Feynman checkerboard

An elementary mathematical model in quantum theory

E. Akhmedova, M. Skopenkov, A. Ustinov, R. Valieva, A. Voropaev

Summary. The course is devoted to the most elementary model of electron motion suggested by R.Feynman. It is a game, in which a checker moves on a checkerboard by certain simple rules, and we count the turnings. It turns out that this simple combinatorial model demonstrates visually many basic ideas of quantum theory. We are going to solve mathematical problems related to the game and discuss their physical meaning; no knowledge of physics is assumed.

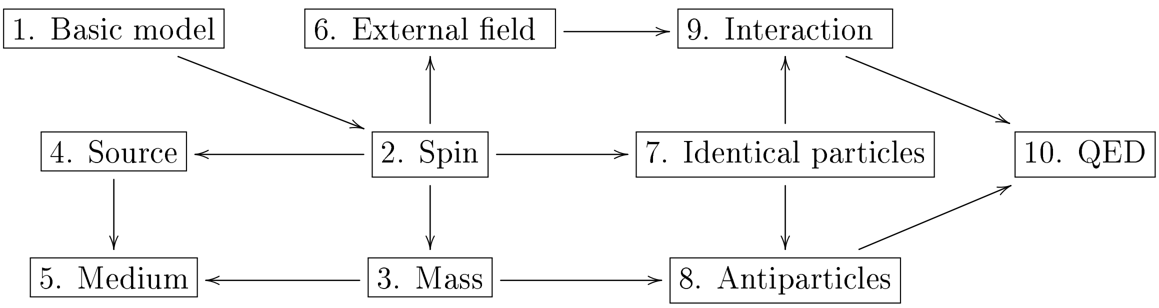
Main results. The main results are explicit formulae for

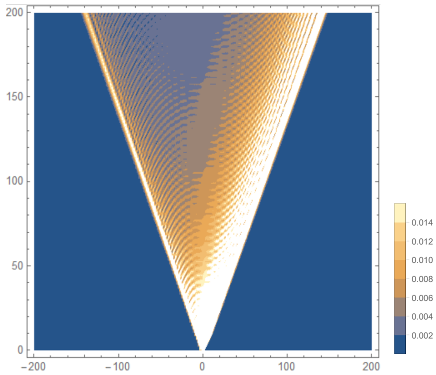
- the percentage of light of given color reflected by a glass plate of given width (Problem 26);
- the probability to find an electron in a given square, if it was emitted from the origin and moves in a fixed plane (Problem 15; see Figure 1).

Although the results are stated in physical terms, they are mathematical theorems, because we provide mathematical models of the physical phenomena in question, with all the objects defined rigorously. More precisely, a sequence of models with increasing precision.

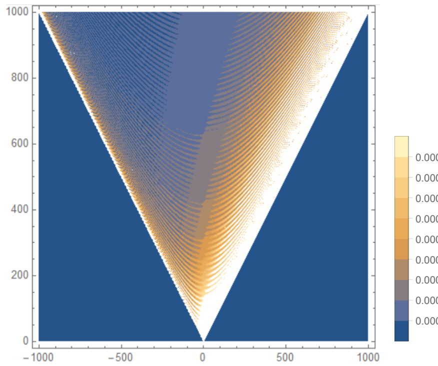
Plan. We start with a basic model and upgrade it step by step in each subsequent section. Before each upgrade, we summarize which physical *question* does it address, which simplifying *assumptions* does it resolve or impose additionally, and which experimental *results* does it explain. Our aim is what is called *2-dimensional quantum electrodynamics* but the last steps on this way (sketched in Sections 8–10) still have not been done. (A *4-dimensional* one can already explain all phenomena in the world except atomic nuclei and gravitation — with a serious proviso — but we do not discuss it.)

The scheme of upgrades dependence might help to choose your way:





Basic model from §1



Model with mass from §3

Figure 1: The probability to find an electron in a small square around a given point (white depicts large oscillations of the probability)

Conventions. If a problem is a statement of an assertion, then it is requested to prove the assertion. A *puzzle* is a problem, in which both a precise statement and a proof are requested. Hard problems are marked with stars; you get *first rank in Feynman checkerboard* for solving one, and *masters in Feynman checkerboard* for solving three from three different sections. If you cannot solve a problem, proceed to the next ones: they may provide hints. Even if you do not reach the top (the proofs of main results), you learn much. You are encouraged to state and try to prove also your own observations and conjectures; you become a *grand-master in Feynman checkerboard* for discovering a new nontrivial one (and maybe even write your own research paper).

