccc} \frac{\partial\mathcal{L}}{\partial(\mathrm{d}\phi)}\otimes \mathrm{d}\phi + \frac{\partial\mathcal{L}}{\partial\phi}\otimes\phi & \in & \bigoplus_{p=0}^{d}\Omega^p(\mathrm{I}^d)^*\otimes\Omega^p(\mathrm{I}^d) & \xrightarrow{\mathrm{d}^*\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{d}} & \bigoplus_{p=0}^{d-1}\Omega^p(\mathrm{I}^d)^*\otimes\Omega^{p+1}(\mathrm{I}^d) \\ \downarrow & & \text{'contraction'}\downarrow & & \text{'contraction'}\downarrow \\ \mathrm{T}^l_k & \in & \mathrm{T}^1_1(\mathrm{I}^d) & \xrightarrow{\text{divergence}} & \mathrm{T}^0_1(\mathrm{I}^d). \end{array}$$

Similarly, (7) discretizes $\frac{\partial\mathcal{L}}{\partial(\mathrm{d}\phi)}\otimes \mathrm{d}\phi + \frac{\partial\mathcal{L}}{\partial\phi}\otimes\phi$ rather than the continuum energy-momentum tensor T^l_k, but the former is usually taken to the latter by the 'contraction' map. Here $\left(\frac{\partial\mathcal{L}}{\partial(\mathrm{d}\phi)}\right)^{m_1\dots m_p} := \frac{\partial\mathcal{L}}{\partial(\mathrm{d}\phi)_{m_1\dots m_p}}$. The former is conserved (i.e. taken to 0 by $\mathrm{d}^*\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{d}$) regardless of symmetries of $\mathcal{L}[\phi]$.

In contrast to continuum theory, type $(0,1)$ tensors are *not* 1-dimensional fields.

Although $I_N^d \times I_N^d$ is naturally identified with I_N^{2d}, the boundary operator on tensors is *not* a restriction of the boundary operator on I_N^{2d}. To avoid confusion, we distinguish between $I_N^d\times I_N^d$ and I_N^{2d} below.

A type $(q,1)$ tensor can be equivalently defined as an element of $C^{d+q-1}(I_N^d\times I_N^{d*};\mathbb{R})$, where I_N^{d*} is the dual grid. Then the boundary operator on tensors is exactly the boundary operator on $I_N^d\times I_N^{d*}$. We avoid working with dual grids for simplicity and for easier generalization to arbitrary spacetimes.

It would be somewhat more conceptual to restrict the domain of a tensor to a "neighborhood of the diagonal" in $I_N^d\times I_N^d$. E.g., type $(0,1)$ tensors can be restricted to the set of faces $e\times f$ such that $e\subset f$: the values at the other faces do not contribute to integration. We avoid such restriction for simplicity.

The set of faces of $I_N^d\times I_N^d$ is naturally mapped to the set of faces of the doubling: to a face $e\times f$ assign the face of the doubling with the center at the midpoint of the segment joining the centers of e and f. Thus informally the values of a tensor are "sitting" on the faces of the doubling; in particular, the ones on the 2-dimensional faces are interpreted as off-diagonal components.

Up to sign and factor $1/2$, the fields W, S, L, P from §2.2 are "induced" by the latter map from $j\times\delta\phi$, $F\times\delta\phi$, $j\times F$, $F\times F$ respectively. These heuristic fields are now replaced by tensors.

Definition 2.11. Let A be a vector-potential generated by a current j, and $F=\delta A$. The *Lorentz force* is the type $(0,1)$ tensor $L=j\times F$. It is supported by faces $e\times f\subset I_N^d\times I_N^d$ such that $\dim e$=1, $\dim f$=2.

The *energy-momentum tensor*, or *stress-energy tensor*, of the electromagnetic field (respectively, of both the field and the current) is the type $(1,1)$ tensor $T'=-\#F\times F$ (respectively, $T=-\#F\times F-j\times A$). The tensor T' is supported by 4-dimensional faces $e\times f\subset I_N^d\times I_N^d$ such that $\dim e=\dim f=2$.

An immediate consequence of these definitions, Maxwell's equations, and charge conservation is

- *Energy and momentum conservation*: $\partial T'=L$ and $\partial T=0$.

Remark 2.9. The latter is a particular case of Theorem 1.3 for the Lagrangian from Corollary 2.2.

In contrast to T', the tensor T has *no* conserved continuum analogue.

More precisely, L and T' discretize the tensors $\mathrm{j}^l\mathrm{F}_{kn}$ and $-\mathrm{F}^{lm}\mathrm{F}_{kn}$, but the latter two are taken to the continuum Lorenz force and energy-momentum tensor by the 'contraction' map from Remark 2.8.

The formula for the discrete energy-momentum tensor T' is even simpler than the continuum analogue. This is achieved at the cost of a rather subtle definition of discrete tensor integration below.

Integral conservation laws

To make discrete tensors at all practical, we define their integration. This allows to get integral forms of the above conservation laws and to compare these tensors with their continuum analogues. The following construction works for any discrete field theory, not just electrodynamics, but only on the grid I_N^d, where $d \geq 2$. In §1.1 (respectively, in Definition 2.8) we have actually applied the construction for $d = 3$, $k = 0$, and the tensor T' (respectively, for $d = 2$, $k = 1, 2$, and the tensor $\delta\phi \times \delta\phi$).

Let us introduce some notation. Let e_k, where $k = 0, \dots, d-1$, be the vector of length $\frac{1}{2}$ pointing in the direction of the axis Ox_k. Each combination of such vectors with coefficients from the set $\{0, 1, \dots, 2N\}$ is the center of a unique face of I_N^d. We use the same notation for a face f and its center. In particular, $f + \mathrm{e}_k$ denotes the face with the center at the point obtained from the center of f by translation by the vector e_k. The dimensions of f and $f + \mathrm{e}_k$ are always different by 1. A *hyperface* is a $(d-1)$-dimensional face of I_N^d.

Definition 2.12. A type $(1, 1)$ tensor is *partially symmetric*, if $T(e \times f) = T(f \times e)$ for each $e \parallel f$ (we set $e \parallel f$, if e and f are vertices). For $k = 0, \dots, d-1$, the *k-th component of the flux* of a partially symmetric tensor T across a nonboundary hyperface $h \perp \mathrm{e}_l$ *in the positive normal direction* is

$$\langle T, h\rangle_k = \frac{1}{2} \sum_{\substack{f : f \subset h, f \ni \max h;\\ f \parallel \mathrm{e}_k \text{ for } h \parallel \mathrm{e}_k}} (-1)^{\dim \mathrm{Pr}(f,k,l)+l+1} \cdot \begin{cases} T((f + \mathrm{e}_l - \mathrm{e}_k) \times f) + T((f + \mathrm{e}_l + \mathrm{e}_k) \times f), & \text{if } h \parallel \mathrm{e}_k;\\ T(f \times f) - T((f + \mathrm{e}_k) \times (f - \mathrm{e}_k)), & \text{if } h \perp \mathrm{e}_k, \end{cases}$$

where the sum is over faces f of arbitrary dimension (we set $f \nparallel \mathrm{e}_k$, if f is a vertex), and $\mathrm{Pr}(f, k, l)$ is the orthogonal projection of f to the linear span of all e_m with $\min\{k, l\} \leq m \leq \max\{k, l\}$.

Assume that $d \geq 2$. Let π be an oriented piecewise-linear hypersurface consisting of nonboundary hyperfaces. For each hyperface $h \subset \pi$ denote

$$\langle h, \pi\rangle = \begin{cases} +1, & \text{if the orientations of } \pi \text{ and } h \text{ agree,}\\ -1, & \text{if the orientations of } \pi \text{ and } h \text{ are opposite.} \end{cases} \tag{8}$$

The latter notation is also used, if π and h have any dimension $p > 0$. The *flux* across π is $\langle T, \pi\rangle_k := \sum_h \langle T, h\rangle_k \langle h, \pi\rangle$. A tensor T is *conserved apart* ∂I_N^d, if $\partial T(e \times f) = 0$ for all faces $e, f \not\subset \partial I_N^d$.

Proposition 2.7 (Integral energy-momentum conservation)**.** *If a partially symmetric type* $(1, 1)$ *tensor is conserved apart the boundary of the grid I_N^d, where $d \geq 2$, then each component of the flux of the tensor across each closed oriented hypersurface consisting of nonboundary hyperfaces vanishes.*

Theorem 2.2 (Integral energy-momentum conservation for a free field)**.** *Let $d \geq 2$. If the Lagrangian is $\mathcal{L}[\phi] = -\#\delta\phi \frown \delta\phi - m^2\phi \frown \phi$ and $\phi \in C^k(I_N^d; \mathbb{R})$ is on shell, then each component of the flux of tensor* (7) *across each closed oriented hypersurface consisting of nonboundary hyperfaces vanishes.*

Remark 2.10. There are many other ways to define a tensor flux; we have chosen the simplest one.

Our definition has the following informal motivation. Values of a tensor are "sitting" on the faces of the doubling; see the paragraph of Remark 2.8 before the last one. The flux across a hyperface is then the sum of these values over the faces adjacent to the hyperface from appropriate "side".

For nonconserved tensors an analogue of the Stokes formula holds; see Proposition 4.2.

Similar results hold for $d = 1$, only oriented hypersurfaces should be replaced by 0-chains.

Unlike continuum theory, the 0-th component of the flux of the energy-momentum tensor T' (see Definition 2.11) across a hyperface $h \perp (1, 0, \dots, 0)$ is not necessarily positive, thus cannot be interpreted as energy. In a sense, this is a higher order effect with respect to the discretization step $1/N$.

We use the notation $\langle T, \pi\rangle_k$, with literally the same definition, even if T is not partially symmetric. This makes no sense in discrete setup but is useful for the continuum limit; see Proposition 2.15.

The energy-momentum tensor T of both the field and the current (see Definition 2.11) is not partially symmetric. In a sense, it still approximates some continuum tensor, but the latter is not conserved. We know neither an integral conservation law nor a conserved continuum analogue for T.

The energy-momentum tensor T' is symmetric in a sense (after "raising an index"). In particular, we shall see that it approximates the symmetric Belinfante–Rosenfeld energy-momentum tensor rather than the nonsymmetric canonical energy-momentum tensor. In other field theories, e.g., for the Dirac field, the discrete energy-momentum tensor approximates the nonsymmetric canonical energy-momentum tensor rather than the Belinfante–Rosenfeld one; see Proposition 2.15.

Let us illustrate analogy between tensor (7) and the continuum canonical energy-momentum tensor

$$\mathrm{T}_k{}^l = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial \mathrm{x}_l)}\frac{\partial\phi}{\partial \mathrm{x}_k} - \delta_k^l \mathcal{L}.$$

Proposition 2.8. *Let $d \geq 2$. Let a local Lagrangian $\mathcal{L}\colon C^0(I_N^d;\mathbb{R}) \to C_0(I_N^d;\mathbb{R})$ be homogeneous quadratic in ϕ and $\delta\phi$. Let ϕ be a 0-dimensional field (not necessarily on shell) and T be the energy-momentum tensor (not necessarily partially symmetric) given by (7). Then for each $0 \leq k, l < d$ and each hyperface $h \perp \mathrm{e}_l$ having maximal vertex v and disjoint with the grid boundary we have*

$$(-1)^l \langle T, h\rangle_k = \frac{1}{2}\left(\frac{\partial\mathcal{L}[\phi]}{\partial(\delta\phi)}(v+\mathrm{e}_l) + \frac{\partial\mathcal{L}[\phi]}{\partial(\delta\phi)}(v+\mathrm{e}_l-2\mathrm{e}_k)\right)\delta\phi(v-\mathrm{e}_k) - \delta_k^l\mathcal{L}[\phi](v).$$

Approximation

The discrete energy-momentum tensor T' indeed approximates the continuum one, as we show now. In continuum theory, an *electromagnetic field* is a continuous antisymmetric matrix field F_{mn} on the unit hypercube I^d. The *(Belinfante–Rosenfeld) energy-momentum tensor* of the field is the matrix field

$$\mathrm{T}_k^l = -\mathrm{F}^{lm}\mathrm{F}_{km} + \tfrac{1}{4}\delta_k^l \mathrm{F}^{mn}\mathrm{F}_{mn},$$

where summation over repeating indices is understood and $\mathrm{F}^{mn} := \begin{cases} -\mathrm{F}_{mn}, & \text{if } m=0 \text{ or } n=0;\\ \mathrm{F}_{mn}, & \text{if } m\neq 0 \text{ and } n \neq 0.\end{cases}$

Let I^d be dissected into N^d equal hypercubes. Given an arbitrary discrete 2-dimensional field F, define the energy-momentum tensor $T' = -\#F \times F$ on the resulting grid literally as on the grid I_N^d.

Remark 2.11. It is somewhat more natural to modify the definition of the operator # by the factor N^{2k-d} because the faces are not unit hypercubes anymore. This leads just to normalization of the energy-momentum tensor T' by a power of N. We avoid such modification for simplicity.

Proposition 2.9 (Approximation property)**.** *Let F_{mn} be a continuous electromagnetic field on I^d. Dissect I^d into N^d equal hypercubes and define a discrete 2-dimensional field F_N on faces f of the resulting grid by the formula*

$$F_N(f) := \mathrm{F}_{mn}(\max f),$$

where the integers $m < n$ are determined by the conditions $\mathrm{e}_m, \mathrm{e}_n \parallel f$. Let T_k^l and $T'_N = -\#F_N \times F_N$ be the continuous and discrete energy-momentum tensor respectively. Take $0 \leq k, l < d$. Then on the set of all hyperfaces $h \perp \mathrm{e}_l$ not intersecting $\partial\mathrm{I}^d$ we have (under the notation before Theorem 2.1)

$$(-1)^l\langle T'_N, h\rangle_k \cong \mathrm{T}_k^l(\max h) \qquad \text{as } N \to \infty.$$

Remark 2.12. Here the fields F_{mn} and F_N do *not* necessarily satisfy the Maxwell equations (and typically F_N cannot, even if F_{mn} does). Approximation of a smooth solution of the Maxwell equations by discrete ones, a standard question of computational electrodynamics, is not discussed in the paper.

2.4 Lattice gauge theory

Classical gauge theory generalizes electrodynamics. It is a basis for quantum gauge theory describing all known interactions except gravity. The idea is simple, as shown by the following toy model; cf.[18].

Toy model

Several cities are connected by roads in the shape of an $M \times N$ grid; see Figure 7. Each city has its own type of goods in an unlimited quantity. E.g., city a has apples and city b has bananas. For two neighboring cities a and b an exchange rate $U(ab) > 0$ is fixed, e.g., 2 banana for an apple. The rate is symmetric, i.e., $U(ba) = U(ab)^{-1}$: one gets back an apple for 2 banana.

A cunning citizen can travel and exchange along a square $abcd$ to multiply his initial amount of goods by a factor of $U(ab)U(bc)U(cd)U(da)$. The total speculation profit is measured by the quantity

$$\mathcal{S}[U] := \sum_{\text{all faces } abcd} \ln^2(U(ab)U(bc)U(cd)U(da)).$$

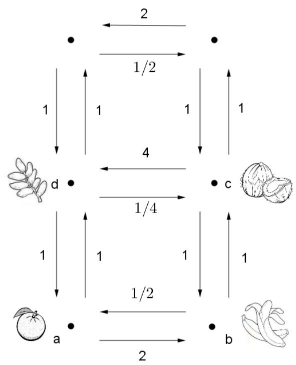


Figure 7: Lattice gauge theory on a 1×2 grid

Here $\ln^2(x)$ is chosen as a function vanishing at $x = 1$ and positive for $x \neq 1$.

The king can set exchange rates except those on the boundary of the grid. He sets them to minimize the quantity $\mathcal{S}[U]$. The resulting collection of rates is an *Abelian gauge group field on shell*.

A gauge group field on shell is far from being unique. For an interior city, one can change the units of measurements, e.g., exchange dozens of apples instead of single ones. Such *gauge transformation* multiplies the rates for all the roads starting from the city by the same value but preserves $\mathcal{S}[U]$.

A similar model on a d-dimensional grid (with an additional minus sign for each summand in $\mathcal{S}[U]$ such that $abcd$ is parallel to $(1, 0, \dots, 0)$) is equivalent to discrete electrodynamics discussed in §2.3. This follows from Corollary 2.2, if one sets $A(ab) = \ln U(ab)$ and $j = 0$; see also Remark 2.13.

Currents

Now modify the model by introducing production of goods. For each pair of neighboring cities a and b fix a production rate $j(ab) \geq 0$: e.g., if a has apples and b has jam, then one produces $j(ab)$ units of jam from one apple. The rate is not at all symmetric: one cannot produce apples from jam. Assume that production always goes in the direction of the coordinate axes.

There is a new way to profit: producing jam and exchanging back to apples, one multiplies the initial amount of apples by $j(ab)U(ba)$. The total profit is now measured by the quantity $S[U, j] = S[U] + \sum_{ab}(j(ab)U(ba) - 1)$. A collection of rates U minimizing $S[U, j]$ for fixed j is called *generated* by j. These rates may not exist, and the total profit can be negative.

These rates satisfy the *conservation law* $-j(\mathbf{1})U(\mathbf{1})^{-1} - j(\mathbf{2})U(\mathbf{2})^{-1} + U(\mathbf{3})^{-1}j(\mathbf{3}) + U(\mathbf{4})^{-1}j(\mathbf{4}) = 0$ for each interior city v, where we use the notation from Figure 3 to the middle (this law is a version of Corollary 2.3). This is a "gauge-invariant" equation, which coincides with the usual charge conservation $\partial j = 0$ in the case when $U = 1$ identically.

