Riemannian geometry

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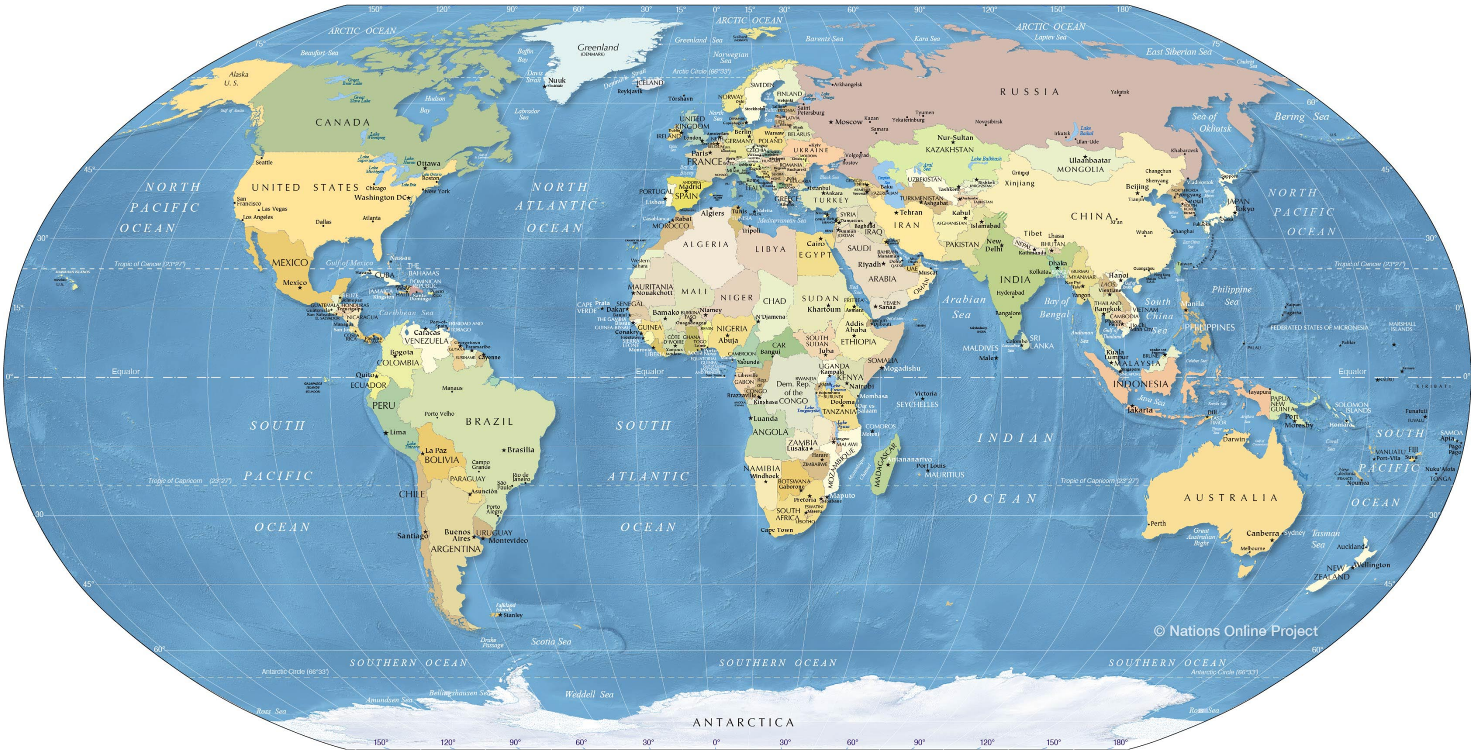
The objective of this course is to introduce various methods of Riemannian geometry. We will solve basic visual problems in Riemannian geometry on a mathematical level of rigor and thus will become competent in its basic notions, tools, general principles, and applications.

Contents

1. Spherical geometry

1.1. a) What oceans does the Beijing-New York plane fly over?

b) Where will you end up if you keep moving northwest?



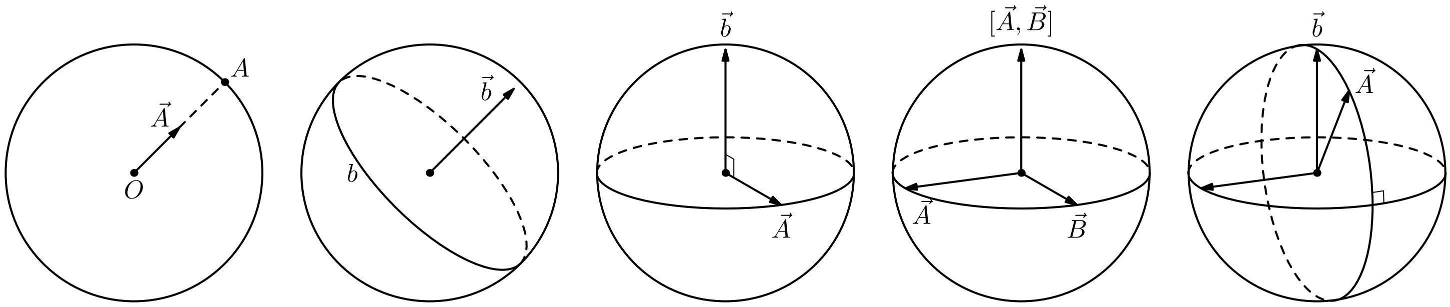
Consider the unit sphere centered at the origin O of three-dimensional space. *A great circle* (*spherical straight line*) is a section of this sphere by an arbitrary plane, passing through O.

1.2. a) Two arcs of great circles join two diametrically-opposite points of a unit sphere and meet at angle α. Find the area bounded by the two arcs.

b) The area of a spherical triangle with angles α, β, γ on a unit sphere equals $\alpha+\beta+\gamma-\pi$.

The *distance* between two points on the sphere is the minimal length of an arc of a great circle joining the points. The *circle of radius R with the center A on the sphere* is the set of points on the sphere at the distance R from A.

1.3. Find the length of a circle of radius R on a unit sphere and the area bounded by it.



a b c d e

Figure 1: Points, straight lines on the sphere, and vectors

To each nonzero space vector assign two objects: a point on the sphere and a great circle. First, to a vector we assign the intersection point of the sphere with the ray emanating from the point O in the direction of the vector (Fig. 1a). Second, we assign the section of the sphere with the plane, passing through the point O and orthogonal to the vector (Fig. 1b). Point (or spherical line) assigned to the given vector is denoted by the same letter as the vector itself, only without vector sign. Denote by $[\vec{A}, \vec{B}]$ the cross product of vectors $\vec{A}$ and $\vec{B}$.