1 Basic model

Question: what is the probability to find an electron in the square (x, y), if it was emitted from the square $(0, 0)$?
Assumptions: no self-interaction, no creation of electron-positron pairs, fixed mass and lattice step, point source; no nuclear forces, no gravitation, electron moves uniformly along the y-axis and does not move along the z-axis.
Results: double-slit experiment, charge conservation.

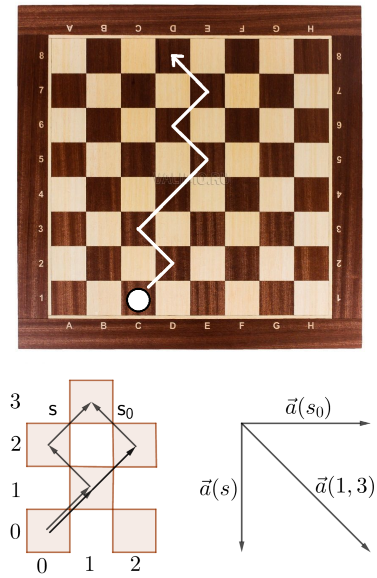


Figure 2: Checker paths

On an infinite checkerboard, a checker moves to the diagonal-neighboring squares, either upwards-right or upwards-left. To each path s of the checker, assign a vector $\vec{a}(s)$ as follows. Start with a vector of length 1 directed upwards. While the checker moves straightly, the vector is not changed, but each time when the checker changes the direction, the vector is rotated through 90° clockwise (independently of the direction the checker turns). In addition, at the very end the vector is divided by $2^{(y-1)/2}$, where y is the total number of moves. The final position of the vector is what we denote by $\vec{a}(s)$. For instance, for the path in Figure 2 to the top, the vector $\vec{a}(s) = (1/8, 0)$ is directed to the right and has length 1/8.

Denote $\vec{a}(x, y) := \sum_s \vec{a}(s)$, where the sum is over all the paths of the checker from the square $(0, 0)$ to the square (x, y), *starting with the upwards-right move*. Set $\vec{a}(x, y) := \vec{0}$, if there are no such paths. For instance, $\vec{a}(1, 3) = (0, -1/2) + (1/2, 0) = (1/2, -1/2)$. The length square of the vector $\vec{a}(x, y)$ is called the *probability*[1] *to find an electron in the square* (x, y), *if it was emitted from the square* $(0, 0)$. Notation: $P(x, y) := |\vec{a}(x, y)|^2$.

In Figure 1 to the top, the color of a point (x, y) with even $x + y$ depicts the value $P(x, y)$. The sides of the apparent angle are *not* the lines $y = \pm x$ (and nobody knows why!).

In what follows squares (x, y) with even and odd $x + y$ are called *black* and *white* respectively.

1. Observations for small y**.** Answer the following questions for each $y = 1, 2, 3, 4$ (and state your own questions and conjectures for arbitrary y): Find the vector $\vec{a}(x, y)$ and the probability $P(x, y)$ for each x. When $P(x, y) = 0$? What is $\sum_{x \in \mathbb{Z}} P(x, y)$ for fixed y? What are the directions of $\vec{a}(1, y)$ and $\vec{a}(0, y)$?

The *probability*[2] *to find an electron in the square* (x, y) *subject to absorption in the square* (x', y') is defined analogously to $P(x, y)$, only the summation is over those paths s that do not pass through (x', y'). The probability is denoted by $P(x, y \text{ bypass } x', y')$.

2. Double-slit experiment. Is it true that $P(x, y) = P(x, y \text{ bypass } 0, 2) + P(x, y \text{ bypass } 2, 2)$? Is it true that $P(x, y) \geq P(x, y \text{ bypass } x', y')$?

3. Find $P(0, 12)$. How to table the values $\vec{a}(x, y)$ quickly without exhaustion of all paths? (The first solution grants *first rank* in Feynman checkers.)

Denote by $a_1(x, y)$ and $a_2(x, y)$ the coordinates of $\vec{a}(x, y)$; see Figure 3.