Non-Abelian gauge theory

In non-Abelian gauge theory the goods become vectors and the rates become matrices. To catch the idea, one can start with the case when $d = 2$, $n = 1$, $G = \{g \in \mathbb{C} : |g| = 1\}$, and drop all #-operators.

Definition 2.13. Denote by $\mathbb{C}^{m \times n}$ the set of matrices with complex entries having m rows and n columns. For $u \in \mathbb{C}^{m \times n}$ denote by $u^* \in \mathbb{C}^{n \times m}$ the conjugate transpose matrix.

A *gauge group* G is a Lie group represented by unitary transformations of $\mathbb{C}^n$. A *gauge group field* U and a *covariant current* j are functions on edges of I_N^d assuming values in G and $\mathbb{C}^{n \times n}$ respectively.

The *operator of parallel transport* along an oriented path π going along the edges and having no self-intersections is

$$U(\pi) := \prod_e U(e)^{\langle e, \pi \rangle},$$

where the product is over all the edges e of the path π, and $\langle e, \pi \rangle = \pm 1$ is given by (8). In particular, the trace $\operatorname{Tr} U(\partial f)$ is a well-defined complex-valued function on 2-dimensional faces f. A gauge group field U *generated* by a covariant current j is a stationary function for the functional (for fixed j)

$$\mathcal{S}[U] = \sum_{\text{faces } f} \#(\operatorname{Re} \operatorname{Tr} U(\partial f) - n) - \sum_{\text{edges } e} \operatorname{Re} \operatorname{Tr} [j^*(e)U(e)]. \tag{9}$$

Since $\mathcal{S}[U]$ is a continuous function on a compact set, we get the following existence theorem.

Proposition 2.10. *For each covariant current there exists a gauge group field generated by it.*

Now we state the Yang–Mills equation (necessary and sufficient for U to be generated by j) and a conservation law. This is a new Corollary 2.3 extending [10, Eq. (4.15)]. It involves projection to certain tangent space of the Lie group G. In gauge theory the role of the (co)boundary is played by the covariant (co)boundary, which is a "gauge covariant" operator equal the (co)boundary for $U = 1$.

Definition 2.14. Fix a gauge group field U. Let j be a $\mathbb{C}^{n\times n}$-valued function on edges. Its *covariant boundary* $D_A^* j$ is a $\mathbb{C}^{n\times n}$-valued function on vertices v given by

$$[D_A^* j](v) = \sum_{e \text{ ending at } v} U(e)^{-1} j(e) - \sum_{e \text{ starting at } v} j(e) U(e)^{-1}. \tag{10}$$

Denote by $D_A^* \# F$ the $\mathbb{C}^{n\times n}$-valued function on edges e given by

$$[D_A^* \# F](e) = \sum_{\text{2-faces } f \supset e} \#(U(e) - U(\partial f - e)), \tag{11}$$

where $\partial f - e$ is the path starting at the vertex $\min e$, consisting of the 3 edges of $\partial f - e$, and ending at $\max e$. E.g., in Figure 7 we have $[D_A^* \# F](dc) = U(dabc) + U(dfec) - 2U(dc)$. (A general conceptual definition is postponed until the end of §2.4, where (10)–(11) become easy propositions.)

Definition 2.15. The *scalar product* of $u, v \in \mathbb{C}^{n\times n}$ is $\langle u, v\rangle := \operatorname{Re}\operatorname{Tr}[u^* v]$. Let $T_u G \subset \mathbb{C}^{n\times n}$ be the linear subspace parallel to the tangent subspace to G at a point $u \in G$. Let $\operatorname{Pr}_{T_u G}\colon \mathbb{C}^{n\times n} \to T_u G$ be the orthogonal projection and $\operatorname{Pr}_{T_U G} j$ be the function on edges e given by $[\operatorname{Pr}_{T_U G} j](e) = \operatorname{Pr}_{T_{U(e)} G} j(e)$. A covariant current j is *conserved*, if $D_A^* \operatorname{Pr}_{T_U G}\, j = 0$.

Corollary 2.3. *A gauge field U generated by a covariant current j satisfies the following equations:*

- *the Yang–Mills equation: $-\operatorname{Pr}_{T_U G} D_A^* \# F = \operatorname{Pr}_{T_U G}\, j$;*
- *Charge conservation law: $D_A^* \operatorname{Pr}_{T_U G}\, j = 0$.*

Remark 2.13. The latter form of change conservation, different from the usual $\partial j = 0$, reflects the fact that non-Abelian gauge fields are themselves charged. In contrast to continuum theory, this remains true even if G is Abelian (the reason is that the cup-product is non-Abelian; cf. Example 1.2). Also, $D_A^* j \neq 0$ in general: e.g., if j vanishes on all edges except one, then $D_A^* j \neq 0$ whatever U is.

However, for the Abelian group $G = \{e^{i\phi} : \phi \in \mathbb{R}\}$ and $d = 2$ the action can be modified so that charge conservation returns to the form $\partial j = 0$ (here $j \in C_1(I_N^2; \mathbb{R})$ is not a covariant current anymore):

$$\mathcal{S}^{\mathrm{Ab}}[U] = -\tfrac{1}{2} \sum_{\text{faces } f} \arccos^2 \operatorname{Re} \# U(\partial f) + i \sum_{\text{edges } e} j(e) \ln U(e).$$

The range of U must be restricted to $\{e^{i\phi} : -\pi/4 < \phi < \pi/4\}$ to keep the action single-valued and differentiable. The resulting theory is equivalent to discrete electrodynamics of §2.3, also with restricted range, because $\mathcal{S}^{\mathrm{Ab}}[e^{i\phi}] = \epsilon\left[-\tfrac{1}{2}\#\delta\phi \frown \delta\phi - j \frown \phi\right]$ for $\phi \in C^1(I_N^2; (-\tfrac{\pi}{4}; \tfrac{\pi}{4}))$.

Connection and curvature

Definition 2.16. Let g and ϕ be G- and $\mathbb{C}^{n\times n}$-valued functions on vertices and k-faces respectively. The *gauge transformation* of ϕ by g is the function $g^* \smile \phi \smile g$ on k-faces f given by (cf. Table 2)

$$[g^* \smile \phi \smile g](f) = g^*(\min f)\, \phi(f)\, g(\max f).$$

Corollary 2.4 (Gauge invariance)**.** *Each simultaneous gauge transformation of U and j by the same element g preserves $\mathcal{S}[U]$. If U is generated by j, then $g^* \smile U \smile g$ is generated by $g^* \smile j \smile g$.*

Table 2: Products of (co)chains of dimension 0 and 1 (where ab denotes an edge with $a < b$).

$\dim\phi = 1, \dim\psi = 0$	$\dim\phi = 0, \dim\psi = 1$	$\dim\phi = \dim\psi = 1$
$[\phi \smile \psi](ab) = \phi(ab)\psi(b)$	$[\phi \smile \psi](ab) = \phi(a)\psi(ab)$	$\phi \smile \psi$ is defined in Figure 3
$[\phi \frown \psi](ab) = \phi(ab)\psi(a)$	$\phi \frown \psi$ is undefined	$[\phi \frown \psi](b) = \sum_{\text{edges } ab: a<b} \phi(ab)\psi(ab)$
$\phi \overset{*}{\frown} \psi$ is undefined	$[\phi \overset{*}{\frown} \psi](ab) = \phi(b)\psi(ab)$	$[\phi \overset{*}{\frown} \psi](b) = \sum_{\text{edges } bc: c>b} \phi(bc)\psi(bc)$

Definition 2.17. The *unit gauge group field* 1 equals the unit $n \times n$ matrix at each edge. For a gauge group field U, the *connection* is the $\mathbb{C}^{n\times n}$-valued function $A[U] = U - 1$. The *curvature* is the $\mathbb{C}^{n\times n}$-valued function on the set of faces given by

$$F[U](abcd) := U(ab)U(bc) - U(ad)U(dc)$$

for each face $abcd$ with the vertices listed counterclockwise starting from the minimal one; see Figure 3.

Remark 2.14. On a grid, a gauge group field U is a gauge transformation of the unit gauge group field, if and only if the curvature $F[U]$ vanishes (this is proved by a standard "homological" argument.)

In contrast to continuum theory, the connection and curvature assume values *not* in the Lie algebra of the Lie group G but in certain other subsets of $\mathbb{C}^{n\times n}$ approximating the Lie algebra in a sense. The fields A and F from §2.2–2.3 are neither connection nor curvature for *no* gauge group field.

Analogously to Proposition 2.9, the tensor $-\mathrm{Re}\,\mathrm{Tr}\,[\#F^* \times F]$ approximates the continuum Belinfante–Rosenfeld energy-momentum tensor. But the former is not conserved and even not gauge invariant.

For a simplicial complex M with fixed vertices ordering, the *curvature* is defined by the formula

$$F[U](abc) = U(ab)U(bc) - U(ac)$$

for each face abc with the vertices listed in increasing order $a < b < c$.

Proposition 2.11. *There is the following expression for the action* (9)*:*

$$\mathcal{S}[U] = \epsilon\,\mathrm{Re}\,\mathrm{Tr}\left[-\tfrac{1}{2}\#F^* \frown F - j^* \frown U\right].$$

Such expression for $\mathcal{S}[U]$ is the one given by the algorithm from §1.3 up to an additive constant.

Covariant differentiation

The covariant (co)boundary is defined in terms of cochain products as follows; cf [10, §IV–V]. Particular cases of the definition shown in Table 2 are sufficient for all our examples.

Definition 2.18. Denote by $C^k(I_N^d; V)$ the set of functions defined on the set of k-dimensional faces and assuming values in a set V. Here V, and hence $C^k(I_N^d; V)$, is a set, not necessarily a group.

Denote by $a \dots b$ the face f such that $\min f = a$, $\max f = b$ (if such face f exists, then it is unique). An ordered triple of faces $a \dots b, b \dots c \subset a \dots c$ of dimensions k, l, $k + l$ respectively is *cooriented* (repectively, *opposite oriented*), if the ordered set consisting of a positive basis in $a \dots b$ and a positive basis in $b \dots c$ is a positive (respectively, negative) basis in $a \dots c$. Write

$$\langle a, b, c\rangle = \begin{cases} +1, & \text{if } a \dots b, b \dots c, a \dots c \text{ are cooriented,} \\ -1, & \text{if } a \dots b, b \dots c, a \dots c \text{ are oppositely oriented.} \end{cases}$$

The *cup-*, *cap-*, and *cop-product* of functions $\Phi \in C^k(I_N^d; \mathbb{C}^{p\times q})$ and $\Psi \in C^l(I_N^d; \mathbb{C}^{q\times r})$ are the $\mathbb{C}^{p\times r}$-valued functions on $(k + l)$-, $(k - l)$-, and $(l - k)$-dimensional faces respectively given by

$$[\Phi \smile \Psi](a \dots c) = \sum_{b:\,\dim(a\dots b)=k, \dim(b\dots c)=l} \langle a, b, c\rangle \Phi(a \dots b)\Psi(b \dots c);$$

$$[\Phi \frown \Psi](b \dots c) = \sum_{a:\,\dim(a\dots c)=k, \dim(a\dots b)=l} \langle a, b, c\rangle \Phi(a \dots c)\Psi(a \dots b);$$

$$[\Phi \overset{*}{\frown} \Psi](a \dots b) = \sum_{c:\,\dim(b\dots c)=k, \dim(a\dots c)=l} \langle a, b, c\rangle \Phi(b \dots c)\Psi(a \dots c),$$

where the sums are over all the vertices such that there exist 3 faces $a \dots b, b \dots c \subset a \dots c$.

For $\Phi \in C^k(I^d_N; \mathbb{C}^{n\times n})$, the *covariant coboundary* and the *covariant boundary* are respectively

$$D_A\Phi := \delta\Phi + A \smile \Phi - (-1)^k \Phi \smile A; \tag{12}$$

$$D^*_A\Phi := \partial\Phi + (\Phi^* \frown A)^* + (-1)^k (A \overset{*}{\frown} \Phi^*)^*. \tag{13}$$

Remark 2.15. For a simplicial complex M the definition requires the following modifications (because a face is not determined by just the minimal and the maximal vertices anymore). Denote by $a_1 a_2 \dots a_{s+1}$ the s-dimensional face with the vertices $a_1 < a_2 < \dots < a_{s+1}$. The value $\langle a, b, c\rangle$ and the "triality" of products is defined by the same formulae, only $a \dots b$, $b \dots c$, $a \dots c$ are replaced by $a_1 \dots a_s b$, $bc_1 \dots c_t$, $a_1 \dots a_s bc_1 \dots c_t$ respectively, summation over b is omitted, and summation over a and c is replaced by summation over all collections $(a_1, \dots, a_s)$ and $(c_1, \dots, c_t)$ respectively.

The definition of the cup-product is equivalent to [25, (22.3)] but not [26, Chapter IX, §14, Eq. (7)].

Up to sign and factors interchange, the cop-product is the cap-product in the same complex but with reversed vertices ordering. The cap- and cop- products vanish for $k < l$ and $k > l$ respectively, and do *not* coincide for $k = l \neq 0$. Usually both are denoted in the same way, which does not lead to a conflict until one identifies chains and cochains (hence the domains of the products). Since we have performed such identification, we need to introduce new notation $\overset{*}{\frown}$ and new term "cop-product".

Proposition 2.12. *For each gauge group field U we have $F = \delta A + A \smile A$, $D_A F = 0$, and (10)–(11).*

Remark 2.16. The results of §2.4 remain true for arbitrary spacetime, if one omits all #-operators. The proofs are analogous, only for a simplicial complex each instance of the fourth vertex "d" of a face $abcd$ is just removed, and a direct checking is used instead of Lemma 4.5.

2.5 The Klein–Gordon field

The classical (not quantum!) Klein–Gordon field does not describe any real physical field but serves as an example for more realistic models. Corollaries 2.6, 2.9, 2.10 and Proposition 2.13 are new.

Basic model

Definition 2.19. Fix a number $m \geq 0$ called *particle mass*. A complex-valued function ϕ on the set of vertices of I^d_N is a *Klein–Gordon field* of mass m, if the following equation holds apart ∂I^d_N:

- *the Klein–Gordon equation:* $\partial \# \delta\phi + m^2\phi = 0$.

Corollary 2.5. *A function $\phi \in C^0(I^d_N; \mathbb{C})$ is a Klein–Gordon field, if and only if among all the functions with the same values at ∂I^d_N, the function ϕ is stationary for the functional $\mathcal{S}[\phi] = \epsilon\mathcal{L}[\phi]$, where*

$$\mathcal{L}[\phi] = -\#\delta\phi \frown \delta\phi^* - m^2\phi \frown \phi^*.$$

Here we impose a boundary condition, because the theory becomes trivial otherwise. The Lagrangian $\mathcal{L}[\phi]$ is *globally gauge invariant*, i.e., $\mathcal{L}[\phi g] = \mathcal{L}[\phi]$ for each $g \in \mathbb{C}$ with $|g| = 1$.

Corollary 2.6 (Charge, energy, momentum conservation)**.** *For a Klein–Gordon field ϕ the current $j[\phi] := 2\mathrm{Im}(\#\delta\phi^* \frown \phi)$ and the tensor $T[\phi] := -2\mathrm{Re}[\#\delta\phi^* \times \delta\phi + m^2\phi^* \times \phi]$ are conserved apart ∂I^d_N.*

Approximation

The resulting current $j[\phi]$ and energy-momentum tensor $T[\phi]$ indeed approximate continuum ones.

In continuum theory, $\mathrm{\phi}$ is a smooth complex-valued function defined on the unit hypercube I^d. (Hereafter *smooth* means C^1, and the derivative at a boundary point of I^d means a one-sided derivative.) The *current* and *energy-momentum tensor* of $\mathrm{\phi}$ are the vector and matrix fields

$$\mathrm{j}^l = 2\mathrm{Im}\left[\mathrm{\phi}\,\partial^l \mathrm{\phi}^*\right] \quad \text{and} \quad \mathrm{T}^l_k = -2\mathrm{Re}\left[\partial^l\mathrm{\phi}^*\partial_k\mathrm{\phi}\right] + \delta^l_k\left[\partial^n\mathrm{\phi}^*\partial_n\mathrm{\phi} + m^2\mathrm{\phi}^*\mathrm{\phi}\right],$$

where summation over n is understood, and we denote $\partial_n\mathrm{\phi} := \frac{\partial\mathrm{\phi}}{\partial \mathrm{x}^n}$, $\partial^n\mathrm{\phi} := \begin{cases} -\partial_n\mathrm{\phi}, & \text{if } n = 0; \\ +\partial_n\mathrm{\phi}, & \text{if } n \neq 0. \end{cases}$

Proposition 2.13 (Approximation property)**.** *Let* $\upphi$ *be a smooth complex-valued field on* I^d, $d \geq 2$. *Dissect* I^d *into* N^d *equal hypercubes and take the discrete field* $\phi_N(v) := \upphi(v)$ *on the vertices* v *of the resulting grid. Let* j^l, T^l_k *be the continuous current and energy-momentum tensor. Define* $j_N = j[\phi_N]$, $T_N = T[\phi_N]$ *by the same formulae as in Corollary 2.6 except that* m *is replaced by* m/N. *Take* $0 \leq k, l < d$. *Then on the set of all edges* $e \parallel \mathrm{e}_l$ *and all hyperfaces* $h \perp \mathrm{e}_l$ *disjoint with* $\partial \mathrm{I}^d$, *we have*

$$N j_N(e) \cong \mathrm{j}^l(\max e) \quad \text{and} \quad (-1)^l N^2 \langle T_N, h\rangle_k \cong \mathrm{T}^l_k(\max h) \qquad \text{as } N \to \infty.$$

Remark 2.17. The fields $\upphi$ and ϕ_N are *not* necessarily Klein–Gordon fields (and typically ϕ_N cannot be such one, even $\upphi$ is). In particular, $j[\phi_N]$ and $T[\phi_N]$ are *not* necessarily conserved.