1.4. a) If the vectors $\vec{A}$ and $\vec{b}$ are orthogonal, then the point A lies on the spherical line b (Fig 1c).
b) The vector $[\vec{A}, \vec{B}]$ corresponds to a spherical line passing through A and B (Fig. 1d).
c) Vector $[\vec{A}, \vec{b}]$ (if nonzero) corresponds to the perpendicular dropped from point A to spherical line b (Fig. 1e).
d) If $\vec{a} + \vec{b} + \vec{c} = 0$, then the spherical lines a, b, c have at least two common points.
e) Prove the *'bac minus cab' identity* : $[\vec{A}, [\vec{B}, \vec{C}]] = \vec{B}(\vec{A}, \vec{C}) - \vec{C}(\vec{A}, \vec{B})$.
f) Prove the *Jacobi identity*: $[\vec{A}, [\vec{B}, \vec{C}]] + [\vec{B}, [\vec{C}, \vec{A}]] + [\vec{C}, [\vec{A}, \vec{B}]] = 0$.
g) (*V.I. Arnold*) Let A, B, and C be vertices of a spherical triangle. What does the Jacobi identity mean geometrically for vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$?

2. Curves

2.1. Draw approximately the following trajectories and curves in the plane or in space. Find their "equations" $r(t) = (x(t), y(t))$ or $r(t) = (\rho(t), \varphi(t))$ in Cartesian or polar coordinates on the plane; $r(t) = (x(t), y(t), z(t))$ or $r(t) = (\rho(t), \varphi(t), z(t))$ in Cartesian or cylindrical coordinates in space. Choose your own coordinate system. All velocities in this problem are assumed to be nonzero.

a) *Parabola* — the set of points in the plane equidistant from the given line (*directrix*) and the given point (*focus*).

b) Ray OA rotates uniformly about its fixed origin O with angular velocity ω. A point M moves uniformly along the ray OA, starting from the point O, with the speed v. The trajectory described by the point M is called the *Archimedes' spiral*.

c) A *helix* is the trajectory of the end of a rod of length $2r$, uniformly with the speed of v falling to the ground, remaining parallel to the ground and simultaneously rotating in a horizontal plane around its midpoint uniformly with the angular velocity ω.

d) A wheel of radius R rolls uniformly without slip along a straight line. The trajectory described by a point on the wheel rim is called a *cycloid*.

A *parametrized regular smooth curve* in space is an infinitely differentiable map $r\colon [a, b] \to \mathbb{R}^3$ such that $r'(t) \neq 0$ for each $t \in [a, b]$.

A *non-parametrized smooth curve*, or just a *curve*, is a subset $\Pi \subset \mathbb{R}^3$ such that for each point $P \in \Pi$ there is a closed neighborhood N_P in $\mathbb{R}^3$ such that $\Pi \cap N_P$ is the image $r([a, b])$ of some injective parametrized regular smooth curve $r\colon [a, b] \to \mathbb{R}^3$.

A *parametrization* of a non-parametrized curve $\Pi \subset \mathbb{R}^3$ is a parametrized curve $r\colon [a, b] \to \mathbb{R}^3$ such that $\Pi = r([a, b])$.

2.2. a) Give an example of two different parametrizations of the same circle.

b) Is the image of a parametrized regular smooth curve always a non-parametrized curve?

The *length* of a parametrized curve $r\colon [a, b] \to \mathbb{R}^3$ is $L(r) := \int_a^b |r'(t)|\, dt$.

2.3. a–d) Calculate the lengths of parameterized curves from $r(a)$ to $r(b)$ for the parameterizations you have chosen in Problem 2.1 (specify the parameterization explicitly!)

The *length* of nonparametrized curve is the length of some its one-to-one parametrization.

2.4. (a) If $r_1, r_2 : [a, b] \to \mathbb{R}^2$ are one-to-one parametrizations of one nonparametrized curve, then the mapping $r_2^{-1} \circ r_1 : [a, b] \to [a, b]$ is one-to-one and monotone.

(b) **Theorem.** The length of a nonparametrized curve is well-defined, i.e. if r_1 and r_2 are two one-to-one parametrizations of the same nonparametrized curve, then $L(r_1) = L(r_2)$.

3. Surfaces

Let D be a rectangle in the plane.

A *parametrized regular smooth surface* is an infinitely differentiable map $r\colon D \to \mathbb{R}^3$ such that the vectors $\partial r/\partial u$ and $\partial r/\partial v$ are linearly independent for each $(u, v) \in D$.

A *non-parametrized smooth surface*, or just a *surface*, is a subset $\Pi \subset \mathbb{R}^3$ such that for each point $P \in \Pi$ there is a closed neighborhood N_P in $\mathbb{R}^3$ such that $\Pi \cap N_P$ is the image $r(D)$ of some injective parametrized smooth surface $r\colon D \to \mathbb{R}^3$.

A *system of coordinates* on a non-parametrized surface $\Pi \subset \mathbb{R}^3$ is a parametrized surface $r\colon D \to \mathbb{R}^3$ such that $r(D) \subset \Pi$.

3.1. Find out which of the following subsets of $\mathbb{R}^3$ are surfaces (and prove your answers):

a) a square in a plane;

b) the lateral surface of a right circular cylinder of hight 1;

c) the lateral surface of a right circular cone of hight 1;

d) the lateral surface of a right circular truncated cone;

e) the boundary of a cube;

f) a sphere;

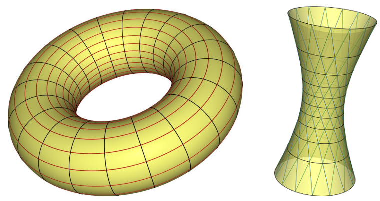
g) a torus, i.e., the result of rotation of a circle about a line lying in the plane of the circle and disjoint with the circle (see Fig. 2a);

h) the hyperboloid $z^2 = x^2 + y^2 - 1$ of one sheet (see Fig. 2b);

i) the surface of revolution of the graph of an infinitely differentiable positive function $\mathbb{R} \to \mathbb{R}$ about the x-axis;

j) the graph of an infinitely differentiable function $D \to \mathbb{R}$;

k) the preimage of 0 under an infinitely differentiable function $\mathbb{R}^3 \to \mathbb{R}$ with nowhere vanishing gradient.



a b

Figure 2: A torus and a hyperboloid of one sheet

The *tangent plane* $T_P\Pi$ to a surface Π at a point P is the union of tangent lines at P to all curves on Π passing through P.

3.2. The tangent plane $T_P\Pi$ is indeed a plane; it contains the vectors $\frac{\partial r}{\partial u}(r^{-1}(P))$ and $\frac{\partial r}{\partial v}(r^{-1}(P))$, where $r\colon D \to \mathbb{R}^3$ is a system of coordinates on Π covering P.

3.3. Prove that the hyperboloid $z^2 = x^2 + y^2 - 1$ of one sheet contains infinitely many lines.

4. Curves on surfaces and isometries

4.1. Express the length of a curve $[a, b] \overset{(u,v)}{\to} D \overset{r}{\to} \mathbb{R}^3$

a) on the cylinder $r(u, v) = (\cos u, \sin u, v)$;

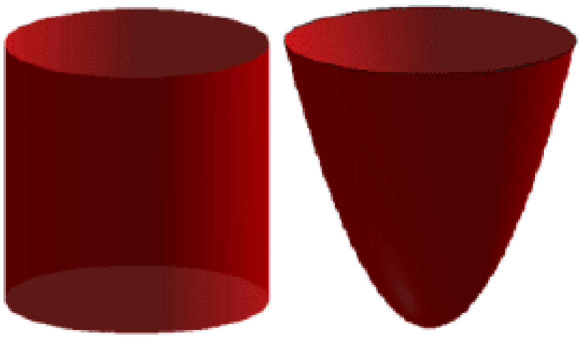
b) on the sphere $r(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$

through the functions u and v (the formula may contain derivatives and integrals; Fig. 3a).