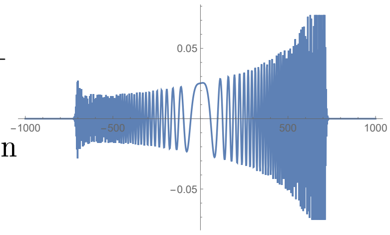


Figure 3: $a_2(x, 1000)$

4. (Puzzle) Draw all the paths giving nonzero contribution to $a_1(1, 5)$. The same for $a_2(1, 5)$.

5. Dirac's equation. Express $a_1(x, y)$ and $a_2(x, y)$ through $a_1(x \pm 1, y - 1)$ and $a_2(x \pm 1, y - 1)$.

6. Probability/charge conservation. For each positive integer y we have $\sum_{x \in \mathbb{Z}} P(x, y) = 1$.

7. Symmetry. How the values $a_1(x, 100)$ for $x < 0$ and for $x \geq 0$ are related with each other (that is, hot to express the ones for $x < 0$ through the ones for $x \geq 0$)? The same for the values $a_1(x, 100) + a_2(x, 100)$.

8. Huygens' principle. What is a fast way to find $\vec{a}(x, 199)$, if we know $\vec{a}(x, 100)$ for all integers x?

9. Using a computer, plot the graphs of the functions $f_y(x) = P(x, y)$ for various y, joining each pair of points $(x, f_y(x))$ and $(x + 2, f_y(x + 2))$ by a segment; cf. Figure 3. The same for the function $a_1(x, y)$.

10.* Find an explicit formula for the vector $\vec{a}(x, y)$ and the probability $P(x, y)$ (it is allowed to use a sum with at most y summands in the answer).

[1] One should think of the value y as fixed, and the squares $(-y, y), (-y + 2, y), \dots, (y, y)$ as all the possible outcomes of an experiment. For instance, the y-th horizontal might be a photoplate detecting the electron.
Familiarity with probability theory is *not* required for solving the presented problems.
If the checker were performing just a random walk (after the first upwards-right move), then $|\vec{a}(s)|^2$ would be the probability of a path s. The latter probability has absolutely *no* sense in our model, but explains the normalization factor $2^{(y-1)/2}$.
Beware that our rule for the probability computation is valid only for the basic model and is slightly changed in the upgrades.

[2] Thus an additional outcome of the experiment is that the electron has been absorbed and has not reached the photoplate.

2 Spin

Question: what is the probability to find a right electron at (x, y), if a right electron was emitted from $(0, 0)$?
Assumptions: the same.
Results: spin reversal.

The trick in the solution of the previous problems has a physical meaning: it is convenient to consider an electron as being in one of the two states: *right-moving* or *left-moving*. We write just '*right*' or '*left*' for brevity[3]. This is not just a convenience but reflects an inalienable electron's property called *spin*[4].

Denote $\vec{a}(x, y, +) := \sum_s \vec{a}(s)$, where the sum is over only those paths from $(0, 0)$ to (x, y), which both start and finish with an upwards-right move. Define $\vec{a}(x, y, -)$ to be an analogous sum over paths which start with an upwards-right move but finish with an upwards-left move.

The length square of the vector $\vec{a}(x, y, +)$ (respectively, $\vec{a}(x, y, -)$) is called the *probability*[5] *to find a right (respectively, left) electron in the square* (x, y)*, if a right electron was emitted from the square* $(0, 0)$. Denote by $P(x, y, +) := |\vec{a}(x, y, +)|^2$ and $P(x, y, -) := |\vec{a}(x, y, -)|^2$ these probabilities.

11. Express $\vec{a}(x, y, +)$ and $\vec{a}(x, y, -)$ through $a_1(x, y)$ and $a_2(x, y)$; $P(x, y)$ through $P(x, y, +)$ and $P(x, y, -)$.

12.* **Spin reversal.** What is the probability $P(y_0, -) := \sum_{x \in \mathbb{Z}} P(x, y_0, -)$ to find a left electron on the line $y = y_0$ (it is allowed to use a sum with at most y_0 summands in the answer)? Find the maximal element and the limit of the sequence $P(1, -), P(2, -), P(3, -), \dots$.

A crash-course in calculus. For this and other problems *with stars*, the following formuli might be useful.

Recall that $\sum_{k=0}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$. Clearly, as n increases, the sum becomes closer and closer to 2. We would like to write $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$; let us give a definition of such an infinite sum. A sequence $a_1, a_2, a_3, \dots$ *has a limit* a, if for each real $\varepsilon > 0$ there is N such that for each integer $n > N$ we have $|a_n - a| < \varepsilon$. Notation: $a = \lim_{n\to\infty} a_n$. For instance, $\lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2$. By definition, put $\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$. Then indeed $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$.