Coupling to a gauge field

Interaction with a gauge field is introduced by replacement of (co)boundary by covariant (co)boundary. Let $U \in C^1(I^d_N; G)$, $A = U - 1$, F be a gauge group field, the connection, and the curvature respectively.

Definition 2.20. The *gauge transformation* $C^k(I^d_N; \mathbb{C}^{1\times n}) \to C^k(I^d_N; \mathbb{C}^{1\times n})$ by $g \in C^0(I^d_N; G)$ is the map

$$\phi \mapsto \phi \smile g.$$

For $\phi \in C^k(I^d_N; \mathbb{C}^{1\times n})$ the *covariant coboundary* and the *covariant boundary* are respectively

$$D_A\phi := \delta\phi - (-1)^k \phi \smile A; \tag{14}$$
$$D^*_A\phi := \partial\phi + (-1)^k (A \overset{*}{\frown} \phi^*)^*. \tag{15}$$

A field $\phi \in C^0(I^d_N; \mathbb{C}^{1\times n})$ is a *Klein–Gordon field interacting with the gauge field*, if apart ∂I^d_N we have

- *the Klein–Gordon equation in a gauge field*: $D^*_A \# D_A\phi + m^2\phi = 0$.

Remark 2.18. Definitions of a gauge transformation and covariant (co)boundary crucially depend on the set of field values (more precisely, on the representation of G): compare (12)–(13) and (14)–(15). For $n = 1$ there is a minor conflict of notation between these pairs of equations, cleared up by context.

Informally, (14)–(15) mean the following. Think of the field value at a face e as sitting at the maximal vertex $\max e$. Then the (co)boundary value at a face v is defined just as the ordinary (co)boundary, but all the involved field values are parallelly transported to the maximal vertex $\max v$.

Corollary 2.7. *A function* $\phi \in C^0(I^d_N; \mathbb{C}^{1\times n})$ *is a Klein–Gordon field interacting with a gauge group field* $U \in C^1(I^d_N; G)$, *if and only if among all the functions with the same values at* ∂I^d_N, *the function* ϕ *is stationary for the functional* $\mathcal{S}[\phi, U] = \epsilon\mathcal{L}[\phi, U]$ *for fixed* U, *where*

$$\mathcal{L}[\phi, U] = -\# D_A\phi \frown (D_A\phi)^* - m^2\phi \frown \phi^* - \tfrac{1}{2}\mathrm{Re}\,\mathrm{Tr}[\# F^* \frown F].$$

Remark 2.19. Using row-vectors ϕ rather than column-vectors is essential to make $\mathcal{L}[\phi, U]$ a local Lagrangian with respect to the gauge group field U as well. The third summand in $\mathcal{L}[\phi, U]$ can be dropped for fixed U but becomes essential for dynamic U in Corollary 2.10.

Corollary 2.8 (Gauge invariance)**.** *The Lagrangian* $\mathcal{L}[\phi, U]$ *from Corollary 2.7 is* gauge invariant, *i.e., for each* $\phi \in C^0(I^d_N; \mathbb{C}^{1\times n})$, $U \in C^1(I^d_N; G)$, $g \in C^0(I^d_N; G)$ *we have* $\mathcal{L}[\phi \smile g, g^* \smile U \smile g] = \mathcal{L}[\phi, U]$.

Corollary 2.9 (Charge conservation)**.** *For a Klein–Gordon field* ϕ *interacting with a gauge group field* U *the covariant current* $j[\phi, U] = 2\phi^* \smile \# D_A\phi \in C^1(I^d_N; \mathbb{C}^{n\times n})$ *is conserved apart* ∂I^d_N, *i.e.,* $D^*_A \mathrm{Pr}_{T_U G}\, j[\phi, U] = 0$ *apart* ∂I^d_N. *(Beware that the product of a column- and a row-vector is a matrix.)*

Corollary 2.10. *A gauge group field* U *is stationary for the functional* $\mathcal{S}[\phi, U]$ *from Corollary 2.7 for fixed* $\phi \in C^0(I^d_N; \mathbb{C}^{1\times n})$, *if and only if* U *is generated by the covariant current from Corollary 2.9.*

2.6 The Dirac field

A classical (not quantum) Dirac field describes the wave function of an electron in quantum-mechanics (not quantum field theory). Our discretization is equivalent to [9, (5.19)] but not to [9, (5.55)]. Corollaries 2.12, 2.15, 2.16, and Proposition 2.15 are new.

Basic model

Definition 2.21. Introduce the *Dirac γ-matrices* (*generators of the Clifford algebra of* $\mathbb{R}^{3,1}$):

$$\gamma^0 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&-1&0&0\\ -1&0&0&0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0&0&0&-i\\ 0&0&i&0\\ 0&i&0&0\\ -i&0&0&0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ -1&0&0&0\\ 0&1&0&0 \end{pmatrix}.$$

The *Dirac chain* $\gamma \in C_1(I_N^4; \mathbb{C}^{4\times 4})$ is given by $\gamma(e) = \gamma^k$ for each edge $e \parallel \mathrm{e}_k$, where $k = 0, 1, 2, 3$. A function $\psi \in C^0(I_N^4; \mathbb{C}^{4\times 1})$ is a *Dirac field* of mass m, if the following equation holds apart ∂I^4:

- *the Dirac equation*: $i\gamma \frown \delta\psi + i\gamma \overset{*}{\frown} \delta\psi - 2m\psi = 0$.

Such form of the equation, with the Dirac chain appearing twice, is forced by the following variational principle and reflects the general lattice *fermion doubling* phenomenon. Denote $\bar\psi := \psi^*\gamma^0$.

Corollary 2.11. *A function $\psi \in C^0(I_N^4; \mathbb{C}^{4\times 1})$ is a Dirac field, if and only if among all the functions with the same values at ∂I_N^4, the function ψ is stationary for the functional $\mathcal{S}[\psi] = \epsilon\mathcal{L}[\psi]$, where*

$$\mathcal{L}[\psi] = \mathrm{Re}\left[\bar\psi \frown (i\gamma \frown \delta\psi - m\psi)\right].$$

Using column-vectors ψ rather than row-vectors is essential to make the expression meaningful.

The *doubling* of the d-dimensional grid I_N^d is defined analogously to Definition 2.7.

Proposition 2.14. *Consider a Dirac field on the doubling of I_N^4. Then the restriction of the field to the initial grid I_N^4 besides the boundary satisfies the Klein–Gordon equation with twice larger mass.*

The Lagrangian $\mathcal{L}[\psi]$ is globally gauge invariant: $\mathcal{L}[\psi g] = \mathcal{L}[\psi]$ for each $g \in \mathbb{C}$ with $|g| = 1$. In the case $m = 0$ there is also a symmetry $\mathcal{L}[e^{i\gamma^5 t}\psi] = \mathcal{L}[\psi]$ for each $t \in \mathbb{R}$, where $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$.

Corollary 2.12 (Current, chiral current, energy, momentum conservation)**.** *For a Dirac field ψ the following current and tensor are conserved apart ∂I_N^4:*

$$j[\psi] = \mathrm{Re}\left[\bar\psi \smile \gamma \smile \psi\right] \quad \text{and} \quad T[\psi] = \mathrm{Re}[(\bar\psi \overset{*}{\frown} i\gamma) \times \delta\psi - (\delta\bar\psi \frown i\gamma + 2m\bar\psi) \times \psi].$$

In the case when $m = 0$ the current $j^5[\psi] = \mathrm{Re}\left[\bar\psi \smile \gamma^5\gamma \smile \psi\right]$ is also conserved apart ∂I_N^4.

Remark 2.20. Unlike continuum theory, $j[\psi](e)$ is not necessarily positive on edges $e \parallel (1, 0, 0, 0)$ (because ψ and $\bar\psi$ are evaluated at distinct endpoints of e) and thus cannot be interpreted as probability.

The tensor $T[\psi]$ is not partially symmetric. Thus we know no integral form of its conservation.

Approximation

The resulting current and energy-momentum tensor indeed approximate the continuum ones.

In continuum theory, $\uppsi\colon \mathrm{I}^4 \to \mathbb{C}^4$ is a smooth function. The *current* and the (*canonical*) *energy-momentum tensor* of $\uppsi$ are the vector and matrix fields

$$\mathrm{j}^l = \mathrm{Re}\left[\bar\uppsi\,\gamma^l\uppsi\right] \quad \text{and} \quad \mathrm{T}_k{}^l = \mathrm{Re}\left[i\bar\uppsi\gamma^l\partial_k\uppsi - \delta_k^l\,(i\bar\uppsi\gamma^n\partial_n\uppsi - m\bar\uppsi\uppsi)\right],$$

where summation over n is understood. In what follows analogues of Remarks 2.11 and 2.17 apply.

Proposition 2.15 (Approximation property)**.** *Let $\uppsi\colon \mathrm{I}^4 \to \mathbb{C}^4$ be a smooth function. Dissect I^4 into N^4 equal hypercubes and define the discrete field $\psi_N(v) := \uppsi(v)$ on the vertices v of the resulting grid. Let j^l, $\mathrm{T}_k{}^l$ be the continuous current and energy-momentum tensor. Define $j_N = j[\phi_N]$, $T_N = T[\phi_N]$ by the same formulae as in Corollary 2.12 except that m is replaced by m/N. Take $0 \le k, l < 4$. Then on the set of all edges $e \parallel \mathrm{e}_l$ and hyperfaces $h \perp \mathrm{e}_l$ not intersecting $\partial\mathrm{I}^4$, we have*

$$j_N(e) \rightrightarrows \mathrm{j}^l(\max e) \quad \text{and} \quad (-1)^l N\langle T_N, h\rangle_k \rightrightarrows \mathrm{T}_k{}^l(\max h) \qquad \text{as } N \to \infty.$$

Coupling to a gauge field

Definition 2.22. Let $U \in C^1(I_N^4; G)$ be a gauge group field. Assume that $n \neq 4$ to avoid confusion. The *covariant coboundary* $D_A\psi$ of $\psi \in C^k(I_N^4; \mathbb{C}^{4\times n})$ is defined literally as for $\psi \in C^k(I_N^4; \mathbb{C}^{1\times n})$. Set

$$\bar{D}_A\psi = (\delta\psi^* + A \smile \psi^*)^*. \qquad (16)$$

A function $\psi \in C^0(I_N^4; \mathbb{C}^{4\times n})$ is a *Dirac field interacting with the gauge field*, if apart ∂I_N^4 we have

- *the Dirac equation in a gauge field*: $i\gamma \frown D_A\psi + i\gamma \overset{*}{\frown} \bar{D}_A\psi - 2m\psi = 0.$

Corollary 2.13. *A function $\psi \in C^0(I_N^4; \mathbb{C}^{4\times n})$ is a Dirac field interacting with a gauge group field $U \in C^1(I_N^4; G)$, if and only if among all functions with the same values at ∂I_N^4, the function ψ is stationary for the functional $\mathcal{S}[\psi, U] = \epsilon\mathcal{L}[\psi, U]$ for fixed U, where*

$$\mathcal{L}[\psi, U] = \mathrm{Re}\,\mathrm{Tr}\left[\bar{\psi} \frown (i\gamma \frown D_A\psi - m\psi) - \tfrac{1}{2}\#F^* \frown F\right].$$

Corollary 2.14 (Gauge invariance)**.** *The Lagrangian $\mathcal{L}[\psi, U]$ from Corollary 2.13 is gauge invariant.*

Corollary 2.15 (Charge conservation)**.** *For a Dirac field ψ interacting with a gauge field U, the covariant current $j[\psi] = -\bar{\psi} \smile i\gamma \smile \psi \in C^1(I_N^4; \mathbb{C}^{n\times n})$ is conserved, i.e., $D_A^*\mathrm{Pr}_{T_U G}\, j[\psi]{=}0$ apart ∂I_N^4.*

Corollary 2.16. *A gauge group field U is stationary for the functional $\mathcal{S}[\psi, U]$ from Corollary 2.13 for fixed $\psi \in C^0(I_N^4; \mathbb{C}^{4\times n})$, if and only if U is generated by the covariant current from Corollary 2.15.*

3 Generalizations

In this section we state the main results of the paper in their full generality, i.e., for nontrivial connections and arbitrary spacetimes. The notions and the results from §1.4 are obtained in the particular case when the gauge group is trivial, i.e., $G = \{1\}$, and the fields assume values in $\mathbb{R}$. Most of the results of §2 are obtained from these general results by substituting specific Lagrangians.

If one replaces the d-dimensional grid I_N^d by an arbitrary spacetime M, then all notions from the middle column of Table 1 except $\#$, γ, $\langle T, h\rangle_k$ are defined literally as in §2; see the right column for definition numbers and Remarks 2.6,2.14,2.15. We do not use and do not define $\#,\gamma,\langle T, h\rangle_k$ for $M{\neq}I_N^d$.

Remark 3.1. Most of the results from §2 remain true for arbitrary M for suitable definition of $\#$.

We do *not* assume that M is a manifold. In fact, faces of M of dimension > 2 have never appeared at all in the examples from §2 (except the identity $D_A F = 0$ which is anyway automatic). The whole ambient spacetime is not that important: think of an electric network lying on a table; is spacetime of the model 1-, 2-, 3- or 4-dimensional? This is why we avoid dual grids and the Hodge star. However dimension-like properties of M like the average vertex degree *are* of course important.

We do *not* distinguish between chains and cochains. This would give no advantage but only complicates theory (perhaps, it *will* become useful for further generalizations). However, to make notation compatible with the commonly used one, we sometimes switch between different notation $C^k(M; V)$ and $C_k(M; V)$ for the same object. Notice that identification of chains and cochains has nothing to do with spacetime metric.

We *do* distinguish between row- and column-vectors. This makes clear, if the product of two vectors is a number or a matrix. Some of our results depend on the type of vectors used as field values.

Let us introduce some notation. For a vertex $v \in M$ denote by $\mathrm{e}_{v,k}$ the set of all k-dimensional faces for which the maximal vertex is v. Order the set $\mathrm{e}_{v,k}$ lexicographically. Denote by $\mathrm{e}_{v,k,j}$ its j-th element. Denote by $p = p(v, k)$ the number of faces in $\mathrm{e}_{v,k}$. Set $q = p(v, k + 1)$. For $\phi \in C^k(M; \mathbb{C}^{1\times n})$ denote $\phi(\mathrm{e}_{v,k}) := (\phi(\mathrm{e}_{v,k,1}), \dots, \phi(\mathrm{e}_{v,k,p}))$. For $\mathrm{f}\colon \mathbb{C}^{m\times n} \to \mathbb{R}$ define $\frac{\partial \mathrm{f}}{\partial z}\colon \mathbb{C}^{n\times m} \to \mathbb{R}$ by $\left(\frac{\partial \mathrm{f}}{\partial z}\right)_{lk} = \frac{\partial \mathrm{f}}{\partial(\mathrm{Re}\, z_{kl})} - i\frac{\partial \mathrm{f}}{\partial(\mathrm{Im}\, z_{kl})}$, where $z = (z_{kl}) \in \mathbb{C}^{m\times n}$. For $M = I_N^d$ denote $g^{ll}(v, k) = \begin{cases} -1, & \text{if } \mathrm{e}_{v,k+1,l} \parallel (1, 0, \dots, 0), \\ +1, & \text{if } \mathrm{e}_{v,k+1,l} \perp (1, 0, \dots, 0). \end{cases}$

Table 3: Partial derivatives of basic Lagrangians $\mathcal{L}[\phi, U]$, $\mathcal{L}[\psi, U]$, or $\mathcal{L}[U]$

	Lagrangian	assumptions	$L_v(\phi_1, \ldots, \phi_p, \phi'_1, \ldots, \phi'_q)$	$\left(\frac{\partial \mathcal{L}}{\partial \phi}\right)^*$ or $\left(\frac{\partial \mathcal{L}}{\partial U}\right)^*$	$\left(\frac{\partial \mathcal{L}}{\partial (D_A\phi)}\right)^*$ or $\left(\frac{\partial \mathcal{L}}{\partial F}\right)^*$
1	$\mathrm{Re}[j \frown \phi^*]$	$j \in C_k(M; \mathbb{C}^{1\times n})$	$\mathrm{Re} \sum_{l=1}^{p} j(\mathrm{e}_{v,k,l})\phi_l^*$	j	0
2	$\phi \frown \phi^*$	-	$\sum_{l=1}^{p} \phi_l \phi_l^*$	2ϕ	0
3	$\#D_A\phi \frown (D_A\phi)^*$	$M = I_N^d$	$\sum_{l=1}^{q} g^{ll}(v,k)\phi'_l(\phi'_l)^*$	0	$2\#D_A\phi$
4	$\mathrm{Re}\,\mathrm{Tr}[\overline{\psi} \frown \psi]$	$M = I_N^4$	$\mathrm{Re}\,\mathrm{Tr}[\psi_1^* \gamma^0 \psi_1]$	$2\gamma^0\psi$	0
5	$\mathrm{Re}\,\mathrm{Tr}[\overline{\psi} \frown (i\gamma \frown D_A\psi)]$	$M = I_N^4$	$\mathrm{Re}\,\mathrm{Tr} \sum_{l=1}^{4} i\psi_1^* \gamma^0 \gamma^{l-1} \psi'_l$	$i\gamma^0\gamma \frown D_A\psi$	$-i\gamma^0\gamma \smile \psi$
6	$\mathrm{Re}\,\mathrm{Tr}[j^* \frown U]$	$j \in C_1(M; \mathbb{C}^{n\times n})$	$\mathrm{Re}\,\mathrm{Tr} \sum_{l=1}^{p} j^*(\mathrm{e}_{v,1,l}) U_l$	j	0
7	$\mathrm{Re}\,\mathrm{Tr}[\#F^* \frown F]$	$M = I_N^d$	$\mathrm{Re}\,\mathrm{Tr} \sum_{l=1}^{q} g^{ll}(v,1)(U')_l^* U'_l$	0	$2\#F$