4.2. Prove that the shortest curve between the north pole N and a point A on the sphere is the arc of the meridian joining N and A. (*Hint*: use spherical coordinates; see Problem 4.1b.)

The *angle* between two intersecting parametrized curves at their common point is the angle between their tangents at the point.

4.3. Find the angle between the images of curves $v = u + 1$ and $v = 3 - u$ under the map $r(u, v) = (v \cos u, v \sin u, v^2)$, which is a system of coordinates on the paraboloid $z = x^2 + y^2$ (Fig. 3b).



a b

Figure 3: A cylinder and a paraboloid

Two surfaces are *intrinsically isometric*, if there is an *intrinsic isometry* between them, i.e., a smooth one-to-one map preserving the lengths of all curves on the surfaces. Two subsets of $\mathbb{R}^3$ are *ambient isometric*, if there is an *ambient isometry* between them, i.e., a map $\mathbb{R}^3 \to \mathbb{R}^3$ preserving the distances in $\mathbb{R}^3$ and taking one subset to the other one.

4.4. An arbitrary rectangle in the plane:

a) is intrinsically isometric to some subset of the lateral surface of arbitrary cylinder with the diameter greater than the side of the rectangle.

b) is not ambient isometric to no subset of the lateral surface of no cylinder.

4.5. a) A sphere and a plane are not intrinsically isometric.

b) A disk in the plane is not intrinsically isometric to no subset of no sphere.

c) A spherical cap is not intrinsically isometric to no subset of a sphere of different radius.

5. Geodesics

The *distance along the surface* Π between points $P, Q \in \Pi$ is the infimum $|P, Q|$ of the lengths of curves lying on the surface and joining P and Q.

A (non-parametrized) curve $\Gamma \subset \Pi$ is called a *geodesic* on Π, if Γ is *locally shortest*, i.e., if each point $P \in \Gamma$ has a neighborhood $N_P \subset \Pi$ such that the distance (along Π) between any two points $P_1, P_2 \in N_P \cap \Gamma$ is equal to the length of a segment of Γ from P_1 to P_2.

5.1. In this problem (similarly defined) geodesics on surfaces of polytopes are considered.

a) Draw a geodesic on a cube joining its opposite vertices.

b) The sum of face angles of a polyhedral angle is not greater than 2π.

c) A geodesic on the surface of a convex polytope does not pass through the vertices (i.e., it can only start and finish there) and crosses the edges according to the law 'the angle of incidence is equal to the angle of reflection'.

5.2. a) The shortest curve on the surface joining two given points is a geodesic.

b) A geodesic is not necessarily the shortest curve.

c) Find at least one geodesic on the hyperboloid $z^2 = x^2 + y^2 - 1$ of one sheet.

d) An intrinsic isometry takes geodesics to geodesics.

e)* During the motion along a geodesic, the left and the right wheels of a car of width ε travel the same length up to $O(\varepsilon^2)$.

f)* Draw an arbitrary curve on the plane. Roll an egg (i.e., a convex body with smooth boundary) along the plane following the curve (without slipping). The resulting curve on the egg is a geodesic if and only if the curve on the plane is a straight line.

5.3. * **Theorem.** The image of a parametrized curve $\gamma\colon [a, b] \to \Pi$ having constant speed is a geodesic on a surface Π, if and only if $\gamma''(t) \perp T_{\gamma(t)}\Pi$ for each $t \in [a, b]$.

A parametrized curve $\gamma\colon [a, b] \to \Pi$ is a *parametrized geodesic* on a surface Π, if $\gamma''(t) \perp T_{\gamma(t)}\Pi$ for each $t \in [a, b]$.

5.4. Find all the geodesics on the surfaces from Problems 3.1abdf (a parametrization is sufficient for d).

5.5. a) A straight line on a surface is a geodesic.

b) A meridian of a surface of revolution is a geodesic.

c) A parallel of a surface of revolution is a geodesic if and only if the tangent to the meridian at each point of the parallel is parallel to the rotation axis.

d)* **Klero's Theorem.** For a parametrized geodesic $\gamma\colon [a, b] \to \Pi$ on a surface Π of revolution, the value $r(t) \sin \angle(\gamma'(t), m(t))$ is constant. Here $r(t)$ is the distance to the rotation axis and $m(t)$ is the meridian through the point $\gamma(t)$.

6. Equation of geodesics and the exponential mapping

6.1. Write a differential equation of the parametrized geodesics on the surface $z = xy$.

6.2. Let $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ and $\gamma = r(x_1, x_2)$ be a parameterized curve on the parametrized surface $r : D \to \mathbb{R}^3$. We will further skip the argument $(x_1(t), x_2(t))$ of the function r and its derivatives. Primes will denote the differentiation with respect to t, and indices of r (but not of Γ and g) will denote the partial differentiation with respect to corresponding variable.

a) The curve γ is a parameterized geodesic if and only if $\gamma'' \cdot r_1 = \gamma'' \cdot r_2 = 0$ for any t.

b) $\gamma' = r_1 x_1' + r_2 x_2'$ is the velocity vector of the curve γ at the point $r(x_1(t), x_2(t))$.

c) $\gamma'' = x_1'' r_1 + x_1' r_1' + x_2'' r_2 + x_2' r_2'$.

d) $r_1' = r_{11} x_1' + r_{12} x_2'$.

e) **Theorem.** The parametrized geodesics on the surface $r(D)$ are exactly the solutions $x_1(t)$ and $x_2(t)$ of the following system of differential equations:

$$\begin{cases} -x_1'' = \Gamma^1_{11}(x_1')^2 + (\Gamma^1_{21} + \Gamma^1_{12})x_1'x_2' + \Gamma^1_{22}(x_2')^2 \\ -x_2'' = \Gamma^2_{11}(x_1')^2 + (\Gamma^2_{21} + \Gamma^2_{12})x_1'x_2' + \Gamma^2_{22}(x_2')^2 \end{cases} \quad \text{or} \quad x_k'' + \sum_{i,j} \Gamma^k_{ij} x_i' x_j' = 0, \quad \text{where}$$

$$g_{ij} = r_i \cdot r_j \quad \text{and} \quad \Gamma^k_{ij} := \frac{g_{3-k,3-k} r_k \cdot r_{ij} - g_{k,3-k} r_{3-k} \cdot r_{ij}}{\det g} \quad \text{are the } \textit{Christoffel symbols}.$$

6.3. a) **Corollary.** Exactly one parametrized geodesic passes through every point in every direction on the surface with any given speed at the starting point.

b) Compute the Christoffel symbols for the spherical coordinate system on the sphere.

Let Π be proper. Define the *(geodesic) exponential map*

$$\exp = \exp_P : T_P\Pi \to \Pi \quad \text{by} \quad \exp(u) := \gamma_{P,u}(1),$$

where $\gamma_{P,u} : [-1, 1] \to \Pi$ is the parametrized geodesic for which $\gamma_{P,u}(0) = P$ and $\gamma'_{P,u}(0) = u$.