The following generalization is called *Newton's binomial theorem* (allowed to use without proof):

$$(1 + x)^r = \sum_{k=0}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1} x^k$$

for each complex x with $|x| < 1$ and each real r, or for $x = 1$ and $r > -1$. In particular, for $r = -1$ and $-\frac{1}{2}$ we get

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{and} \qquad \frac{1}{\sqrt{1-x}} = \sum_{k=0}^{\infty} \frac{2k(2k-1)\cdots(k+1)}{k(k-1)\cdots 1} \frac{x^k}{4^k}.$$

The following *Stirling formula* allows to estimate the summands in Newton's binomial theorem:

$$\sqrt{2\pi}\, k^{k+1/2} e^{-k} \le k(k-1)\cdots 1 \le e\, k^{k+1/2} e^{-k}.$$

Here e denotes $\lim_{n\to\infty}(1 + 1/n)^n$. It is an irrational number between 2.71 and 2.72.

3 Mass

Question: what is the probability to find a right electron of mass m at (x, y), if it was emitted from $(0, 0)$?
Assumptions: the mass and the square side are now arbitrary.
Results: a formula for the probability for small length of the square side.

To check our model against experiment we need the following generalization.

Fix $\varepsilon > 0$ and $m \ge 0$ called *side of the square* and *particle mass* respectively. Assume that the centers of the squares have the coordinates $(k\varepsilon, l\varepsilon)$, where k and l are integers; see Figure 4. To each path s of the checker from the square with the center $(0, 0)$ to the square with the center (x, y), assign a vector $\vec{a}(s, m\varepsilon)$

[3]Beware that in 3 or more dimensions '*right*' and '*left*' mean something very different from the movement direction. Although often visualized as the direction of the electron rotation, these states cannot be explained in nonquantum terms.

[4]And *chirality*; beware that the term *spin* usually refers to a property, not related to the movement direction at all.

[5]Thus an experiment outcome is a pair (final x-coordinate, last-move direction), whereas the final y-coordinate is fixed. These are the fundamental probabilities, whereas $P(x, y)$ should in general be *defined* by the formula from the solution of Problem 11 rather than the above formula $P(x, y) = |\vec{a}(x, y)|^2$ (being a coincidence).

as follows. Start with the vector $(0,1)$. While the checker moves straightly, the vector is not changed, but each time when the checker changes the direction, the vector is rotated through $90°$ clockwise and multiplied by $m\varepsilon$. In addition, at the very end the vector is divided by $(1+m^2\varepsilon^2)^{(y/\varepsilon-1)/2}$, where y/ε is the total number of moves. The final position of the vector is what we denote by $\vec{a}(s, m\varepsilon)$. The vectors $\vec{a}(x, y, m, \varepsilon, \pm)$ and the numbers $P(x, y, m, \varepsilon, \pm)$ are defined analogously to $\vec{a}(x, y, \pm)$ and $P(x, y, \pm)$, only $\vec{a}(s)$ is replaced by $\vec{a}(s, m\varepsilon)$. For instance, $P(x, y, 1, 1, +) = P(x, y, +)$. In Figure 1 to the bottom, the color of a point (x, y) depicts the value $P(x, y, 1, 0.2, +) + P(x, y, 1, 0.2, -)$ (once $(x + y)/0.2$ is even).