Definition 3.1. (Cf. Definition 2.1.) A *local Lagrangian* is a differentiable function

$$\mathcal{L}\colon C^k(M; \mathbb{C}^{1\times n}) \times C^1(M; \mathbb{C}^{n\times n}) \to C_0(M; \mathbb{R})$$

such that

$$\mathcal{L}[\phi, U](v) = L_v(\phi(\mathrm{e}_{v,k}), [D_{A[U]}\phi](\mathrm{e}_{v,k+1})) \tag{17}$$

for some differentiable function $L_v(\phi_1, \ldots, \phi_p, \phi'_1, \ldots, \phi'_q)$ not depending on U. Define

$$\frac{\partial \mathcal{L}}{\partial \phi} \in C_k(M; \mathbb{C}^{n\times 1}) \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial (D_A\phi)} \in C_{k+1}(M; \mathbb{C}^{n\times 1})$$

by the formulae

$$\frac{\partial \mathcal{L}}{\partial \phi}(\mathrm{e}_{v,k,j}) := \frac{\partial L_v}{\partial \phi_j}(\phi(\mathrm{e}_{v,k}), [D_A\phi](\mathrm{e}_{v,k+1})), \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial (D_A\phi)}(\mathrm{e}_{v,k+1,j}) := \frac{\partial L_v}{\partial \phi'_j}(\phi(\mathrm{e}_{v,k}), [D_A\phi](\mathrm{e}_{v,k+1})). \tag{19}$$

A field $\phi \in C^k(M; \mathbb{C}^{1\times n})$ is *on shell*, if it is stationary for the functional $\mathcal{S}[\phi, U] = \epsilon\mathcal{L}[\phi, U]$ for given fixed $U \in C^1(M; G)$. For $\phi \in C^k(M; \mathbb{R})$ or $\psi \in C^0(I_N^4; \mathbb{C}^{4\times n})$ the definition is analogous; in the former case $\mathcal{L}[\phi, 1]$ is called a *local Lagrangian* $C^k(M; \mathbb{R}) \to C_0(M; \mathbb{R})$.

Proposition 3.1. *For fixed j, each of the Lagrangians in Table 3 to the left is local and the partial derivatives are given by the two columns to the right, under the assumptions in the third column.*

Theorem 3.1 (the Euler–Lagrange equation)**.** *Let $\mathcal{L}\colon C^k(M; \mathbb{C}^{1\times n}) \times C^1(M; \mathbb{C}^{n\times n}) \to C_0(M; \mathbb{R})$ be a local Lagrangian, $A \in C^1(M; \mathbb{C}^{n\times n})$ be a connection. Then $\phi \in C^k(M; \mathbb{C}^{1\times n})$ is on shell, if and only if*

$$D_A^* \left(\frac{\partial \mathcal{L}[\phi]}{\partial (D_A\phi)}\right)^* + \left(\frac{\partial \mathcal{L}[\phi]}{\partial \phi}\right)^* = 0. \tag{20}$$

A *local* Lagrangian $\mathcal{L}\colon C^1(M; \mathbb{C}^{n\times n}) \to C_0(M; \mathbb{R})$ and the partial derivatives $\frac{\partial \mathcal{L}}{\partial U} \in C_1(M; \mathbb{C}^{n\times n})$, $\frac{\partial \mathcal{L}}{\partial (F[U])} \in C_2(M; \mathbb{C}^{n\times n})$ are defined analogously to Definition 3.1, only the fields ϕ and $D_A\phi$ are replaced by a gauge group field U and the curvature $F[U]$ respectively ($F[U] \neq D_A U$). A gauge group field U is *on shell*, if it is stationary for the functional $\mathcal{S}[U] = \epsilon\mathcal{L}[U]$ under the constraint $U \in C^1(M; G)$.

Theorem 3.2 (the Euler–Lagrange equation)**.** *Let $\mathcal{L}\colon C^1(M; \mathbb{C}^{n\times n}) \to C_0(M; \mathbb{R})$ be a local Lagrangian. Then a gauge group field $U \in C^1(M; G)$ is on shell, if and only if*

$$\mathrm{Pr}_{T_U G}\left[D_A^* \left(\frac{\partial \mathcal{L}[U]}{\partial (F[U])}\right)^* + \left(\frac{\partial \mathcal{L}[U]}{\partial U}\right)^*\right] = 0. \tag{21}$$

Theorem 3.3 (Noether's theorem)**.** *If a local Lagrangian $\mathcal{L}[\phi, U]$ satisfies (5) for some $\Delta \in C^k(M; \mathbb{C}^{1\times n})$ and $U \in C^1(M; G)$, then for each field ϕ on shell the edgewise scalar product of the covariant current $j[\phi, U] = \left(\frac{\partial \mathcal{L}[\phi, U]}{\partial (D_A\phi)} \frown \Delta\right)^*$ with the gauge group field U is conserved, i.e. $\partial \langle j[\phi, U], U\rangle = 0$.*

For gauge invariant Lagrangians the numerous Noether currents are combined together as follows.

Theorem 3.4 (Charge conservation)**.** *If a local Lagrangian $\mathcal{L}[\phi, U]$ is gauge invariant, then for each field ϕ on shell and each gauge group field U the following covariant current is conserved:*

$$j[\phi, U] = \left(\frac{\partial \mathcal{L}[\phi, U]}{\partial (D_A \phi)} \frown \phi\right)^* = \left(\frac{\partial \mathcal{L}[\phi, U]}{\partial U}\right)^* .$$

Theorem 3.5 (Charge conservation)**.** *Let $\mathcal{L}[U] = \mathcal{L}'[U] - \mathrm{Re}\,\mathrm{Tr}\,[j^* \frown U]$ be a local Lagrangian, where $j \in C_1(M; \mathbb{C}^{n\times n})$ is fixed, $\mathcal{L}'[U]$ is gauge invariant and does not depend on j. Then for each gauge group field U on shell the covariant current j is conserved, i.e., $D_A^* \mathrm{Pr}_{T_U G} j = 0$.*

The last three theorems are not completely obvious even if spacetime is a 1×1 grid. The gauge invariance (defined in Corollary 2.8 and crucial here) is usually guaranteed by the following result.

Proposition 3.2 (Gauge covariance, see [10])**.** *For each $U \in C^1(M; G)$, $\Phi \in C^k(M; \mathbb{C}^{n\times n})$, $\phi \in C^k(M; \mathbb{C}^{1\times n})$, $g \in C^0(M; G)$ we have:*

$$\begin{aligned}
A[g^* \smile U \smile g] &= g^* \smile A[U] \smile g + g^* \smile \delta g \qquad && (= g^* \smile A[U] \smile g - \delta g^* \smile g);\\
F[g^* \smile U \smile g] &= g^* \smile F[U] \smile g;\\
D_{A[g^* \smile U \smile g]}(g^* \smile \Phi \smile g) &= g^* \smile (D_{A[U]}\Phi) \smile g; && D_{A[g^* \smile U \smile g]}(\phi \smile g) = (D_{A[U]}\phi) \smile g;\\
D^*_{A[g^* \smile U \smile g]}(g^* \smile \Phi \smile g) &= g^* \smile (D^*_{A[U]}\Phi) \smile g; && D^*_{A[g^* \smile U \smile g]}(\phi \smile g) = (D^*_{A[U]}\phi) \smile g.
\end{aligned}$$

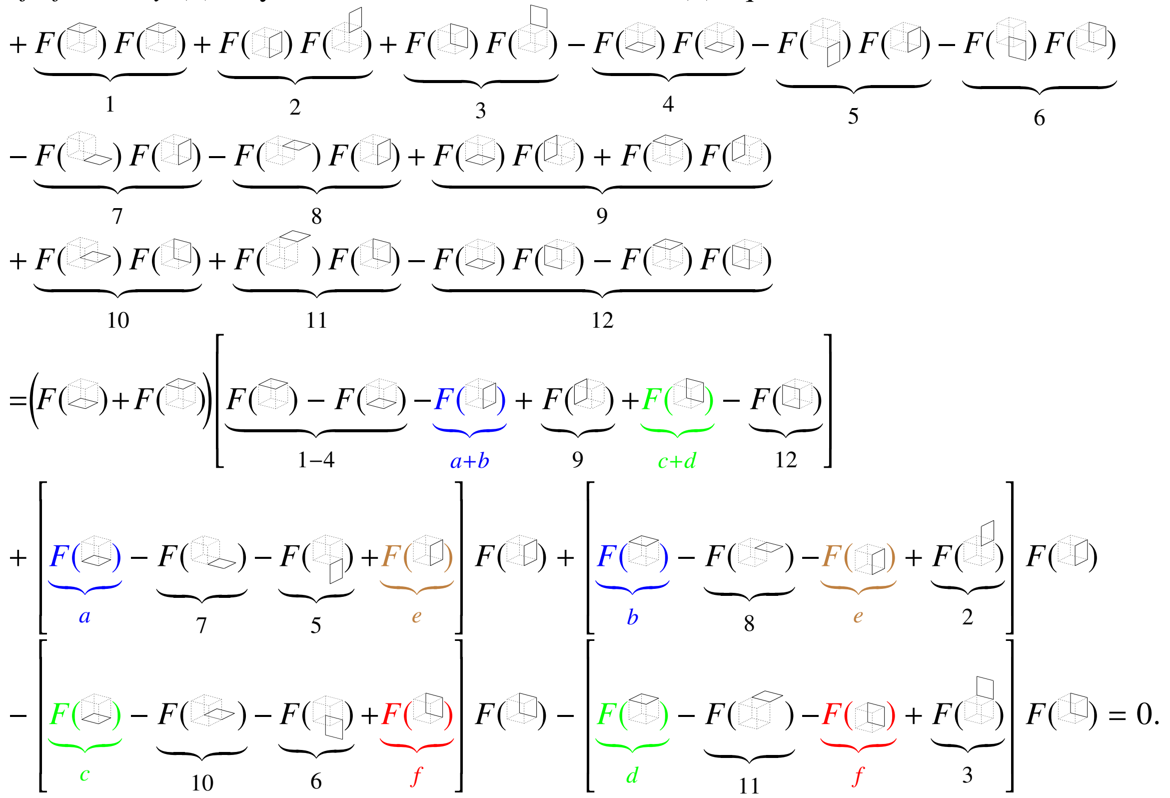
All the Lagrangians in the left column of Table 3 not involving j are gauge invariant.

4 Proofs

4.1 Basic results

First we prove the results of §1. We start with a heuristic elementary proof of the result of §1.1.

Proof of identity (3). By definition the left-hand side of (3) equals



Here the terms labeled by letters cancel each other; the terms in square brackets vanish by (1)–(2). □

Now we prove the results of §1.4. Here the fields are $\mathbb{R}$-valued and the connection $A = 0$.

Lemma 4.1 (Lagrangian functional derivative)**.** *For a local Lagrangian $\mathcal{L}\colon C^k(M;\mathbb{R}) \to C_0(M;\mathbb{R})$ and arbitrary fields $\phi, \Delta \in C^k(M;\mathbb{R})$ we have*

$$\left.\frac{\partial \mathcal{L}[\phi + t\Delta]}{\partial t}\right|_{t=0} = \left(\frac{\partial \mathcal{L}[\phi]}{\partial \phi} + \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)}\right) \frown \Delta - (-1)^k \partial \left(\frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \frown \Delta\right).$$

Proof. Take a vertex $v \in M$. Starting with (17)–(19), where $D_A\phi = \delta\phi$ because $A = 0$, then applying Definition 2.4, and finally the well-known 'integration by parts' identity [25]

$$\partial(\phi \frown \psi) = (-1)^{\dim\psi}(\partial\phi \frown \psi - \phi \frown \delta\psi) \tag{22}$$

we get

$$\begin{aligned}
\left.\frac{\partial \mathcal{L}[\phi + t\Delta]}{\partial t}(v)\right|_{t=0} &= \left.\frac{\partial}{\partial t} L_v([\phi + t\Delta](\mathrm{e}_{v,k}), [\delta\phi + \delta\, t\Delta](\mathrm{e}_{v,k+1}))\right|_{t=0} \\
&= \sum_{j=1}^{p(v,k)} \frac{\partial}{\partial \phi_j} L_v(\phi(\mathrm{e}_{v,k}), \delta\phi(\mathrm{e}_{v,k+1}))\, \left.\frac{\partial}{\partial t}[\phi + t\Delta](\mathrm{e}_{v,k,j})\right|_{t=0} \\
&+ \sum_{j=1}^{p(v,k+1)} \frac{\partial}{\partial \phi'_j} L_v(\phi(\mathrm{e}_{v,k}), \delta\phi(\mathrm{e}_{v,k+1}))\, \left.\frac{\partial}{\partial t}[\delta\phi + \delta\, t\Delta](\mathrm{e}_{v,k+1,j})\right|_{t=0} \\
&= \sum_{j=1}^{p(v,k)} \frac{\partial \mathcal{L}[\phi]}{\partial \phi}(\mathrm{e}_{v,k,j})\Delta(\mathrm{e}_{v,k,j}) + \sum_{j=1}^{p(v,k+1)} \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)}(\mathrm{e}_{v,k+1,j})\delta\Delta(\mathrm{e}_{v,k+1,j}) \\
&= \left[\frac{\partial \mathcal{L}[\phi]}{\partial \phi} \frown \Delta + \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \frown \delta\Delta\right](v) \\
&= \left[\left(\frac{\partial \mathcal{L}[\phi]}{\partial \phi} + \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)}\right) \frown \Delta - (-1)^k \partial \left(\frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \frown \Delta\right)\right](v).
\end{aligned}$$

□

Lemma 4.2. *Let $\phi \in C_k(M;\mathbb{R})$. If $\epsilon[\phi \frown \Delta] = 0$ for each $\Delta \in C^k(M;\mathbb{R})$, then $\phi = 0$.*

Proof. Take $\Delta = \phi$. Then by Definition 2.4 we have $0 = \epsilon[\phi \frown \phi] = \sum_f \phi(f)^2$, where the sum is over all the k-dimensional faces f of M. Thus $\phi = 0$. □

Proof of the Euler–Lagrange Theorem 1.1. A field ϕ is on shell, if and only if for each field Δ we have

$$0 = \epsilon \left.\frac{\partial \mathcal{L}[\phi + t\Delta]}{\partial t}\right|_{t=0} = \epsilon \left[\left(\frac{\partial \mathcal{L}[\phi]}{\partial \phi} + \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)}\right) \frown \Delta\right] - (-1)^k \epsilon\partial \left[\frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \frown \Delta\right] = \epsilon \left[\left(\frac{\partial \mathcal{L}[\phi]}{\partial \phi} + \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)}\right) \frown \Delta\right].$$

The latter two equalities follow from Lemma 4.1 and the obvious identity $\epsilon\partial = 0$ respectively. Since Δ is arbitrary, by Lemma 4.2 the resulting equation is equivalent to (4). □

Proof of the Noether Theorem 1.2. By Lemma 4.1 and Theorem 1.1 for a field ϕ on shell we get

$$\left.\frac{\partial \mathcal{L}[\phi + t\Delta]}{\partial t}\right|_{t=0} = \left(\frac{\partial \mathcal{L}[\phi]}{\partial \phi} + \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)}\right) \frown \Delta - (-1)^k \partial \left(\frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \frown \Delta\right) = -(-1)^k \partial j[\phi].$$

Thus $j[\phi]$ is a conserved current, if and only if the left-hand side vanishes. □

Proof of Theorem 1.3. By Theorem 1.1, Definition 2.10, and the known identity $\partial\partial = \delta\delta = 0$ we have

$$\partial T[\phi] = \partial \left(\frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \times \delta\phi - \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \times \phi\right) = \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \times \delta\delta\phi + \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \times \delta\phi - \partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \times \delta\phi - \partial\partial \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \times \phi = 0.$$

□

4.2 Integral conservation laws

Now we prove the completely new results of the subsubsection "Integral conservation laws" of §2.3. We need to integrate tensors defined on $I_N^d \times I_N^d$ over the faces of the *doubling*. For a vertex f of the doubling, define $f_0, \dots, f_{d-1} \in \mathbb{Z}$ by the formula $f = f_0 e_0 + \dots + f_{d-1} e_{d-1}$. The face of the initial grid with the center f is denoted by f as well.