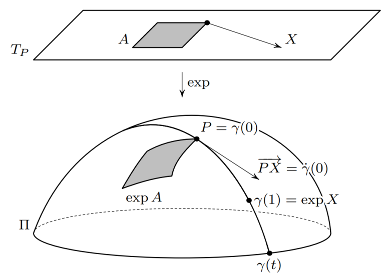


Figure 4: Exponential map

6.4. a) For the unit sphere S^2, the exponential mapping $T_{(0,0,1)} \to S^2$ sends polar coordinates to spherical ones:

$$\exp(\rho\cos\varphi, \rho\sin\varphi) = (\sin\rho\cos\varphi, \sin\rho\sin\varphi, \cos\rho).$$

b) Generalize this result for surfaces of revolution.

6.5. a) The exponential mapping $\exp_P$ takes each line passing through P to a geodesic.

b) Does the exponential mapping $\exp_P$ always take an arbitrary line to a geodesic?

c) Does the exponential mapping $\exp_P$ always take a circle (in T_P) of radius R centered at P to a circle (on the surface) of radius R centered at P?

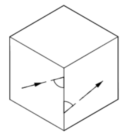
d) **Gauss lemma.** The exponential mapping $\exp_P$ takes any line passing through P and any circle (in T_P) centered at P to two orthogonal curves.

6.6. a) There is $r_P > 0$ such that (the restriction of) the exponential mapping $\exp_P$ from the disk of radius r_P centered at P in T_P onto the analogous disk on the surface is one-to-one.

b) For each point Q within distance at most r_P from P, the shortest curve joining P and Q on the surface is the image of the parametrized geodesic. (Return to Problem 5.3 here).

7. Parallel transport

7.1. Take a cube. Two vectors lying in adjacent faces are *parallel*, if they form equal "vertical" angles with the common side of the faces (i.e., their directions become the same, if one "unfolds" the two faces around the common side; see the figure). Let f_1, f_2, f_3 be 3 faces with a common vertex. Take any vector $\vec{e}_1 \subset f_1$, then a vector $\vec{e}_2 \subset f_2$ parallel to $\vec{e}_1$, then $\vec{e}_3 \subset f_3$ parallel to $\vec{e}_2$, finally $\vec{e}_4 \subset f_1$ parallel to $\vec{e}_3$. What is the angle between $\vec{e}_4$ and $\vec{e}_1$?



Let $\Pi \subset \mathbb{R}^3$ be a surface and $\gamma\colon [a,b] \to \Pi$ be a parametrized curve. A vector field $v\colon [a,b] \to \mathbb{R}^3$ is *parallel along the curve* γ (in the sense of Levi–Civita), if $v(t)$ is tangent to and $v'_t(t)$ is perpendicular to the surface at the point $\gamma(t)$ for any t.

The vector $v(b)$ is *obtained from the vector* $v(a)$ *by translation along the given curve.*

7.2. a) Which tangent vectors to a plane in space are obtained from each other by translation?

b) The velocity vector field of a parameterized curve on the surface is parallel along the curve if and only if the curve is a parametrized geodesic.

c) Parallelism along a parametrized curve *with given image* does not depend on the curve.

d) Result of parallel translation along curve *with given endpoints* may depend on the curve.

A vector field on a surface is *parallel along a nonparametrized curve*, if it is parallel along any of its parametrizations (see Problem 7.2c).

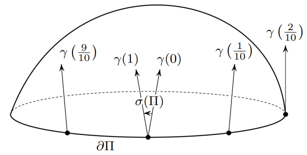


Figure 5: Parallel field

7.3. On a meridian of a surface of revolution, a continuous family of unit vectors tangent to parallels is parallel along the meridian.

7.4. If both vector fields $u(t)$ and $v(t)$ are parallel along the same parametrized curve, then

a) $|v(a)| = |v(b)|$. b) $u(a)\cdot v(a) = u(b)\cdot v(b)$. c) $\angle(u(a), v(a)) = \angle(u(b), v(b))$.

d) The vector fields $u+v$ and $3u$ are parallel along the same parametrized curve.

7.5. a) **Theorem.** Parallel translation along a given curve on a given surface defines an orthogonal mapping of tangent spaces.

b) A vector field $v(t)$ is parallel along a parametrized geodesic $\gamma(t)$ if and only if both $|v(t)|$ and $\angle(v(t), \gamma'(t))$ are constant along the geodesic.

7.6. Let $v(t) = a_1(t) r_1\big(x_1(t), x_2(t)\big) + a_2(t) r_2\big(x_1(t), x_2(t)\big)$ be a vector tangent to the surface at the point $r(x_1(t), x_2(t))$. Use notation of Problem 6.2.

a) $v' = a_1' r_1 + a_1 r_1' + a_2' r_2 + a_2 r_2'$.

b) $v(t)$ is parallel along the curve $\gamma = r(x_1, x_2)$ if and only if $v'\cdot r_1 = v'\cdot r_2 = 0$.

c) **Theorem.** A vector field $v(t)$ is parallel along the curve $\gamma = r(x_1, x_2)$ if and only if

$$\begin{cases} -a_1' = (\Gamma^1_{11} x_1' + \Gamma^1_{21} x_2') a_1 + (\Gamma^1_{12} x_1' + \Gamma^1_{22} x_2') a_2 \\ -a_2' = (\Gamma^2_{11} x_1' + \Gamma^2_{21} x_2') a_1 + (\Gamma^2_{12} x_1' + \Gamma^2_{22} x_2') a_2 \end{cases} \qquad \text{or} \quad a_k' + \sum_{i,j} \Gamma^k_{ij} x_i' a_j = 0.$$

d) Any vector can be translated along any curve.

7.7. What angle does a tangent vector rotate through during the parallel translation along the parallels on

a) a cylinder; b) the cone $z^2 = x^2 + y^2$; c) the unit sphere?

8. Area and scalar curvature

Let $r\colon D \to \Pi$ be a 1–1 parametrization of a nonparametrized surface $\Pi \subset \mathbb{R}^3$. The *area* of Π is $S(r(D)) := \int\int_D |r_u \times r_v| du dv$.

8.1. a) Find the area of the intersection of the cylinder $x^2 + y^2 \leq 1$ and the surface $z = xy$.

b) The area of $r(D)$ does not depend on the choice of a 1–1 parametrization r.

c) The area of the surface formed by rotating the graph of a function $f\colon [a, b] \to [0, +\infty)$ around the Ox axis is equal to $2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$.

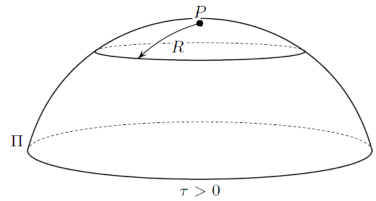
d) *Hulden's First Theorem.* If the surface is formed by a rotation around the axis of some curve lying in one plane with the axis entirely on one side of the axis, then the surface area is equal to the product of the length of the curve and the length of the circle described by the center of mass of the curve.

The *disc* $B_{\Pi,P}(R)$ *on the surface* Π *with the center* $P \in \Pi$ *and the radius* R is the set of points of Π within distance R from P. Denote by $S_{\Pi,P}(R)$ the area of the disc and by $L_{\Pi,P}(R)$ the length of its boundary *circle*.

8.2. a) A circle of sufficiently small radius on Π is a curve (hence its length is well-defined).

b) A disk of sufficiently small radius on a surface is a surface (hence its area is well-defined).

c) For sufficiently small R we have $L_{\Pi,P}(R) = S'_{\Pi,P}(R)$.



The *scalar curvature* of the surface Π at an interior point P is

$$\tau = \tau_{\Pi,P} := 6 \lim_{R\to 0} \frac{2\pi R - L_{\Pi,P}(R)}{\pi R^3}.$$

8.3. Find the scalar curvature at points of surfaces from Problem 3.1abdf.

8.4. a) How does the scalar curvature change under spatial homothety?

b) $\tau = 24 \lim_{R\to 0} \dfrac{\pi R^2 - S_{\Pi,P}(R)}{\pi R^4}$.

c) An intrinsic isometry preserves scalar curvature.

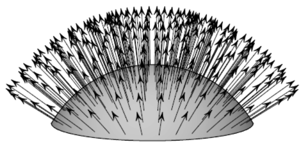
d)* The limit defining the scalar curvature exists.

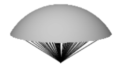
9. Gaussian curvature

A *coorientation* of a surface Π is the family of unit vectors $n(P)$ normal to the surface (that is, perpendicular to $T_P\Pi$) and continuously depending on the point $P \in \Pi$.

The mapping $n\colon \Pi \to S^2 \subset \mathbb{R}^3$ is called *spherical* or *Gaussian.* The set $n(\Pi) \subset S^2 \subset \mathbb{R}^3$ is called the *spherical* or *Gaussian image* of the cooriented surface Π.

9.1. Equip the surfaces in Problem 3.1abdfg with coorientations. Find their spherical images.





If $n\colon \Pi \to n(\Pi)$ is 1–1, then *the area of the spherical mapping (with a sign)* or *total Gaussian curvature* $K(\Pi)$ is the area of the spherical image with a plus (respectively, a minus) sign, if when going around the boundary of the surface clockwise (viewed from to the normals), the boundary of the spherical image is passed clockwise (respectively, counterclockwise).

If $n\colon \Pi \to n(\Pi)$ is 1–1 outside the boundary, then the sign of the area is defined analogously, only we go around a curve on Π close to the boundary. If Π can be decomposed into pieces $\Pi_1, \dots, \Pi_m$ with $n\colon \Pi_i \to n(\Pi_i)$ being 1–1 outside the boundary, then set $K(\Pi) := K(\Pi_1) + \dots + K(\Pi_m)$. If $S(n(\Pi)) = 0$, then set $K(\Pi) := 0$.

9.2. Find the total Gaussian curvatures of the surfaces in Problem 3.1abdfg.

9.3. The total Gaussian curvature $K(\Pi)$ does not depend on n, if $n\colon \Pi \to n(\Pi)$ is 1–1.

In general, the *area* of a parametrized surface (not necessarily 1–1!) with coorientation $n\colon r(D) \to \mathbb{R}^3$ is $S(r,n) := \int\int_D r_u \wedge r_v \wedge n\, dudv$, where we write n instead of $n(r(u,v))$ hereafter. The *total Gaussian curvature* $K(\Pi)$ of the surface $\Pi = r(D)$ is the area of the parametrized surface $n \circ r\colon D \to S^2$ with the coorientation n, where $n = r_u \times r_v / |r_u \times r_v|$.

9.4. a) This number $S(r,n)$ can be different for different r with the same $r(D)$ and n.

b) The vector $n(r(u,v))$ is perpendicular to the surface $n(r(D))$ at its point $n(r(u,v))$. I.e., the normal field $\tilde{n}(n(r(u,v))) := n(r(u,v))$ defines a coorientation of the surface $n(r(D))$.

c) The area $S(n \circ r, n)$ for $n = \frac{r_u \times r_v}{|r_u \times r_v|}$ does not depend on a 1–1 parametrization $r\colon D \to \Pi$.

d) Ambient isometries preserve the total Gaussian curvature.

e)* **Gauss's Theorem Egregium.** Intrinsic isometries keep the total Gaussian curvature.

The *Gaussian curvature* of a surface Π at a point P is $K := K_{\Pi,P} := \lim\limits_{R\to 0} \frac{K(B_{\Pi,P}(R))}{S(B_{\Pi,P}(R))}$, where $B_{\Pi,P}(R)$ is the disc on Π with the radius R and the center P.

9.5. Find the Gaussian curvature at the points of the surfaces in Problem 3.1abdfgi.

9.6. a) **Theorem.** We have

$$K = \frac{n_u \wedge n_v \wedge n}{|r_u \times r_v|} = f_{xx}f_{yy} - f_{xy}^2 = \frac{(r_{uu} \wedge r_u \wedge r_v)(r_{vv} \wedge r_u \wedge r_v) - (r_{uv} \wedge r_u \wedge r_v)^2}{|r_u \times r_v|^4},$$

where $n = r_u \times r_v / |r_u \times r_v|$, and the second formula holds at the origin for the surface $z = f(x,y)$ tangent to the Oxy plane at the origin.

In particular, the limit defining the Gaussian curvature exists.

b)* **Theorem.** For a two-dimensional surface in $\mathbb{R}^3$ we have $\tau = 2K$.

10. Sectional curvature

The *total sectional curvature* $\sigma(\Pi)$ of a cooriented two-dimensional surface Π with smooth boundary $\partial\Pi$ is the couterclockwise angle between the tangent vector at a boundary point and the vector obtained from it by a parallel translation along the curve $\partial\Pi$ in the counterclockwise direction.

If $\partial\Pi$ is piecewise smooth, then the definition is analogous.

10.1. The total sectional curvature is:

a) preserved by an ambient isometry;

b) additive, i.e. $\sigma(\Pi_1 \cup \Pi_2) = \sigma(\Pi_1) + \sigma(\Pi_2)$, if Π_1 and Π_2 are surfaces bounded by closed piecewise smooth curves such that $\Pi_1 \cap \Pi_2 \subset \partial\Pi_1 \cap \partial\Pi_2$ is a connected curve and $\Pi_1 \cup \Pi_2$ is a surface (on Π_1 and Π_2, we take coorientations obtained by restricting some coorientation on $\Pi_1 \cup \Pi_2$);

c) independent of the coorientation.

10.2. a) Find the total sectional curvature of a spherical triangle with the angles α, β, γ.

b) How the total sectional curvature and the area of a polygon on a unit sphere are related?

c) The same for a part of the unit sphere bounded by a closed curve.

10.3. Let $n\colon \Pi \to S^2$ be the spherical mapping.

a) A vector tangent to the surface Π at a point P is tangent to the sphere S^2 at $n(P)$.

b) If a vector field $v(t)$ is parallel along a curve $\gamma(t)$ on Π and $n(\gamma(t))$ is a curve, then $v(t)$ is parallel along $n(\gamma(t))$ on S^2.

c) $\sigma(\Pi) = \sigma(n(\Pi))$.

d) **Gauss–Bonnet theorem.** $\sigma(\Pi) = K(\Pi) \mod 2\pi$.