Definition 4.1. Let T be a partially symmetric type $(1,1)$ tensor, g be a nonboundary hyperface of the doubling, $e_l \perp g$, $f = \max g$. The *k-th component of the flux* of T across g *in positive normal direction* is

$$\langle T, g\rangle_k = \frac{1}{2}(-1)^{\left(l+1+\sum\limits_{\min\{k,l\}\le m\le \max\{k,l\}} f_m\right)} \cdot \begin{cases} -T((f-e_k)\times(f+e_l)), & \text{if } l\ne k,\ 2\nmid f_k,\ 2\nmid f_l;\\ T((f+e_l-e_k)\times f), & \text{if } l\ne k,\ 2\nmid f_k,\ 2\mid f_l;\\ T(f\times(f+e_l-e_k)), & \text{if } l\ne k,\ 2\mid f_k,\ 2\nmid f_l;\\ -T((f+e_l)\times(f-e_k)), & \text{if } l\ne k,\ 2\mid f_k,\ 2\mid f_l;\\ T(f\times f)-T((f+e_l)\times(f-e_k)), & \text{if } l=k. \end{cases}$$

The *flux* across an oriented hypersurface π consisting of nonboundary faces of the doubling is the sum of the fluxes across all the hyperfaces g of π with the coefficients $\langle g, \pi\rangle$ given by (8).

Let L be a type $(0,1)$ tensor, g be a d-dimensional face of the doubling, $f = \max g$. Denote

$$\langle L, g\rangle_k := \frac{1}{2}(-1)^{1+\sum_{m<k} f_m} \cdot \begin{cases} L(f\times(f-e_k)), & \text{if } 2\mid f_k;\\ L((f-e_k)\times f), & \text{if } 2\nmid f_k. \end{cases}$$

Proposition 4.1. *The flux of a partially symmetric type* $(1,1)$ *tensor across a hyperface h of the initial grid (see Definition 2.12) is the sum of fluxes across all the hyperfaces of the doubling contained in h.*

Proof. Compare the k-th components of the fluxes. Take $e_l \perp h$. Consider the 2 cases: $l = k$ and $l \ne k$.

For $l = k$, the map $g \mapsto \max g$ is a 1–1 map between the set of hyperfaces of the doubling contained in h and the set of faces of the initial grid I_N^d contained in h and containing $\max h$. (Recall that the vertex $\max g$ is identified with the face of the initial grid with the center at $\max g$.) Since $\dim \Pr(f,k,k) = 0 = f_k \pmod 2$, by Definitions 2.12 and 4.1 the case $l = k$ follows.

For $l \ne k$, the map $g \mapsto \begin{cases} \max g, & \text{if } 2\nmid(\max g)_k\\ \max g - e_k, & \text{if } 2\mid(\max g)_k; \end{cases}$ is a 2–1 map between the set of hyperfaces of the doubling in h and the set of faces f of the initial grid I_N^d such that $f\subset h$, $f \ni \max h$, $f \parallel e_k$. The contribution of a pair of hyperfaces mapped to the same face f to the sum of fluxes is

$$\begin{aligned} &\tfrac{1}{2}(-1)^{\left(l+1+\sum\limits_{\min\{k,l\}\le m\le\max\{k,l\}} f_m\right)} T((f+e_l-e_k)\times f)+\\ &+\tfrac{1}{2}(-1)^{\left(l+1+\sum\limits_{\min\{k,l\}\le m\le\max\{k,l\}} (f_m+\delta_{mk})\right)}[-T((f+e_k+e_l)\times(f+e_k-e_k))] =\\ &=\tfrac{1}{2}(-1)^{\dim\Pr(f,k,l)+l+1}[T((f+e_l-e_k)\times f)+T((f+e_l+e_k)\times f)] \end{aligned}$$

because $2\nmid f_k$ and $2\mid f_l$. Summation over all such pairs proves the case $l\ne k$. □

Now let us prove an analogue of the Stokes formula; cf. (3) and §4.1. For that we need a lemma.

Lemma 4.3. *For each k-dimensional face f of the d-dimensional grid I_N^d denote by $[f]\in C^k(I_N^d;\mathbb{R})$ the field, which equals 1 at f, and equals 0 at all the other faces. Then*

$$\partial[f] = \sum_{l:2\nmid f_l} (-1)^{\sum_{0\le m\le l} f_m}\cdot([f-e_l]-[f+e_l]);$$

$$\delta[f] = \sum_{l:2\mid f_l} (-1)^{\sum_{0\le m\le l} f_m}\cdot([f-e_l]-[f+e_l]).$$

Proof. This is a direct computation using Definition 2.9. It suffices to prove that f and $f - \mathrm{e}_l$ are cooriented, if and only if $2 \mid \sum_{0\le m\le l} f_m$. Assume that $2 \mid f_l$; the opposite case is analogous. A positive basis in f is the sequence formed by all the vectors e_m such that $2 \nmid f_m$ in a natural order. A positive basis in $f - \mathrm{e}_l$ is obtained by insertion of e_l into the sequence. Adding the outer normal to the former basis means adding e_l at the beginning of the sequence instead. Since moving e_l to the beginning of the sequence requires $\sum_{0\le m<l} f_m \pmod 2$ transpositions, the lemma follows. □

Proposition 4.2 (the Stokes Formula)**.** *Let $0 \le k < d \ge 2$. For each partially symmetric type* $(1,1)$ *tensor T and each d-dimensional face g of the doubling of I_N^d we have $\langle T, \partial g\rangle_k = \langle \partial T, g\rangle_k$.*

Proof. This is a straightforward computation; a technical difficulty is signs. Set $f = \max g$. Assume that $2 \mid f_k$; the opposite case is discussed at the end of the proof. For any fields ϕ and ψ denote $T(\psi\times\phi) = \sum_{e,f} T(e\times f)\psi(e)\phi(f)$. Then $\partial T(e\times f) = T([e]\times\partial[f]) + T(\delta[e]\times[f])$ and by Lemma 4.3

$$\begin{aligned}\partial T(f\times(f-\mathrm{e}_k)) &= T([f]\times\partial[f-\mathrm{e}_k]) + T(\delta[f]\times[f-\mathrm{e}_k])\\ &= \sum_{l:2\nmid f_l-\delta_{kl}} (-1)^{\sum_{m\le l}(f_m-\delta_{mk})}\cdot[T(f\times(f-\mathrm{e}_k-\mathrm{e}_l)) - T(f\times(f-\mathrm{e}_k+\mathrm{e}_l))]\\ &+ \sum_{l:2\mid f_l}(-1)^{\sum_{m\le l} f_m}\cdot[T((f-\mathrm{e}_l)\times(f-\mathrm{e}_k)) - T((f+\mathrm{e}_l)\times(f-\mathrm{e}_k))]\,.\end{aligned}$$

It remains to show that here the l-th summand multiplied by $(-1)^{1+\sum_{m<k} f_m}$ equals twice the difference of the fluxes across the two opposite hyperfaces of g orthogonal to e_l multiplied by $(-1)^l$. (The latter sign factor is required to get the right contribution of the two faces into the whole flux across ∂g in the positive normal direction; see Lemma 4.3 for $k = d$). Denote $f' = f - \mathrm{e}_l$, $k' = \min\{k,l\}$, $l' = \max\{k,l\}$. Denote by $g + \mathrm{e}_l/2$ and $g - \mathrm{e}_l/2$ the hyperfaces of g orthogonal to e_l such that $\max(g+\mathrm{e}_l/2) = f$ and $\max(g-\mathrm{e}_l/2) = f'$ respectively.

Consider the following 3 cases: 1) $l = k$; 2) $l \ne k$ and $2 \mid f_l$; 3) $l \ne k$ and $2 \nmid f_l$.

For $l = k$ (hence $2 \mid f_k = f_l$) the l-th summands in the two sums multiplied by $(-1)^{1+\sum_{m<k} f_m}$ add up to

$$\begin{aligned}&(-1)^{1+\sum_{m<k} f_m}(-1)^{\sum_{m\le k}(f_m-\delta_{mk})}\cdot[T(f\times(f-2\mathrm{e}_k)) - T(f\times f)]+\\ +&(-1)^{1+\sum_{m<k} f_m}(-1)^{\sum_{m\le k} f_m}\cdot[T((f-\mathrm{e}_k)\times(f-\mathrm{e}_k)) - T((f+\mathrm{e}_k)\times(f-\mathrm{e}_k))] =\\ =&(-1)^{f_k+1}\cdot[T(f\times f) - T((f+\mathrm{e}_k)\times(f-\mathrm{e}_k))]-\\ -&(-1)^{f_k}\cdot[T((f-\mathrm{e}_k)\times(f-\mathrm{e}_k)) - T((f-\mathrm{e}_k+\mathrm{e}_k)\times(f-2\mathrm{e}_k))] =\\ =&(-1)^k\cdot(-1)^{k+1+f_k}\cdot[T(f\times f) - T((f+\mathrm{e}_k)\times(f-\mathrm{e}_k))]-\\ -&(-1)^k\cdot(-1)^{k+1+f'_k}\cdot[T(f'\times f') - T(f'+\mathrm{e}_k)\times(f'-\mathrm{e}_k))] =\\ =&(-1)^k 2\langle T, g+\mathrm{e}_k/2\rangle_k - (-1)^k 2\langle T, g-\mathrm{e}_k/2\rangle_k;\end{aligned}$$

see Definition 4.1 applied for $l = k$. We have found the contribution of the l-th summands for $l = k$.

For $l \ne k$ and $2 \nmid f_l$ the l-th summand multiplied by $(-1)^{1+\sum_{m<k} f_m}$ is

$$\begin{aligned}&(-1)^{1+\sum_{m<k} f_m}(-1)^{\sum_{m\le l}(f_m-\delta_{mk})}\cdot[T(f\times(f-\mathrm{e}_k-\mathrm{e}_l)) - T(f\times(f-\mathrm{e}_k+\mathrm{e}_l))] =\\ \overset{(*)}{=}&(-1)^{1+\sum_{k'\le m\le l'} f_m}\cdot[T(f\times(f+\mathrm{e}_l-\mathrm{e}_k)) - T((f-\mathrm{e}_l+\mathrm{e}_l)\times(f-\mathrm{e}_l-\mathrm{e}_k))] =\\ =&(-1)^l\cdot(-1)^{l+1+\sum_{k'\le m\le l'} f_m}\cdot T(f\times(f+\mathrm{e}_l-\mathrm{e}_k))-\\ -&(-1)^l\cdot(-1)^{l+1+\sum_{k'\le m\le l'} f'_m}\cdot[-T((f'+\mathrm{e}_l)\times(f'-\mathrm{e}_k))] =\\ =&(-1)^l 2\langle T, g+\mathrm{e}_l/2\rangle_k - (-1)^l 2\langle T, g-\mathrm{e}_l/2\rangle_k;\end{aligned}$$

see Definition 4.1 applied for $l \ne k$, $2 \mid f_k$, $2 \nmid f_l$ and $l \ne k$, $2 \mid f'_k$, $2 \mid f'_l$. Here (*) follows from

$$1 + \sum_{m<k} f_m + \sum_{m\le l}(f_m - \delta_{mk}) = \begin{cases}\sum_{k\le m\le l} f_m, & \text{if } k<l;\\ \sum_{l<m<k} f_m + 1, & \text{if } k>l;\end{cases} = \sum_{k'\le m\le l'} f_m \pmod 2,$$

where we used the conditions $2 \nmid f_l$ and $2 \mid f_k$ to change the range of summation over m.

For $l \neq k$ and $2 \mid f_l$ the l-th summand multiplied by $(-1)^{1+\sum_{m<k} f_m}$ is

$$
\begin{aligned}
&(-1)^{1+\sum_{m<k} f_m}(-1)^{\sum_{m\le l} f_m} \cdot [T((f-\mathrm{e}_l)\times(f-\mathrm{e}_k)) - T((f+\mathrm{e}_l)\times(f-\mathrm{e}_k))] = \\
=&(-1)^{\sum_{k'\le m\le l'} f_m} \cdot [T((f+\mathrm{e}_l)\times(f-\mathrm{e}_k)) - T((f-\mathrm{e}_l)\times(f-\mathrm{e}_l+\mathrm{e}_l-\mathrm{e}_k))] = \\
=&(-1)^l \cdot (-1)^{l+1+\sum_{k'\le m\le l'} f_m} \cdot [-T((f+\mathrm{e}_l)\times(f-\mathrm{e}_k))] - \\
-&(-1)^l \cdot (-1)^{l+1+\sum_{k'\le m\le l'} f'_m} \cdot T(f'\times(f'+\mathrm{e}_l-\mathrm{e}_k)) = \\
=&(-1)^l 2\langle T, g+\mathrm{e}_l/2\rangle_k - (-1)^l 2\langle T, g-\mathrm{e}_l/2\rangle_k.
\end{aligned}
$$

Summation of the expressions obtained in the three cases completes the proof in the case when $2 \mid f_k$.

For $2 \nmid f_k$ the proof is analogous and starts from the evaluation of $\partial T((f-\mathrm{e}_k)\times f)$. For $l = k$ one ends up with an expression involving $T((f-\mathrm{e}_k)\times(f+\mathrm{e}_k))$ rather than $T((f+\mathrm{e}_k)\times(f-\mathrm{e}_k))$. But the latter two values are equal because T is partially symmetric. □

Proof of Proposition 2.7. This follows directly from Propositions 4.1 and 4.2. □

Proof of Theorem 2.2. Clearly, tensor (7) is partially symmetric for this particular Lagrangian $\mathcal{L}[\phi]$; cf. rows 2–3 of Table 3. Thus the corollary follows directly from Theorem 1.3 and Proposition 2.7. □

Proof of Proposition 2.8. Consider the cases when $l \neq k$ and $l = k$ separately.

For $l \neq k$ the only nonvanishing contribution to the flux of T comes from the edge $f = v - \mathrm{e}_k$. We have $\dim \Pr(f, k, l) = 1$. Thus by (7) we get the required expression

$$
\begin{aligned}
(-1)^l\langle T, h\rangle_k &= \tfrac{1}{2}(-1)^l(-1)^l\,[T((v+\mathrm{e}_l)\times(v-\mathrm{e}_k)) + T((v+\mathrm{e}_l-2\mathrm{e}_k)\times(v-\mathrm{e}_k))] \\
&= \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v+\mathrm{e}_l) + \frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v+\mathrm{e}_l-2\mathrm{e}_k)\right]\delta\phi(v-\mathrm{e}_k).
\end{aligned}
$$

For $l = k$ the contribution to the flux comes from $f = v$ and $f = v - \mathrm{e}_m$ for each $m \neq k$. Thus

$$
\begin{aligned}
(-1)^k\langle T, h\rangle_k &= \frac{1}{2}(-1)^k(-1)^{k+1}\left[T(v\times v) - T((v+\mathrm{e}_k)\times(v-\mathrm{e}_k)) + \sum_{m\neq k} T((v-\mathrm{e}_m)\times(v-\mathrm{e}_m))\right] \\
&= -\frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial\phi}(v)\phi(v) - \frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v+\mathrm{e}_k)[\delta\phi](v-\mathrm{e}_k) + \sum_{m\neq k}\frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v-\mathrm{e}_m)[\delta\phi](v-\mathrm{e}_m)\right] \\
&= \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v+\mathrm{e}_k) + \frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v-\mathrm{e}_k)\right]\delta\phi(v-\mathrm{e}_k) \\
&\quad - \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial\phi}(v)\phi(v) + \sum_m \frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v-\mathrm{e}_m)\delta\phi(v-\mathrm{e}_m)\right] \\
&= \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v+\mathrm{e}_k) + \frac{\partial\mathcal{L}}{\partial(\delta\phi)}(v+\mathrm{e}_k-2\mathrm{e}_k)\right]\delta\phi(v-\mathrm{e}_k) - \mathcal{L}[\phi](v).
\end{aligned}
$$

The latter equality is proved as follows. Since $\mathcal{L}$ is homogeneous quadratic, it follows that $\frac{\partial L_v}{\partial\phi_1}\phi_1 + \frac{\partial L_v}{\partial\phi'_1}\phi'_1 + \dots + \frac{\partial L_v}{\partial\phi'_d}\phi'_d = 2L_v(\phi_1, \phi'_1, \dots, \phi'_d)$. Hence $\frac{\partial\mathcal{L}}{\partial\phi} \frown \phi + \frac{\partial\mathcal{L}}{\partial(\delta\phi)} \frown \delta\phi = 2\mathcal{L}[\phi]$, as required. □

4.3 Identities

For the sequel we need several identities for cochain operations, most of which are well-known.

Definition 4.2. The *pairing* of fields $\phi, \psi \in C^k(M; \mathbb{C}^{m\times n})$, where $m = 1$ or $m = n$, is defined by

$$\langle\phi,\psi\rangle = \operatorname{Re}\operatorname{Tr}\sum_{k\text{-dimensional faces } f}\phi(f)\psi^*(f) = \epsilon\operatorname{Re}\operatorname{Tr}[\phi\frown\psi^*] = \epsilon\operatorname{Re}\operatorname{Tr}[\phi\overset{*}{\frown}\psi^*].$$

Given $U \in C^1(M;G)$, denote by $C^1(M;T_UG)$ the set of all $\Delta \in C^1(M;\mathbb{C}^{n\times n})$ such that $\Delta(e)$ belongs to the tangent space $T_{U(e)}G$ for each edge e. For $\phi \in C_k(M;\mathbb{C}^{n\times m})$, where $m = 1$ or $m = n$, denote

$$\check{D}^*_A\phi = (D^*_A\phi^*)^* = \partial\phi + (-1)^k A \overset{*}{\frown} \phi + \delta_{mn} \cdot \phi \frown A. \tag{23}$$

Lemma 4.4 (Nondegeneracy of the pairing)**.** *Let $\phi \in C_k(M;\mathbb{C}^{m\times n})$, $\psi \in C_0(M;\mathbb{C}^{n\times n})$, $\chi \in C_1(M;\mathbb{C}^{n\times n})$.*
If $\langle\phi,\Delta\rangle = 0$ for each $\Delta \in C^k(M;\mathbb{C}^{m\times n})$, then $\phi = 0$.
If $\langle\psi,\Delta\rangle = 0$ for each $\Delta \in C^0(M;T_1G)$, then $\mathrm{Pr}_{T_1G}\psi = 0$.
If $\langle\chi,\Delta\rangle = 0$ for each $\Delta \in C^1(M;T_UG)$, then $\mathrm{Pr}_{T_UG}\chi = 0$.