The *sectional curvature* of the surface Π at the point P is

$$\sigma(P) := \lim_{R\to 0} \frac{\sigma(B_{\Pi,P}(R))}{S(B_{\Pi,P}(R))}.$$

10.4. How does the sectional curvature at a point changes under spatial homothety?

To summarize, we shall see that for a two-dimensional surface in $\mathbb{R}^3$, the scalar, sectional, and Gaussian curvatures coincide (up to a factor of 2).

11. Riemannian metric and isometries

A *Riemannian metric* on Π (induced from $\mathbb{R}^3$) or the *first quadratic form* of Π is the family of bilinear forms

$$g_P\colon T_P\Pi \times T_P\Pi \to \mathbb{R}, \quad \text{where } P \in \Pi, \qquad \text{defined by} \quad g_P(a,b) := a \cdot b.$$

11.1. a) The length of the image of the curve $\gamma\colon [a,b] \to \Pi$ is $\int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt$.

b) The cosine of the angle between parametrized curves $\gamma, \beta : [-1,1] \to \Pi$ on Π at a point $P = \gamma(0) = \beta(0)$ equals $\dfrac{g_P(\gamma',\beta')}{\sqrt{g_P(\gamma',\gamma')g_P(\beta',\beta')}}$. Here $\gamma' := \gamma'(0)$ and $\beta' := \beta'(0)$.

11.2. The Riemannian metric is symmetric and positive definite.

11.3. a) **Theorem.** The matrix of the Riemannian metric of the surface $r(D)$ at a point $P = r(x_1,x_2)$ in the standard basis (r_1,r_2) is $g_{ij} = r_i \cdot r_j$. (See notation in Problem 6.2.)

b) Calculate the matrix of the Riemannian metric of $r(D)$ at the point $r(u,v)$ in the standard basis for parameterized surface $r(u,v) = (u,v,f(u,v))$ (in terms of the function f and its partial derivatives).

c) The same for the parametrized surface $r(u,v) = (x(u,v), y(u,v), z(u,v))$ (in terms of the functions x, y, z and their partial derivatives).

d) **Theorem.** For two parametrizations $r, \widetilde{r}\colon D \to \mathbb{R}^3$ of one nonparametrized surface, the matrices $G, \widetilde{G}$ of the Riemannian metric of the surface $r(D) = \widetilde{r}(D)$ at the point $r(u_0,v_0) = \widetilde{r}(\widetilde{u_0},\widetilde{v_0})$ in bases (r_u,r_v) and $(\widetilde{r}_u,\widetilde{r}_v)$ are related via $\widetilde{G} = J^T G J$, where $J = (r^{-1} \circ \widetilde{r})'$.

11.4. Theorem. The following 3 conditions on the mapping between surfaces are equivalent:

(I) the mapping is an intrinsic isometry;

(R) the mapping (more precisely, its derivative) takes the Riemannian metric on the first surface to the Riemannian metric on the second one;

(D) the mapping preserves distances.

11.5. a) An intrinsic isometry (more precisely, its derivative) preserves the lengths of the tangent vectors.

b) An intrinsic isometry preserves angles between curves.

c) For a surface Π and a point $P \in \Pi$, define the function $f\colon \Pi \to \mathbb{R}$ by $f(X) = |P,X|^2/2$, where $|P,X|$ is the distance along the surface. Then $f'(P) = 0$ and $f''(P)$ is the Riemannian metric. We define $f''(P)$ as the symmetric bilinear form $F\colon T_P\Pi \times T_P\Pi \to \mathbb{R}$ such that $F(\gamma'(0),\gamma'(0)) = f(\gamma(t))''|_{t=0}$ for each parametrized curve $\gamma\colon [-\varepsilon,\varepsilon] \to \Pi$ with $\gamma(0) = P$.

11.6. a) Given a parametrized surface $r\colon D \to \mathbb{R}^3$, express $r_k \cdot r_{ij}$ through $g_{ij} := r_i \cdot r_j$ and their derivatives. (See notation in Problem 6.2.)

b) If $f\colon \Pi_1 \to \Pi_2$ is an intrinsic isometry, then the Christoffel symbols in a coordinate system $r\colon D \to \Pi_1$ equal the Christoffel symbols in the coordinate system $f \circ r\colon D \to \Pi_2$.

c) **Theorem.** An intrinsic isometry takes parametrized geodesics to parametrized geodesics.

d) **Theorem.** An intrinsic isometry of surfaces takes a family of vectors parallel along some curve to a family of vectors parallel along its image. (Return to Problem 9.4.e here).

12. Riemannian metric in a coordinate system

A *Riemannian metric* in a coordinate system $r\colon D \to \mathbb{R}^3$ on Π (or *the inverse r-image of the Riemannian metric on $r(D)$*) is the family of matrices $G = g_{ij}$ of the following bilinear forms in the standard basis $(0,1),(1,0)$:

$$\overline{g}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \quad \text{for each } X \in D, \quad \text{defined by} \quad \overline{g}(a,b) := r'(X)a \cdot r'(X)b.$$

Let $\det G$ be the determinant of the *matrix* g_{ij}.

12.1. a) The length of the image of the curve $\gamma = (u_1, u_2)\colon [a,b] \to D$ on a parametrized surface $r\colon D \to \mathbb{R}^3$ equals

$$\int_a^b \sqrt{\sum_{i,j} g_{ij} u_i' u_j'}\, dt = \int_a^b \sqrt{\overline{g}(\gamma', \gamma')}\, dt = \int_a^b \sqrt{g_{r(\gamma)}(r'\gamma', r'\gamma')}\, dt.$$

Here the arguments t of the functions γ, u_1, u_2 and the argument $(u_1(t), u_2(t))$ of the functions r', g_{ij} are missing.

b) **Theorem.** $g_{ij} = r_i \cdot r_j$.

c) $\det G = g_{11}g_{22} - g_{12}^2 > 0$ at any point.

d) **Theorem.** For two parametrizations $r, \widetilde{r}$ of one nonparametrized surface the matrices $G, \widetilde{G}$ are related via $\widetilde{G} = J^T G J$, where $J = (r^{-1} \circ \widetilde{r})'$.

e) Let $\gamma, \beta\colon [-1,1] \to D$ be parameterized curves, where $\gamma(0) = \beta(0) = X$. Denote $(a_1, a_2) := \gamma'(0)$ and $(b_1, b_2) := \beta'(0)$. Then the cosine of the angle between the curves $r \circ \gamma$ and $r \circ \beta$ (on the parameterized surface $r\colon D \to \mathbb{R}^3$) at the point $r(X)$ equals

$$\frac{\sum_{i,j} g_{ij} a_i b_j}{\sqrt{(\sum_{i,j} g_{ij} a_i a_j)(\sum_{i,j} g_{ij} b_i b_j)}} = \frac{\overline{g}(\gamma', \beta')}{\sqrt{\overline{g}(\gamma', \gamma')\overline{g}(\beta', \beta')))}} = \frac{g_r(r'\gamma', r'\beta')}{\sqrt{g_r(r'\gamma', r'\gamma')g_r(r'\beta', r'\beta')}}.$$

Here γ' and β' are evalated at the point 0, and r, r', g_{ij} are evalated at the point X.

f) $|r_1 \times r_2|^2 = \det G$.

g) The area of the surface $r(D)$ is $\int\int_D \sqrt{\det G}\, du_1 du_2$.