Proof. First, take $\Delta = \phi$. Then $0 = \langle\phi,\phi\rangle = \sum_f \mathrm{Re\,Tr}[\phi^*(f)\phi(f)] = \sum_f \sum_{i,j=1}^{m,n} |\phi_{ij}(f)|^2$. Thus $\phi = 0$.

For the third assertion, take $\Delta = \mathrm{Pr}_{T_UG}\chi$. Then $0 = \langle\chi, \mathrm{Pr}_{T_UG}\chi\rangle = \sum_e \langle\chi(e), \mathrm{Pr}_{T_{U(e)}G}\chi(e)\rangle = \sum_e \langle \mathrm{Pr}_{T_{U(e)}G}\chi(e), \mathrm{Pr}_{T_{U(e)}G}\chi(e)\rangle$, where the sums are over all edges e, because $\mathrm{Pr}_{T_{U(e)}G}$ is an orthogonal projection. Since the pairing $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{n\times n}$ is nondegenerate, it follows that $\mathrm{Pr}_{T_UG}\chi = 0$.

The second assertion is proved analogously. □

Lemma 4.5. *In a* cubical *complex M, for each $U \in C^1(M;G)$ and $\Phi \in C^k(M;\mathbb{C}^{n\times n})$ we have*

$$D_A\Phi = U \smile \Phi - (-1)^k \Phi \smile U; \qquad F = U \smile U$$
$$\check{D}^*_A\Phi = \Phi \frown U + (-1)^k U \overset{*}{\frown} \Phi.$$

The two identities in the 1st column hold for a simplicial *complex M for $k = 0$ and $k = 1$ respectively.*

Proof. By Definitions 2.9 and 2.18 it follows that

$$[\delta\Phi](a\dots c) = \sum_{\substack{b:\dim(a\dots b)=1,\\ \dim(b\dots c)=k}} \langle a,b,c\rangle\Phi(b\dots c) - (-1)^k \sum_{\substack{b:\dim(a\dots b)=k,\\ \dim(b\dots c)=1}} \langle a,b,c\rangle\Phi(a\dots b) = 1 \smile \Phi - (-1)^k\Phi \smile 1;$$

$$[\partial\Phi](b\dots c) = \sum_{\substack{a:\dim(a\dots b)=1,\\ \dim(a\dots c)=k}} \langle a,b,c\rangle\Phi(a\dots c) + (-1)^k \sum_{\substack{d:\dim(b\dots d)=k,\\ \dim(c\dots d)=1}} \langle b,c,d\rangle\Phi(b\dots d) = \Phi \frown 1 + (-1)^k 1 \overset{*}{\frown} \Phi,$$

where 1 is the unit gauge group field and the sums are over the vertices such that there exist faces $a\dots b, b\dots c \subset a\dots c$ or $b\dots c, c\dots d \subset b\dots d$. Using (12)–(13), we get the required identities. □

Lemma 4.6. *(Cf. [10]) For each $\phi \in C^k(M;\mathbb{C}^{p\times q})$, $\psi \in C^l(M;\mathbb{C}^{q\times r})$, $\chi \in C^m(M;\mathbb{C}^{r\times s})$ we have*

$$\delta\delta = 0; \quad \delta(\phi \smile \psi) = (\delta\phi) \smile \psi + (-1)^{\dim\phi}\phi \smile \delta\psi; \quad (\phi \smile \psi) \smile \chi = \phi \smile (\psi \smile \chi);$$
$$\partial\partial = 0; \quad \partial(\phi \frown \psi) = (-1)^{\dim\psi}(\partial\phi \frown \psi - \phi \frown \delta\psi); \quad (\phi \frown \psi) \frown \chi = \phi \frown (\psi \smile \chi);$$
$$\epsilon\partial = 0; \quad \partial(\phi \overset{*}{\frown} \psi) = \phi \overset{*}{\frown} \partial\psi + (-1)^{\dim\psi-\dim\phi}\delta\phi \overset{*}{\frown} \psi; \quad \phi \overset{*}{\frown} (\psi \overset{*}{\frown} \chi) = (\phi \smile \psi) \overset{*}{\frown} \chi;$$
$$(\phi \smile \psi)^* = \begin{cases} \psi^* \frown \phi^*, & \text{if } \dim\phi = 0; \\ \psi^* \overset{*}{\frown} \phi^*, & \text{if } \dim\psi = 0; \end{cases} \qquad (\phi \overset{*}{\frown} \psi) \frown \chi = \phi \overset{*}{\frown} (\psi \frown \chi).$$

For each $\phi \in C_{k+1}(M;\mathbb{C}^{n\times m})$, $\psi \in C^k(M;\mathbb{C}^{m\times n})$, $U \in C^1(M;G)$, where $m = 1$ or $m = n$, we have

$$D_AD_A\psi = -\psi \smile F + \delta_{mn} \cdot F \smile \psi; \qquad D_A(\phi \smile \psi) = D_A\phi \smile \psi + (-1)^{\dim\phi}\phi \smile D_A\psi;$$
$$\check{D}^*_A\check{D}^*_A\phi = -F \overset{*}{\frown} \phi + \delta_{mn} \cdot \phi \frown F; \qquad \check{D}^*_A(\phi \frown \psi) = (-1)^{\dim\psi}(\check{D}^*_A\phi \frown \psi - \phi \frown D_A\psi);$$
$$\mathrm{Re\,Tr}\,\epsilon\,\check{D}^*_A\phi = 0, \text{ if } m = n \text{ and } \dim\phi = 1; \qquad \check{D}^*_A(\phi \overset{*}{\frown} \psi) = \phi \overset{*}{\frown} \check{D}^*_A\psi + (-1)^{\dim\psi-\dim\phi}D_A\phi \overset{*}{\frown} \psi.$$

For each $\phi \in C^k(M;\mathbb{C}^{m\times n})$, $\psi \in C^l(M;\mathbb{C}^{n\times n}$ or $\mathbb{C}^{m\times m})$, $\chi \in C_{k+l}(M;\mathbb{C}^{m\times n})$, $U \in C^1(M;G)$, where $m = 1$ or $m = n$ (and $l = 1$ for the identities in the 1st and 3rd column below), we have:

$$\langle\chi,\delta\phi\rangle = \langle\partial\chi,\phi\rangle; \qquad \langle\chi,\psi \smile \phi\rangle = \langle(\chi^* \frown \psi)^*,\phi\rangle; \qquad \mathrm{Re\,Tr}\,D^*_A\psi = \partial\,\mathrm{Re\,Tr}\,[U^* \cdot \psi];$$
$$\langle\chi,D_A\phi\rangle = \langle D^*_A\chi,\phi\rangle; \qquad \langle\chi,\phi \smile \psi\rangle = \langle(\psi \overset{*}{\frown} \chi^*)^*,\phi\rangle; \qquad \mathrm{Pr}_{T_1G}\,D^*_A\psi = D^*_A\mathrm{Pr}_{T_UG}\,\psi.$$

In the 3rd column, "$\cdot$" is the edgewise product, i.e., $[U^* \cdot \psi](e) := U^*(e)\psi(e)$ for each edge e.

Proof. The identities involving neither the cop-product nor covariant (co)boundary are well-known in the case when the functions assume values in a commutative ring; cf. [10]. Without the commutativity the proof is literally the same. Let us prove the remaining identities.

For an ordered 4-ple of faces $a \dots b, b \dots c, c \dots d \subset a \dots d$ write $\langle a, b, c, d\rangle = +1$, if the ordered set consisting of positive bases in $a \dots b$, $b \dots c$, $c \dots d$ is a positive basis in $a \dots d$. Otherwise write $\langle a, b, c, d\rangle = -1$. Clearly, $\langle a, b, c, d\rangle = \langle a, b, c\rangle\langle a, c, d\rangle = \langle a, b, d\rangle\langle b, c, d\rangle$. Thus by Definition 2.18

$$
\begin{aligned}
[\phi \overset{*}{\frown} (\psi \overset{*}{\frown} \chi)](a \dots b) &= \sum_{c:\dim(b\dots c)=k,\dim(a\dots c)=m-l} \langle a, b, c\rangle \phi(b \dots c)[\psi \overset{*}{\frown} \chi](a \dots c) \\
&= \sum_{c,d:\dim(b\dots c)=k,\dim(c\dots d)=l,\dim(a\dots d)=m} \langle a, b, c\rangle\langle a, c, d\rangle \phi(b \dots c)\psi(c \dots d)\chi(a \dots d) \\
&= \sum_{c,d:\dim(b\dots c)=k,\dim(c\dots d)=l,\dim(a\dots d)=m} \langle a, b, d\rangle\langle b, c, d\rangle \phi(b \dots c)\psi(c \dots d)\chi(a \dots d) \\
&= [(\phi \smile \psi) \overset{*}{\frown} \chi](a \dots b).
\end{aligned}
$$

Setting $m = k + l$, changing the notation χ to χ^*, and applying the operator $\epsilon \operatorname{Re}\operatorname{Tr}$, we obtain $\langle(\psi \overset{*}{\frown} \chi^*)^*, \phi\rangle = \langle\chi, \phi \smile \psi\rangle$. Taking $\psi = A$, $\phi \in C^k(M; \mathbb{C}^{m\times n})$, $\chi \in C_{k+1}(M; \mathbb{C}^{m\times n})$, multiplying by $(-1)^{\dim\phi} = -(-1)^{\dim\chi}$, adding the known identity $\langle\partial\chi, \phi\rangle = \langle\chi, \delta\phi\rangle$ (and for $m = n$ also the known identity $\langle(\chi \frown \psi^*)^*, \phi\rangle = \langle\chi, \phi \smile \psi\rangle$), and using (12)–(15), we get $\langle D_A^*\chi, \phi\rangle = \langle\chi, D_A\phi\rangle$.

The formula for $(\phi \overset{*}{\frown} \psi) \frown \chi$ is proved analogously.

Next, the formula for $\check{D}_A^*(\phi \frown \psi)$ for a cubical complex and $m = n$ follows from

$$
\begin{aligned}
(-1)^l \check{D}_A^*(\phi \frown \psi) &= (-1)^l(-1)^{k-l} U \overset{*}{\frown} (\phi \frown \psi) + (-1)^l(\phi \frown \psi) \frown U \\
&= (-1)^k(U \overset{*}{\frown} \phi) \frown \psi + (\phi \frown U) \frown \psi - \phi \frown (U \smile \psi) + (-1)^l\phi \frown (\psi \smile U) \\
&= (\check{D}_A^*\phi) \frown \psi - \phi \frown D_A\psi,
\end{aligned}
$$

where we used Lemma 4.5 and the identities not involving (covariant) (co)boundary. Alternatively, the formula for $\check{D}_A^*(\phi \frown \psi)$ can be deduced from the formula for $\delta(\phi \smile \psi)$ by pairing with an arbitrary field Δ and applying Lemma 4.4 and the identities from the paragraph before the previous one; this works for a simplicial complex and for $m = 1$ as well.

The formulae for $D_A(\phi \smile \psi)$, $\check{D}_A^*(\phi \overset{*}{\frown} \psi)$, $D_A D_A$, $\check{D}_A^*\check{D}_A^*$ are proved analogously.

Finally, for each vertex v by Lemma 4.5 we have (where $\langle U, \psi\rangle$ is the edgewise scalar product)

$$
[\operatorname{Re}\operatorname{Tr} D_A^*\psi](v) = \operatorname{Re}\operatorname{Tr}[\psi^* \frown U - U \overset{*}{\frown} \psi^*]^*(v) = \sum_{e:\max e=v} \langle\psi(e), U(e)\rangle - \sum_{e:\min e=v} \langle\psi(e), U(e)\rangle = [\partial\langle U, \psi\rangle](v);
$$

$$
D_A^*\operatorname{Pr}_{T_U G}\psi = ((\operatorname{Pr}_{T_U G}\psi)^* \frown U - U \overset{*}{\frown} (\operatorname{Pr}_{T_U G}\psi)^*)^* = \operatorname{Pr}_{T_1 G}(\psi^* \frown U)^* - \operatorname{Pr}_{T_1 G}(U \overset{*}{\frown} \psi^*)^* = \operatorname{Pr}_{T_1 G} D_A^*\psi.
$$

Applying the operator ϵ we get $\operatorname{Re}\operatorname{Tr}\epsilon\,\check{D}_A^*\psi = \epsilon\operatorname{Re}\operatorname{Tr} D_A^*\psi^* = \epsilon\,\partial\operatorname{Re}\operatorname{Tr}[U \cdot \psi] = 0$. □

4.4 Generalizations

Now we proceed to the proof of the results of §3. The argument is parallel to that of §4.1.

Proof of Proposition 3.1. This is a straightforward computation using the explicit expression for the function L_v given in the middle part of Table 3. In row 5 we use the identity $(\gamma^0\gamma)^* = \gamma^0\gamma$. □

Lemma 4.7 (Lagrangian functional derivative)**.** *For a local Lagrangian* $\mathcal{L}\colon C^k(M; \mathbb{C}^{1\times n}) \times C^1(M; \mathbb{C}^{n\times n}) \to C_0(M; \mathbb{R})$ *and arbitrary fields* $\phi, \Delta \in C^k(M; \mathbb{C}^{1\times n})$, $U \in C^1(M; G)$ *we have*

$$
\left.\frac{\partial \mathcal{L}[\phi + t\Delta, U]}{\partial t}\right|_{t=0} = \operatorname{Re}\operatorname{Tr}\left[\left(\frac{\partial \mathcal{L}[\phi, U]}{\partial \phi} + \check{D}_A^* \frac{\partial \mathcal{L}[\phi, U]}{\partial (D_A\phi)}\right) \frown \Delta - (-1)^k \check{D}_A^* \left(\frac{\partial \mathcal{L}[\phi, U]}{\partial (D_A\phi)} \frown \Delta\right)\right].
$$

Proof. This is proved literally as Lemma 4.1 with δ and ∂ replaced by D_A and $\check{D}_A^*$ respectively, and $\operatorname{Re}\operatorname{Tr}$ applied to each summand. Instead of (22) use the formula for $\check{D}_A^*(\phi \frown \psi)$ from Lemma 4.6. □

Proof of Theorem 3.1. A field ϕ is on shell, if and only if $\frac{\partial S[\phi+t\Delta,U]}{\partial t}\Big|_{t=0} = 0$ for each $\Delta \in C^k(M,\mathbb{C}^{1\times n})$. By Lemmas 4.7 and 4.4 this is equivalent to (20) because $\epsilon \operatorname{Re}\operatorname{Tr} \check{D}^*_A = 0$ by Lemma 4.6. □

Lemma 4.8 (Lagrangian functional derivative)**.** *For a local Lagrangian $\mathcal{L}\colon C^1(M;\mathbb{C}^{n\times n}) \to C_0(M;\mathbb{R})$ and arbitrary fields $U \in C^1(M;G)$, $\Delta \in C^1(M;T_U G)$ we have*

$$\frac{\partial \mathcal{L}[U+t\Delta]}{\partial t}\bigg|_{t=0} = \operatorname{Re}\operatorname{Tr}\left[\left(\frac{\partial \mathcal{L}[U]}{\partial U} + \check{D}^*_A \frac{\partial \mathcal{L}[U]}{\partial (F[U])}\right) \frown \Delta + \check{D}^*_A\left(\frac{\partial \mathcal{L}[U]}{\partial (F[U])} \frown \Delta\right)\right].$$

Proof. This is proved analogously to Lemma 4.1 with ϕ and $\delta\phi$ replaced by U and $F = \delta A + A \smile A$, using that

$$\begin{aligned}\frac{\partial}{\partial t} F[U+t\Delta]\bigg|_{t=0} &= \frac{\partial}{\partial t}[\delta(U+t\Delta-1) + (U+t\Delta-1)\smile(U+t\Delta-1)]\bigg|_{t=0} \\ &= \delta\Delta + (U-1)\smile\Delta + \Delta\smile(U-1) = D_A\Delta.\end{aligned}$$

□

Proof of Theorem 3.2. A gauge group field U is on shell, if and only if $\frac{\partial S[U+t\Delta]}{\partial t}\Big|_{t=0} = 0$ for each $\Delta \in C^1(M,T_U G)$. By Lemmas 4.8, 4.6, and 4.4 this is equivalent to (21). □

Proof of Theorem 3.3. This follows from $\partial\langle j[\phi,U],U\rangle = \operatorname{Re}\operatorname{Tr} D^*_A j[\phi,U] = \frac{\partial}{\partial t}\mathcal{L}[\phi+t\Delta,U]\big|_{t=0} = 0$. Here the 1st equality is given by Lemma 4.6. The 2nd one is proved as in the proof of Theorem 1.2 with δ, ∂ replaced by D_A, $\check{D}^*_A$, and Re Tr applied to each summand. The 3rd one is (5). □

Remark 4.1. If (5) holds in a subset of M, then the current $\langle j[\phi,U],U\rangle$ is conserved on the subset.