12.2. a) The coordinate system of the exponential mapping is *Euclidean* at the point P, i.e. in an orthonormal basis in the tangent plane,

$$g_{ij}(P) = \delta_{ij} := \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad g_{ij}'(P) = 0.$$

(Return to Problem 8.4.d here).

b)* Express the scalar curvature τ through $r\colon D \to \mathbb{R}^3$ and its partial derivatives.

13. Ricci bilinear form

The Ricci and the Riemann curvature tensors are interesting for higher-dimensional surfaces. For two-dimensional ones, they essentially reduce to the scalar (or Gaussian) curvature. However, we continue to work with two-dimensional surfaces by default and just mention modifications required for the higher-dimensional ones. It is suggested to try solving the problems of the following sections in the higher-dimensional case as well (unless a problem indicates explicitly that a two-dimensional surface is considered).

The *Ricci bilinear form (tensor)* of Π at $P \in \Pi$ is a symmetric bilinear form $\rho = \rho_{\Pi,P}\colon T_P\Pi \times T_P\Pi \to \mathbb{R}$ such that for each bounded surface $A \subset T_P$ containing P we have

$$S(\exp(A)) = S(A) - \frac{1}{6}\int_A \rho(u,u)du + O(\mathrm{diam}(A)^5).$$

(For an n-dimensional surface Π, the surface A must also be n-dimensional, and the area S is replaced by the n-dimensional volume V.)

13.1. a) Such a symmetric bilinear form exists and is unique.

b) In the coordinate system of the exponential mapping, the matrix of the Ricci form is

$$\rho_{kl} = -3\sum_i \frac{\partial^2 g_{ii}}{\partial u_k \partial u_l}.$$

c) An intrinsic isometry preserves the Ricci bilinear form.

d) Find the matrix ρ_{ij} of the Ricci form for a surface $r\colon D \to \mathbb{R}^3$ in the basis (r_1, r_2).

13.2. For a symmetric bilinear form $\omega\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

a) There exists a unique $\widetilde{\omega}\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\widetilde{\omega}(u)\cdot v = \omega(u,v)$ for each $u, v \in \mathbb{R}^2$.

b) We have $\int_B \omega(u,u)du = (\pi/4)\mathrm{tr}\,\widetilde{\omega} = (\pi/4)\sum_i \omega(e_i,e_i)$, where B is the unit disk in $\mathbb{R}^2$ and (e_1,e_2) is the standard basis in $\mathbb{R}^2$. (In the n-dimensional case, $\pi/4$ should be replaced by $V(B)/(n+2)$.)

13.3. a) **Theorem.** $\tau = \mathrm{tr}\,\widetilde{\rho}$, where the operator $\widetilde{\rho}\colon T_P\Pi \to T_P\Pi$ is defined by $\widetilde{\rho}(u)\cdot v = \rho(u,v)$ for each $u, v \in T_P\Pi$.

b)* **Theorem.** For a two-dimensional surface Π, the Ricci bilinear form is proportional to the Riemannian metric with the coefficient $\tau/2$: $2\rho(u,v) = \tau u \cdot v$. In other words, for each bounded surface $A \subset T_P\Pi$ containing P we have

$$S(\exp(A)) = S(A) - \frac{\tau}{12}\int_A |u|^2 du + O(\mathrm{diam}(A)^5).$$

14. Sectional-curvature operator

Let $A \subset T_P\Pi$ be a surface bounded by an oriented piecewise smooth curve ∂A containing the point P. Denote by $\sigma(A)\colon T_P\Pi \to T_P\Pi$ the linear operator taking a vector $x \in T_P\Pi$ to the vector obtained from x by the parallel translation along the oriented curve $\exp_P(\partial A)$. We assume that ∂A is contained in the domain of $\exp_P$.

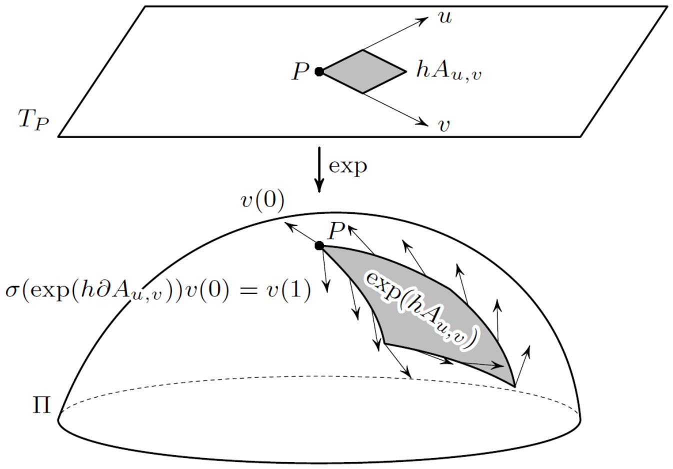


Figure 6: The $\sigma(A)$ operator and the sectional-curvature operator

The *sectional-curvature operator* of the surface Π at the point $P \in \Pi$, corresponding to a pair $u, v \in T_P\Pi$ of linearly independent vectors, is a linear operator $R(u,v) = R(u,v)_P\colon T_P\Pi \to T_P\Pi$ such that for the parallelogram $A_{u,v}$ spanned by u, v, we have

$$\sigma(hA_{u,v}) = \mathrm{Id} + h^2 R(u,v) + o(h^2) \quad \text{as} \quad h \to 0.$$

Here the oriented curve $\partial(hA_{u,v})$ leaves P in the direction of u. If u, v are linearly dependent, then set $R(u,v) = 0$. (For a higher-dimensional surface Π, the definition is the same.)

Denote by $R^{\alpha}_{PQ}\colon T_P\Pi \to T_P\Pi$ the counterclockwise (if viewed from Q) rotation through an angle α about a vector $PQ \perp T_P\Pi$.

14.1. For a two-dimensional surface Π with a coorientation $n\colon \Pi \to S^2$ and two vectors $u, v \in T_P\Pi$ such that $u \wedge v \wedge n > 0$ we have:

a) $\sigma(hA_{u,v}) = R_n^{\sigma S(\exp(hA_{u,v}))+o(h^2)}$ as $h \to 0$, where σ is the sectional curvature at P.

b) $\sigma(hA_{u,v}) = R_n^{h^2\sigma S(A_{u,v})+o(h^2)}$ as $h \to 0$.

c) $R(u,v) = \sigma\, u \wedge v \wedge n\, R_n^{\pi/2}$.

14.2. a)* The sectional-curvature operator $R(u,v)$ exists and is unique.

b) It depends linearly on u, v.

c) An intrinsic isometry preserves the sectional-curvature operator.

14.3. Riemann-tensor symmetry theorem.

a) $R(u,v)$ is skew-symmetric, that is, $[R(u,v)x] \cdot y = -[R(u,v)y] \cdot x$.

b) $R(u,v)$ is skew-symmetric in u, v: $R(u,v) = -R(v,u)$.

c) *The Bianchi identity (algebraic).* $R(u,v)x + R(v,x)u + R(x,u)v = 0$.

d) $[R(u,v)x] \cdot y = [R(x,y)u] \cdot v$.