Lemma 4.9 (Lagrangian functional derivative)**.** *For a local Lagrangian $\mathcal{L}\colon C^k(M;\mathbb{C}^{1\times n})\times C^1(M;\mathbb{C}^{n\times n}) \to C_0(M;\mathbb{R})$ and arbitrary fields $\phi \in C^k(M;\mathbb{C}^{1\times n})$ and $U,\Delta \in C^1(M;\mathbb{C}^{n\times n})$ we have*

$$\frac{\partial \mathcal{L}[\phi,U+t\Delta]}{\partial t}\bigg|_{t=0} = \operatorname{Re}\operatorname{Tr}\left[\left(\frac{\partial \mathcal{L}[\phi,U]}{\partial (D_A\phi)} \frown \phi\right) \frown \Delta\right] \quad \textit{and} \quad \frac{\partial \mathcal{L}[\phi,U]}{\partial U} = \frac{\partial \mathcal{L}[\phi,U]}{\partial (D_A\phi)} \frown \phi.$$

Proof. Analogously to the proof of Lemma 4.1 using (14) and Lemma 4.6 we get

$$\begin{aligned}\frac{\partial \mathcal{L}[\phi,U+t\Delta]}{\partial t}\bigg|_{t=0} &= \operatorname{Re}\operatorname{Tr}\left[\frac{\partial \mathcal{L}[\phi,U]}{\partial \phi} \frown \frac{\partial \phi}{\partial t} + \frac{\mathcal{L}[\phi,U]}{\partial (D_A\phi)} \frown \frac{\partial (D_{A[U+t\Delta]}\phi)}{\partial t}\bigg|_{t=0}\right] \\ &= 0 + \operatorname{Re}\operatorname{Tr}\left[\frac{\mathcal{L}[\phi,U]}{\partial (D_A\phi)} \frown \frac{\partial[\delta\phi + \phi\smile(U-1+t\Delta)]}{\partial t}\bigg|_{t=0}\right] \\ &= \operatorname{Re}\operatorname{Tr}\left[\frac{\mathcal{L}[\phi,U]}{\partial (D_A\phi)} \frown (\phi\smile\Delta)\right] = \operatorname{Re}\operatorname{Tr}\left[\left(\frac{\mathcal{L}[\phi,U]}{\partial (D_A\phi)} \frown \phi\right) \frown \Delta\right].\end{aligned}$$

A local Lagrangian $\mathcal{L}[\phi,U]$ is also local with respect to U and does not depend on $F[U]$. Since $\Delta \in C^1(M;\mathbb{C}^{n\times n})$ is arbitrary, by Lemmas 4.8 and 4.4 it follows that $\frac{\partial \mathcal{L}[\phi,U]}{\partial U} = \frac{\partial \mathcal{L}[\phi,U]}{\partial (D_A\phi)} \frown \phi$. □

Lemma 4.10 (Infinitesimal form of gauge invariance)**.** *For each gauge invariant differentiable function $\mathcal{L}\colon C^k(M;\mathbb{C}^{1\times n}) \times C^1(M;\mathbb{C}^{n\times n}) \to C_0(M;\mathbb{R})$ and each $\Delta \in C^0(M,T_1 G)$ we have*

$$\frac{\partial}{\partial t}\mathcal{L}[\phi + t\phi\smile\Delta, U + tD_A\Delta]\bigg|_{t=0} = 0.$$

Proof. Since $\mathcal{L}[\phi,U]$ is gauge invariant and differentiable, by Lemma 4.5 up to first order in t

$$\begin{aligned}\mathcal{L}[\phi,U] &= \mathcal{L}[\phi\smile\exp(t\Delta), \exp(-t\Delta)\smile U\smile\exp(t\Delta)] \\ &= \mathcal{L}[\phi + t\phi\smile\Delta, U + t(U\smile\Delta - \Delta\smile U)] + o(t) \\ &= \mathcal{L}[\phi + t\phi\smile\Delta, U + tD_A\Delta] + o(t) \qquad \text{as } t \to 0.\end{aligned}$$

Differentiating with respect to t and setting $t = 0$, we get the required result. □

Lemma 4.11 (Local covariant constants)**.** *For each $U \in C^1(M;G)$, $g_0 \in T_1G$, and each vertex v there is $g \in C^0(M;T_1G)$ such that $g(v) = g_0$ and $[D_A g](uv) = 0$ for each neighbor u of v.*

Proof. Set $g(v) = g_0$, $g(u) = U(uv)g(v)U(vu)$ at each neighbor u of v, and let g be arbitrary at the other vertices. Then by Lemma 4.5 we have $[D_A g](uv) = U(uv)g(v) - U(uv)g(v)U(vu)U(uv) = 0$. □

Proof of Theorem 3.4. Take an arbitrary vertex v and $g_0 \in T_1G$. Let $g \in C^0(M;T_1G)$ be given by Lemma 4.11. Apply Lemma 4.10 for $\Delta = g$. Since $D_A g(uv) = 0$ for each neighbor u of v, we obtain that equation (5) holds at the vertex v with $\Delta = \phi \smile g$ (notice that the connection in (5) does not depend on t). By Theorem 3.3, Remark 4.1, and Lemma 4.6, we have

$$0 = \partial \mathrm{Re\,Tr}\left[\left(\frac{\partial \mathcal{L}[\phi,U]}{\partial(D_A\phi)} \frown (\phi \smile g)\right)\cdot U\right](v) = \mathrm{Re\,Tr}\left[\check{D}_A^*\left(\left(\frac{\partial \mathcal{L}[\phi,U]}{\partial(D_A\phi)} \frown \phi\right) \frown g\right)\right](v)$$
$$= \mathrm{Re\,Tr}\left[\check{D}_A^*\left(\frac{\partial \mathcal{L}[\phi,U]}{\partial(D_A\phi)} \frown \phi\right) \frown g - \left(\frac{\partial \mathcal{L}[\phi,U]}{\partial(D_A\phi)} \frown \phi\right) \frown D_A g\right](v) = \mathrm{Re\,Tr}\left[D_A^*\left(\frac{\partial \mathcal{L}[\phi,U]}{\partial(D_A\phi)} \frown \phi\right)^*(v)\cdot g_0\right].$$

Here we used that $[D_A g](uv) = 0$ for each edge uv containing v. Since the vertex v and $g_0 \in T_1G$ are arbitrary, by Lemma 4.4 it follows that $\mathrm{Pr}_{T_1G}\, D_A^*\left(\frac{\partial \mathcal{L}[\phi,U]}{\partial(D_A\phi)} \frown \phi\right)^* = 0$. By Lemma 4.9 we have $\frac{\partial \mathcal{L}[\phi,U]}{\partial(D_A\phi)} \frown \phi = \frac{\partial \mathcal{L}[\phi,U]}{\partial U} = j[\phi,U]^*$. By Lemma 4.6 we have $D_A^*\mathrm{Pr}_{T_UG}\, j[\phi,U] = \mathrm{Pr}_{T_1G}\, D_A^* j[\phi,U] = 0$, i.e., the covariant current $j[\phi,U]$ is conserved. □

Proof of Theorem 3.5. Denote $\mathcal{S}[U] = \epsilon\mathcal{L}[U]$ and $\mathcal{S}'[U] = \epsilon\mathcal{L}'[U]$. Take arbitrary $\Delta \in C^0(M,T_1G)$. By Lemmas 4.10 (with $\mathcal{L}[\phi,U]$ replaced by $\mathcal{L}'[U]$) and 4.6 we get

$$\frac{\partial}{\partial t}\mathcal{S}[U + tD_A\Delta]\bigg|_{t=0} = \frac{\partial}{\partial t}\left(\mathcal{S}'[U + tD_A\Delta] + \langle j, U + tD_A\Delta\rangle\right)\bigg|_{t=0} = 0 + \langle j, D_A\Delta\rangle = \langle D_A^* j, \Delta\rangle.$$

For a gauge group field U on shell the left-hand side vanishes, because $D_A\Delta = U \smile \Delta - \Delta \smile U \in C^1(M,T_UG)$ is a possible variation of U. Thus $\langle D_A^* j, \Delta\rangle = 0$ for arbitrary $\Delta \in C^0(M,T_1G)$. By Lemmas 4.4 and 4.6 we get $0 = \mathrm{Pr}_{T_1G} D_A^* j = D_A^*\mathrm{Pr}_{T_UG}\, j$, as required. □

Proof of Proposition 3.2. Let us present the proof for a cubical complex. For a simplicial complex the argument is literally the same, only each instance of the fourth vertex "d" is just removed.

Since the group G consists of unitary matrices, for each edge uv and each face $abcd$ we have

$$\begin{aligned} A[g^* \smile U \smile g](uv) &= g^*(u)U(uv)g(v) - 1 \\ &= g^*(u)(U(uv) - 1)g(v) + g^*(u)(g(v) - g(u)) = [g^* \smile A[U] \smile g + g^* \smile \delta g](uv); \\ F[g^* \smile U \smile g](abcd) &= [g^* \smile U \smile g](abc) - [g^* \smile U \smile g](adc) \\ &= g^*(a)U(ab)g(b)g^*(b)U(bc)g(c) - g^*(a)U(adc)g(c) = [g^* \smile F[U] \smile g](abcd). \end{aligned}$$

Now, using (14)–(15) and Lemma 4.6 we get

$$\begin{aligned} D_{A[g^*\smile U\smile g]}(\phi \smile g) &= \delta(\phi \smile g) - (-1)^k\phi \smile g \smile [g^* \smile A[U] \smile g + g^* \smile \delta g] \\ &= (\delta\phi) \smile g + (-1)^k\phi \smile \delta g - (-1)^k\phi \smile (g \smile g^*) \smile [A[U] \smile g + \delta g] \\ &= (D_{A[U]}\phi) \smile g; \\ \left(D^*_{A[g^*\smile U\smile g]}(\phi \smile g)\right)^* &= \partial(\phi \smile g)^* + (-1)^k[g^* \smile A[U] \smile g - \delta g^* \smile g] \overset{*}{\frown} (\phi \smile g)^* \\ &= \partial(g^* \overset{*}{\frown} \phi^*) + (-1)^k[g^* \smile A[U] \smile g - \delta g^* \smile g] \overset{*}{\frown} (g^* \overset{*}{\frown} \phi^*) \\ &= g^* \overset{*}{\frown} \partial\phi^* + (-1)^k\delta g^* \overset{*}{\frown} \phi^* + (-1)^k(g^* \smile A[U] - \delta g^*) \overset{*}{\frown} (g \overset{*}{\frown} (g^* \overset{*}{\frown} \phi^*)) \\ &= g^* \overset{*}{\frown} (\partial\phi^* + (-1)^k A[U] \overset{*}{\frown} \phi^*) = \left(D^*_{A[U]}\phi \smile g\right)^*. \end{aligned}$$

The formulae involving $\Phi \in C^k(M;\mathbb{C}^{n\times n})$ are proved analogously. Gauge invariance of the Lagrangians not involving j in Table 3 is a straightforward consequence. □

4.5 Proofs of examples

Now we apply the general results of §3 to prove particular results of §2 (except those proved in §4.2).

Proof of Corollary 2.1. This follows directly from Proposition 2.7 applied to the boundary hypersurface of a face and the tensor $T[\phi] = \delta\phi \times \delta\phi$, which is conserved by Theorem 1.3; cf. Remark 2.4. □

Proof of Theorem 2.1. First let us prove the "convergence" of F_N to F. It is convenient to modify the grid slightly. Consider the auxiliary grid M obtained by dissection of I^2 into $(N+1)^2$ equal squares and its dual $N \times N$ grid M' with the vertices at face-centers of M. Consider all the discrete fields in question as defined on M' instead of the initial $N \times N$ grid; this does not affect approximation.

Let F'_N be the function on vertices of M such that $\partial\delta F'_N = 0$ apart $\partial\mathrm{I}^2$ and $F'_N = \mathrm{F}$ on $\partial\mathrm{I}^2$. The restriction of F'_N to nonboundary vertices can be considered as a function on faces of M'. Actually, it is a magnetic field on M' generated by the source s_N (in particular, it exists by Proposition 2.6). Indeed, the condition $\partial\delta F'_N = 0$ implies that it is a magnetic field generated by *some* source. The source is exactly s_N because for each boundary vertex v of the initial $N \times N$ grid we have $F'_N(v_+) - F'_N(v_-) = \mathrm{F}(v_+) - \mathrm{F}(v_-) = \int_{v_-v_+} \mathrm{s}\,d\mathrm{l} = s_N(v)$, where v_-, v, v_+ are in the counterclockwise order along $\partial\mathrm{I}^2$. By Propositions 2.4 and 2.6 the function $F'_N - F_N$ on faces of M' is a constant (depending on N).

By [5, Proposition 3.3] on the set of vertices at distance $\geq r$ from $\partial\mathrm{I}^2$, we have $F'_N(v) \cong \mathrm{F}(v)$ as $N \to \infty$. In particular, for one of the faces f_N closest to $c := (\frac12, \frac12)$ we have $F'_N(f_N) \to \mathrm{F}(c) = 0 = F_N(f_N)$ as $N \to \infty$. Since $F'_N - F_N$ is a constant, it follows that $F_N(f) \cong F'_N(f) \cong \mathrm{F}(\max f) \cong N^2 \int_f \mathrm{F}\,d\mathrm{S}$.

The convergence of $j_N = -\partial F_N$ follows immediately from the second part of [5, Proposition 3.3].

To prove the convergence of ϕ_N, join a vertex v with the vertex u closest to c such that $\phi_N(u) = 0$ by a shortest grid path uv. By the convergence of j_N we get $\phi_N(v) = \sum_{e\subset uv} \langle uv, e\rangle j_N(e) \cong \int_{cv} \vec{\mathrm{j}}\cdot d\vec{\mathrm{l}} = \upphi(v)$.

The convergence of the other fields is a straightforward consequence. For instance, let $e = uv$ be a horizontal edge with the midpoint e' and $f \supset e$ be a face with the center f'. Then

$$NL_N(e'f') = \tfrac{N}{2} j_N(e)F_N(f) \cong F_N(f)\tfrac{N}{2}\int_e \vec{\mathrm{j}}\cdot d\vec{\mathrm{l}} \cong \mathrm{F}(e')\vec{\mathrm{j}}(e')\cdot \tfrac{N}{2}\int_e d\vec{\mathrm{l}} = *\vec{\mathrm{j}}(e')\mathrm{F}(e')\cdot N\int_{e'f'} d\vec{\mathrm{l}} \cong N\int_{e'f'}\vec{\mathrm{L}}\cdot d\vec{\mathrm{l}},$$

$$N^2\sigma_{N,2}(uv) = \tfrac{N^2}{2}[\delta\phi(uv)^2 - \delta\phi(v_-v)\delta\phi(vv_+)] \cong \tfrac12\tfrac{\partial\upphi}{\partial \mathrm{x}_1}(v)^2 - \tfrac12 \tfrac{\partial\upphi}{\partial \mathrm{x}_2}(v)^2 = \upsigma_{22}(v) \cong N\int_e\left(\upsigma_{22}\,d\mathrm{x}^1 - \upsigma_{21}\,d\mathrm{x}^2\right),$$

as required (in the latter formula the notations v_+ and v_- from Definition 2.8 are used). □

Proof of Corollary 2.2. This follows from Theorem 1.1 and Proposition 3.1 for the particular case when $\phi, j \in C^1(I^d_N; \mathbb{R})$, $n = 1$, $U = 1$, hence $D_A\phi = \delta\phi$; see rows 1 and 3 of Table 3. □

Proof of Proposition 2.9. First note that $F_N(f) = \mathrm{F}_{mn}(\max f) \cong \mathrm{F}_{mn}(\max h)$ on the set of all pairs (f, h) having common vertices, because F_{mn} is continuous on I^d, hence uniformly continuous.

Consider the cases when $l = k$ and $l \neq k$ separately.

Assume that $l = k$. For a 1- or 2-dimensional face $f \subset h \perp \mathrm{e}_k$ we have $\dim \mathrm{Pr}(f, k, k) = 0$. Thus

$$\begin{aligned}
(-1)^k\langle T'_N, h\rangle_k &= -\frac12\left[\sum_{f: f\subset h, f\ni \max h, \dim f = 2} T'_N(f\times f) - \sum_{f: f\subset h, f\ni \max h, \dim f = 1} T'_N((f+\mathrm{e}_k)\times(f-\mathrm{e}_k))\right] \\
&= \frac12\left[\sum_{f: f\subset h, f\ni \max h, \dim f = 2} \# F_N(f)F_N(f) - \sum_{f: f\subset h, f\ni \max h, \dim f = 1} \# F_N(f+\mathrm{e}_k)F_N(f-\mathrm{e}_k)\right] \\
&\cong \frac12\left[\sum_{m,n\neq k: m<n} \mathrm{F}^{mn}\mathrm{F}_{mn} - \sum_{m\neq k}\mathrm{F}^{km}\mathrm{F}_{km}\right](\max h) = \left[\frac14\sum_{m,n}\mathrm{F}^{mn}\mathrm{F}_{mn} - \sum_{m}\mathrm{F}^{km}\mathrm{F}_{km}\right](\max h) \\
&= \mathrm{T}^k_k(\max h).
\end{aligned}$$

Assume that $l \neq k$. For a 2-dimensional face $f \parallel \mathrm{e}_k, \mathrm{e}_m$, where $m \neq k, l$, we have $\dim \Pr(f, k, l) = 2$ or 1 depending on if m is between k and l or not. Thus

$$\begin{aligned}(-1)^l \langle T'_N, h\rangle_k &= \frac{1}{2} \sum_{\substack{f: f\subset h, f\ni \max h,\\ \dim f = 2, f\parallel \mathrm{e}_k}} (-1)^{\dim \Pr(f,k,l)} \left[\# F_N(f + \mathrm{e}_l - \mathrm{e}_k)) + \# F_N(f + \mathrm{e}_l + \mathrm{e}_k)\right] F_N(f) \\ &\cong -\sum_{m\neq k} \operatorname{sgn}(m-k)\operatorname{sgn}(m-l)\mathrm{F}^{\min\{l,m\},\max\{l,m\}}(\max h)\mathrm{F}_{\min\{k,m\},\max\{k,m\}}(\max h) \\ &= -\sum_{m\neq k} \mathrm{F}^{lm}(\max h)\mathrm{F}_{km}(\max h) = \mathrm{T}^l_k(\max h). \end{aligned}$$ □

Proof of Corollary 2.3. The Yang–Mills equation follows from Theorem 3.2, Propositions 2.12 and 3.1; see rows 6–7 of Table 3 and Eq. (11). Proposition 3.2 and Theorem 3.5 imply charge conservation. □

Proof of Corollary 2.4. This follows directly from Proposition 3.2 because $\operatorname{Re}\operatorname{Tr}[j^* \frown U]$ is preserved under simultaneous gauge transfornation of U and j; see line 7 of Table 3. □

Proof of Proposition 2.11. Let $abcd$ be a face with the vertices listed in the order compatible with the positive orientation of the face boundary, starting from the minimal vertex. Then

$$\begin{aligned}\operatorname{Re}\operatorname{Tr}\left[\# F^*(abcd)F(abcd)\right] &= \#\operatorname{Re}\operatorname{Tr}\left[(U(abc) - U(adc))^*(U(abc) - U(adc))\right] \\ &= \#\operatorname{Re}\operatorname{Tr}\left[U(cbabc) - U(cdabc) - U(cbadc) + U(cdadc)\right] \\ &= \#\operatorname{Re}\operatorname{Tr}\left[1 - U(abcda) - U(abcda)^* + 1\right] \\ &= 2\#(n - \operatorname{Re}\operatorname{Tr} U(abcda)).\end{aligned}$$

Multiplying by $-1/2$ and summing over all the faces $abcd$, we get the required expression. □

Proof of Proposition 2.12. By the formulas of Lemma 4.5 for F and for $D_A\Phi$ in the case when $U = 1$ and $\Phi = A$, we get $F = (1 + A) \smile (1 + A) = 0 + D_0 A + A \smile A = \delta A + A \smile A$. By Lemma 4.5 and the associativity of the cup-product, $D_A F = U \smile F - F \smile U = U \smile (U \smile U) - (U \smile U) \smile U = 0$. By Lemma 4.5 and the 3rd column of Table 2 we get (10).

Let us prove (11). By Definition 1.1 for each $f \supset e$ we have either $\min f = \min e$ or $\max f = \max e$. Consider a face $f = abcd$ containing $e = ab$ such that $\min f = a$. Then $U(e) - U(\partial f - e) = U(ab) - U(adcb) = (F(abcd)^* U(bc))^*$. Applying # and summing the obtained expression over all such faces f, we get $(\# F^* \frown U)^*$. Analogous sum over all the faces f such that $\max f = b$ gives $(U \overset{*}{\frown} \# F^*)^*$. Then Lemma 4.5 implies (11). □

Proof of Corollary 2.5. This follows from a version of Theorem 1.1 for complex-valued fields and nonfree boundary conditions and the case $U = 1, n = 1$ of Proposition 3.1; see rows 2–3 of Table 3. □

Proof of Corollary 2.6. Since $\mathcal{L}[\phi]$ is globally gauge invariant, it follows that (5) holds for $\Delta = i\phi$. By the versions of Theorems 1.2 and 1.3 for complex-valued fields and nonfree boundary conditions, it follows that the real parts of (6) and (7) are conserved apart the boundary, as required. □

Proof of Proposition 2.13. Since $\upphi$ is C^1, we get $N[\delta\phi_N](e) \cong \partial_l \upphi(\max e)$, $\phi_N(\min e) \cong \upphi(\max e)$,

$$N j_N(e) = 2N \operatorname{Im}[\#\delta\phi_N \frown \phi_N](e) \cong 2 \operatorname{Im}\left[\partial^l \upphi^*(\max e)\upphi(\max e)\right] = \mathrm{j}^l(\max e).$$

For $v := \max h$, by a version of Proposition 2.8 for $\mathbb{C}$-valued fields and rows 2–3 of Table 3, we get

$$\begin{aligned}(-1)^l N^2 \langle T_N, h\rangle_k &= -N^2 \operatorname{Re}\left[\left(\#\delta\phi^*_N(v + \mathrm{e}_l) + \#\delta\phi^*_N(v + \mathrm{e}_l - 2\mathrm{e}_k)\right)\delta\phi_N(v - \mathrm{e}_k)\right] + \\ &\quad + N^2 \updelta^l_k \left[\#\delta\phi_N \frown \delta\phi^*_N + \tfrac{m^2}{N^2}\phi_N \frown \phi^*_N\right](v) \\ &\cong -2\operatorname{Re}\left[\partial^l \upphi^* \partial_k \upphi\right](v) + \updelta^l_k \left[\partial^n \upphi^* \partial_n \upphi + m^2 \upphi^* \upphi\right](v) = \mathrm{T}^l_k(v).\end{aligned}$$ □

Proof of Corollary 2.7. Drop the last term (not depending on ϕ) from the Lagrangian $\mathcal{L}[\phi, U]$. Then by a version of Theorem 3.1 and rows 2–3 of Table 3 the corollary follows. □

Proof of Corollary 2.8. This follows directly from Proposition 3.2; see rows 2–3 of Table 3. □

Proof of Corollary 2.9. This follows from Corollary 2.8, a version of Theorem 3.4 for nonfree boundary conditions, row 3 of Table 3, and the formula for $(\phi\smile\psi)^*$ from Lemma 4.6. □

Proof of Corollary 2.10. For fixed $\phi\in C^0(I_N^d;\mathbb{C}^{1\times n})$ the Lagrangian $\mathcal{L}[\phi,U]=:\mathcal{L}[U]$ from Corollary 2.7 is local with respect to U. By Lemma 4.9 and row 7 of Table 3 we get $\frac{\partial\mathcal{L}[U]}{\partial U}=j[\phi,U]^*$ and $\frac{\partial\mathcal{L}[U]}{\partial(F[U])}=\#F^*$, where $j[\phi,U]$ is given by Corollary 2.9. Let U_0 be stationary for the functional $\mathcal{S}[\phi,U]=\epsilon\mathcal{L}[U]$. By Theorem 3.2 U_0 satisfies the Yang–Mills equation from Corollary 2.3 with $j=j[\phi,U_0]$. Then again by Theorem 3.2 U_0 is stationary for $\mathcal{S}[U]$ from Definition 2.17, where $j=j[\phi,U_0]$ is fixed (i.e., one keeps $j=j[\phi,U_0]$ rather than $j=j[\phi,U]$ under a variation of U). Thus U_0 is generated by $j[\phi,U_0]$. The reciprocal assertion is proved analogously. □

Proof of Corollaries 2.11 and 2.13. Let us prove Corollary 2.13; 2.11 is a particular case. Drop the last term (not depending on ψ) from the Lagrangian $\mathcal{L}[\psi,U]$. By a version of Theorem 3.1, a field $\psi\in C^0(I_N^4;\mathbb{C}^{4\times n})$ is stationary for the functional $\mathcal{S}[\psi]$, if and only if the following expression vanishes:

$$D_A^*\left(\frac{\partial\mathcal{L}}{\partial(D_A\psi)}\right)^*+\left(\frac{\partial\mathcal{L}}{\partial\psi}\right)^* = D_A^*(-i\gamma^0\gamma\smile\psi)+i\gamma^0\gamma\frown D_A\psi-2m\gamma^0\psi = i\gamma^0\gamma\overset{*}{\frown}\bar{D}_A\psi+i\gamma^0\gamma\frown D_A\psi-2m\gamma^0\psi.$$

Left-multiplying by $(\gamma^0)^{-1}$, we get the Dirac equation in a gauge field. Here the 1st equality is obtained by rows 4–5 of Table 3 and the 2nd one follows from

$$(D_A^*(\gamma\smile\psi))^*=\partial(\psi^*\overset{*}{\frown}\gamma^*)-A\overset{*}{\frown}(\psi^*\overset{*}{\frown}\gamma^*)=\psi^*\overset{*}{\frown}\partial\gamma^*-\delta\psi^*\overset{*}{\frown}\gamma^*-(A\smile\psi^*)\overset{*}{\frown}\gamma^*=-(\gamma\overset{*}{\frown}\bar{D}_A\psi)^*,$$

where we used the obvious identity $\partial\gamma^*=0$, equations (15)–(16), and Lemma 4.6. □

Proof of Proposition 2.14. Let the *Dirac operator* on the doubling act by $\not\partial\psi=\gamma\frown\delta\psi+\gamma\overset{*}{\frown}\delta\psi$ for each $\mathbb{C}^{4\times 1}$-valued field ψ on the vertices of the doubling. Then the Dirac equation is $i\not\partial\psi-2m\psi=0$. Applying the operator $i\not\partial+2m$ to the left-hand side and canceling the $\pm im\not\partial$-terms we get $\not\partial\not\partial\psi+4m^2\psi=0$. It remains to prove the identity $\not\partial\not\partial=\partial_{\mathrm{initial}}\#\delta_{\mathrm{initial}}$, where $\partial_{\mathrm{initial}}$ and $\delta_{\mathrm{initial}}$ are the boundary and coboundary operators respectively on the initial grid I_N^4.

Take a nonboundary vertex v of I_N^4. By the identity $\gamma^k\gamma^l+\gamma^l\gamma^k=-8\,\mathrm{sgn}\langle \mathrm{e}_k,\mathrm{e}_l\rangle$ we get

$$\begin{aligned}[\not\partial\not\partial\psi](v)&=\sum_{k=0}^{3}\gamma^k[\not\partial\psi(v+\mathrm{e}_k)-\not\partial\psi(v-\mathrm{e}_k)]\\&=\sum_{k,l=0}^{3}\gamma^k\gamma^l[\psi(v+\mathrm{e}_k+\mathrm{e}_l)-\psi(v+\mathrm{e}_k-\mathrm{e}_l)-\psi(v-\mathrm{e}_k+\mathrm{e}_l)+\psi(v-\mathrm{e}_k-\mathrm{e}_l)]\\&=\sum_{k=0}^{3}(-4)\mathrm{sgn}\langle\mathrm{e}_k,\mathrm{e}_k\rangle[\psi(v+2\mathrm{e}_k)-2\psi(v)+\psi(v-2\mathrm{e}_k)]=[\partial_{\mathrm{initial}}\#\delta_{\mathrm{initial}}\psi](v).\end{aligned}$$ □

Remark 4.2. The actions from Corollaries 2.11 and 2.13 can be written as $\mathcal{S}[\psi,U]=\epsilon\mathcal{L}'[\psi,U]$, where

$$\mathcal{L}'[\psi,U]=\mathrm{Re}\,\mathrm{Tr}\left[\bar{\psi}\frown(i\not{D}_A\psi-m\psi)-\tfrac{1}{2}\#F^*\frown F\right]$$

and $\not{D}_A\psi:=\gamma\frown D_A\psi+\gamma\overset{*}{\frown}\bar{D}_A\psi$ so that $\not{D}_0=\not\partial$. But in contrast to $\mathcal{L}[\psi,U]$, the Lagrangian $\mathcal{L}'[\psi,U]$ is nonlocal with respect to the gauge group field U.

Proof of Corollary 2.12. Since $\mathcal{L}[e^{-it}\psi]=\mathcal{L}[\psi]$ for each $t\in\mathbb{R}$, we have (5) with $\Delta=-i\psi$. Then by a version of Theorem 1.2 for complex-valued fields, row 5 of Table 3 for $U=1$, the identity $(\gamma^0\gamma)^*=\gamma^0\gamma$, and the formula for $(\phi\smile\psi)^*$ from Lemma 4.6 we have the conserved current

$$j[\psi]=\mathrm{Re}\left[\frac{\partial\mathcal{L}[\psi]}{\partial(\delta\psi)}\frown\Delta\right]=\mathrm{Re}\left[(-i\gamma^0\gamma\smile\psi)^*\frown(-i\psi)\right]=\mathrm{Re}\left[(\bar{\psi}\smile\gamma\smile\psi)^*\right]=\mathrm{Re}\left[\bar{\psi}\smile\gamma\smile\psi\right].$$

The conservation of $j^5[\psi]$ is proved analogously, only take $\Delta=i\gamma^5\psi$ and apply the identities $(\gamma^5)^*=\gamma^5$ and $\gamma^5\gamma^0=-\gamma^0\gamma^5$. The conservation of $T[\psi]$ follows from Theorem 1.3 and rows 4–5 of Table 3. □

Proof of Proposition 2.15. Since ψ is C^1, we get $\psi(\min e) \cong \psi(\max e)$, $N\delta\psi_N(e) \cong \partial_l\psi(\max e)$, and

$$j_N(e) = \mathrm{Re}[\bar\psi_N \smile \gamma \smile \psi_N](e) = \mathrm{Re}\left[\bar\psi(\min e)\gamma^l\psi(\max e)\right] \cong \mathrm{Re}\left[\bar\psi(\max e)\gamma^l\psi(\max e)\right] = \mathrm{j}^l(\max e).$$

For $v = \max h$, by a version of Proposition 2.8 for vector-valued fields and rows 4–5 of Table 3,

$$\begin{aligned}(-1)^l N\langle T_N, h\rangle_k &= \tfrac{N}{2}\mathrm{Re}\left[([-i\gamma_0\gamma \smile \psi_N]^*(v+\mathrm{e}_l) + [-i\gamma_0\gamma \smile \psi_N]^*(v+\mathrm{e}_l-2\mathrm{e}_k))\right]\delta\psi_N(v-\mathrm{e}_k)\\ &\quad - N\delta^l_k\mathrm{Re}\left[\bar\psi_N \frown \left(i\gamma \frown \delta\psi_N - \tfrac{m}{N}\psi_N\right)\right](v)\\ &\cong \mathrm{Re}\left[i\bar\psi\gamma^l\partial_k\psi - \delta^l_k\left(i\bar\psi\gamma^n\partial_n\psi - m\bar\psi\psi\right)\right](v) = \mathrm{T}_k{}^l(v).\end{aligned}$$ □

Proof of Corollary 2.14. This follows directly from Proposition 3.2; see rows 4,5,7 of Table 3. □

Proof of Corollary 2.15. By Corollary 2.14 and Theorem 3.4 we get the conserved covariant current

$$j[\psi] = \left(\frac{\partial\mathcal{L}[\psi,U]}{\partial(D_A\psi)} \frown \psi\right)^* = \left((-i\gamma^0\gamma \smile \psi)^* \frown \psi\right)^* = -\bar\psi \smile i\gamma \smile \psi.$$ □

Proof of Corollary 2.16. For fixed $\psi \in C^0(I_N^d; \mathbb{C}^{4\times n})$ the Lagrangian $\mathcal{L}[\psi, U] =: \mathcal{L}[U]$ from Corollary 2.13 is local with respect to U. By Lemma 4.9 and row 7 of Table 3 we get $\frac{\partial\mathcal{L}[U]}{\partial U} = j[\psi]^*$ and $\frac{\partial\mathcal{L}[U]}{\partial(F[U])} = \#F^*$, where $j[\psi]$ is given by Corollary 2.15. Let U be stationary for $\mathcal{S}[\psi, U] = \epsilon\mathcal{L}[U]$. By Theorem 3.2 U satisfies the Yang–Mills equation from Corollary 2.3 with $j = j[\psi]$. Again by Theorem 3.2 U is stationary for $\mathcal{S}[U]$ from Proposition 2.11, i.e., U is generated by $j = j[\psi]$. The reciprocal assertion is proved analogously. □

5 Open problems

- Expand the suggested discretization algorithm to:
  - quantum field theories via path integral formalism;
  - general relativity via discretizing the raising index operator ♯ for nonflat spacetimes;
  - hydrodynamics via discretizing the fluid energy-momentum tensor.
- Extend the suggested discretization algorithm to involve the following conservation laws:
  - energy conservation in nontrivial connection via making the cross-product gauge invariant;
  - angular momentum conservation via discretizing the radius vector;
  - integral-form energy conservation in general complexes via discretizing tensor integration.
- Prove the conservation of the discrete covariant chiral current. Generally, is the covariant current from Theorem 3.3 times i conserved for each gauge invariant Lagrangian satisfying (5)?
- Prove analogous conservation laws in statistical field theory. E.g., is the expectation of a covariant current conserved, if the gauge group field is random with the probability density proportional to the exponential of the action from Definition 2.17?
- Apply the discretization algorithm to characteristic classes to obtain invariants of piecewise-linear homeomorphisms or rational homotopy type.
- Constuct a "second-generation" discretization algorithm for field theories, in which not only spacetime, but also the set of field values becomes discrete; e.g., as in the Feymann checkerboard.
- Prove that the discussed discrete field theories approximate continuum ones in a sense. Even no analogue of Theorem 2.1 for planar graphs with faces not being inscribed is known [5, 24].
- State and prove a "reciprocal Noether theorem" giving a symmetry of the continuum limit for each discrete conservation law.

- Find an experimentally measurable quantity in our discretization not converging to the continuum counterpart; this would make the discretization *falsifiable* against the continuum theory.

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