15. Riemann curvature tensor

The *Riemann curvature tensor* of Π at P is the trilinear mapping

$$R\colon (T_P\Pi)^3 \to T_P\Pi, \quad \text{defined by formula} \quad R(u,v,x) := R(u,v)x.$$

Given a coordinate system $r\colon D \to \Pi$, denote $R(r_i, r_j, r_k) =: \sum_l R^l_{ijk} r_l$.

15.1. a) Find R^l_{ijk} on a sphere in spherical coordinates.

b) **Theorem.** For a two-dimensional surface Π with a coorientation $n\colon \Pi \to S^2$ the Riemann tensor is expressed through the sectional curvature: $R(u,v,x) = \sigma\, u \wedge v \wedge n R_n^{\pi/2}(x)$.

(Or, in the exponential mapping coordinate system, $R^1_{122} = \sigma \det(g_{ij})$; the remaining components are zero or equal to $\pm R^1_{122}$ due to the symmetries of the Riemann tensor.)

c) Express R^l_{ijk} through the Christoffel symbols and their derivatives.

15.2. Consider the coordinate system of the exponential mapping $\exp_P$. Let u^k be the k-th coordinate of a point.

a) For each point and each k we have $\sum_{ij} \Gamma^k_{ij} u^i u^j = 0$.

Prove the following identities at the point P for each i, j, k, l:

b) $\Gamma^k_{ij} = 0$.

c) $\frac{\partial \Gamma^l_{ij}}{\partial u^k} + \frac{\partial \Gamma^l_{jk}}{\partial u^i} + \frac{\partial \Gamma^l_{ki}}{\partial u^j} = 0$.

d) $\frac{\partial \Gamma^l_{ij}}{\partial u^k} - \frac{\partial \Gamma^l_{jk}}{\partial u^i} = R^l_{ikj}$.

e) $3\frac{\partial \Gamma^l_{ij}}{\partial u^k} = R^l_{ikj} - R^l_{kji} = R^l_{ikj} + R^l_{jki}$.

f) $R^l_{ikj} + R^i_{lkj} = 3\frac{\partial^2 g_{il}}{\partial u^j \partial u^k}$. (Hint: use Problem 14.3.)

g) **Theorem.** The bilinear Ricci form is equal to the contraction of the Riemann tensor: $\rho(u,v) = \sum_i R(u, e_i)v \cdot e_i$ and $\rho_{kl} = \sum_i R^i_{kil}$. (Return to Problems 9.6.b and 13.3.b here.)

16. Riemann manifolds

A *(local) Riemannian manifold* is a pair (D, g), where $D \subset \mathbb{R}^2$ is open and g is a family of positive-definite symmetric bilinear maps $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ smoothly depending on $P \in D$.

The family is called the *Riemannian metric.* In other words, it is a collection of smooth functions $g_{ij}\colon D \to \mathbb{R}$ with $1 \le i, j \le 2$ such that $g_{ij} = g_{ji}$, $g_{ii} > 0$, and $g_{ii}g_{jj} - (g_{ij})^2 > 0$. The Riemannian metric is often denoted by $\sum_{i,j} g_{ij}(x^1, x^2)\, dx^i dx^j$, where x^1, x^2 are the Cartesian coordinates and dx^1, dx^2 is just a notation for independent variables.

Curve lengths, angles, and areas on (D, g) are defined in terms of the Riemannian metric by the formulae from Problems 12.1.aeg. *Intrinsic isometries, (non-parametrized) geodesics, distance, circles, and scalar curvature* on (D, g) are defined literally as for surfaces in $\mathbb{R}^3$.

16.1. Poincaré model of the hyperbolic plane. Let (H, g^H) be the half-plane $y > 0$ with $g^H := \frac{(dx)^2+(dy)^2}{y^2}$. Let (D^2, g^D) be the unit circle $x^2 + y^2 < 1$ with $g^D := 4\,\frac{(dx)^2+(dy)^2}{(1-x^2-y^2)^2}$.

a) For any two intersecting curves on (H, g^H), the angle between them on (H, g^H) equals the angle between them in the Euclidean plane.

b) The map $(x, y) \mapsto (ax \pm b, ay)$ is an intrinsic isometry of (H, g^H) for each $a, b > 0$.

c) The map $(x, y) \mapsto \frac{(2x, x^2+y^2-1)}{x^2+(1+y)^2}$ is an intrinsic isometry from (H, g^H) to (D^2, g^D).

d) Find the shortest curve and the distance between $(0, 0)$ and (x, y) on (D^2, g^D).

e) Find the length of the circle of radius R centered at $(0, 0)$ on (D^2, g^D) and the scalar curvature at $(0, 0)$. (Hint: this circle is a circle of some radius $r(R)$ in the Euclidean plane.)

f) **Theorem.** The scalar curvature at each point of (H, g^H) and (D^2, g^D) equals -2.

A *parametrized geodesic* on (D, g) is a curve $\gamma(t) = (x^1(t), x^2(t))$ satisfying the equation

$$x_k'' + \sum_{i,j} \Gamma_{ij}^k x_i' x_j' = 0, \quad \text{where} \quad \Gamma_{ij}^k := \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right) \tag{1}$$

and g^{kl} are the entries of the inverse matrix of g_{ij}.

16.2. a) Exactly one parametrized geodesic passes through every point in every direction with any given speed at the starting point.

b) An intrinsic isometry takes parametrized geodesics to parametrized geodesics.

c) Each parametrized geodesic $\gamma(t)$ has constant speed: $g(\gamma'(t), \gamma'(t)) = \text{const}$.

16.3. Find the parametrized geodesics on: a) (H, g^H) (hint: use the previous two problems);

b) the half-plane $y > 0$ with the Riemannian metric $y(dx)^2 + y(dy)^2$.

The *tangent plane $T_P D$ at a point $P \in D$* is the plane $\mathbb{R}^2$. The *exponential mapping* $\exp_P$ is defined literally as for surfaces in $\mathbb{R}^3$. To a point with the polar coordinates (r, ϕ) in $\mathbb{R}^2$, it assigns the point $x(r, \phi) := \gamma(r)$, where $\gamma(t)$ is the solution of (1) with the initial conditions $\gamma(0) = P$, $|\gamma'(0)| = 1$, $\angle(\gamma'(0), (1, 0)) = \phi$ (the latter is the angle on (D, g)).

16.4. The exponential mapping $x(r, \phi)$ satisfies the following identities for each $r \ne 0$:

a) $g(x_r, x_r) = 1$; b) $\frac{\partial}{\partial \phi} g(x_r, x_r) = \frac{\partial}{\partial r} g(x_r, x_\phi)$; c) $x_\phi \to 0$ and $g(x_r, x_\phi) \to 0$ as $r \to 0$;

d) $g(x_r, x_\phi) = 0$. e) **Corollary.** The image of a parametrized geodesic is a geodesic.

17. Literature

The literature is not required to solve the problems unless otherwise explicitly indicated.

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[2] V. Toponogov, Differential geometry of curves and surfaces: A concise guide. 2006

[3] A. Skopenkov, Basic differential geometry as a sequence of interesting problems, arXiv:0801.1568