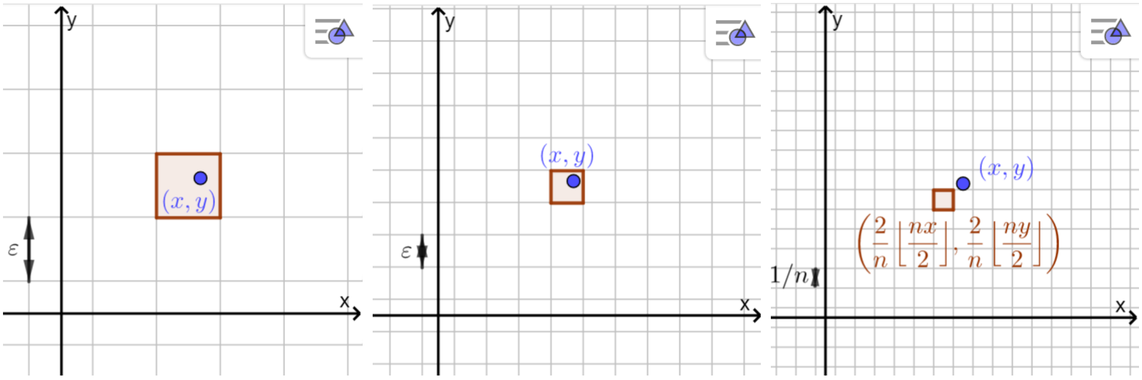


Figure 4: The point (x, y) stays fixed while the square side tends to zero.

13. (Puzzle) **Massless and heavy particles.** Find $P(x,y,0,\varepsilon,+)$ and define $P(x,y,\infty,\varepsilon,+)$ for each x,y,ε.

14. Solve analogues of Problems 5, 6, and 10 for $m, \varepsilon \neq 1$.

To fit an experiment, we take sidelength $\varepsilon = 1/n$ to be small, that is, tend n to infinity. We fix a point (x, y) in the plane and approximate it by black squares of side $1/n$ with the center $\left(\frac{2}{n}\left\lfloor\frac{nx}{2}\right\rfloor, \frac{2}{n}\left\lfloor\frac{ny}{2}\right\rfloor\right)$; see Figure 4. This leads to the following problem.[6]

15.* (First main problem) **Continuum limit.** For each real x, y, m find $\lim_{n\to\infty} n\,\vec{a}\left(\frac{2}{n}\left\lfloor\frac{nx}{2}\right\rfloor, \frac{2}{n}\left\lfloor\frac{ny}{2}\right\rfloor, m, \frac{1}{n}, -\right)$ and $\lim_{n\to\infty} n\,\vec{a}\left(\frac{2}{n}\left\lfloor\frac{nx}{2}\right\rfloor, \frac{2}{n}\left\lfloor\frac{ny}{2}\right\rfloor, m, \frac{1}{n}, +\right)$ In the answer, it is allowed to use the following expressions[7]:

$$J_0(z) := \sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{2k}}{(k!)^2} \qquad \text{and} \qquad J_1(z) := \sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{2k+1}}{k!(k+1)!}.$$

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4 Source

Question: what is the probability to find a right electron at (x, y), if it was emitted by a source of wave length λ?
Assumptions: the source is now realistic.
Results: wave propagation, dispersion relation.

A realistic source does not produce electrons localized at $x = 0$ (as in our game) but a rather wide wave impulse instead. For our game, this means that the checker can start from an arbitrary black square on the horizontal line $y = 0$ (not too far from the origin), but the initial direction of the vector $\vec{a}(s)$ is rotated through an angle proportional to the distance from the starting square to the origin; see Figure 5.

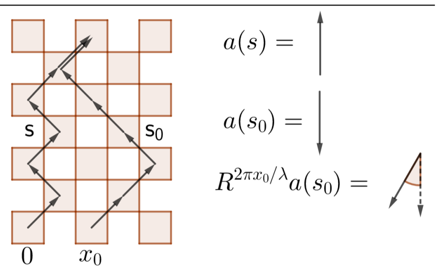

Figure 5: Checker paths may now start at distinct squares

Formally, fix real $\lambda > 0$ and odd Δ called *wave length*, and *impulse width* respectively. Denote by $R^\alpha\,\vec{a}$ the rotation of a vector $\vec{a}$ through the angle $|\alpha|$, which is counterclockwise for $\alpha \geq 0$ and clockwise for $\alpha < 0$. For each integers x and y define the vector[8]

$$\vec{a}(x, y, \lambda, \Delta, +) := \frac{1}{\sqrt{\Delta}} \sum_{\substack{x_0=1-\Delta \\ x_0 \text{ even}}}^{\Delta-1} \sum_{s} R^{2\pi x_0/\lambda}\,\vec{a}(s),$$

[6] In the limits, the normalization factor of n is a bit harder to explain; we not discuss it.
[7] Called *Bessel functions*, which are almost as well-studied as sine and cosine; but familiarity with them is not required.
[8] The notation $\vec{a}(x, y, \lambda, \Delta, \pm)$ should not be confused with $\vec{a}(x, y, m, \varepsilon, \pm)$.

where the second sum is over all checker paths s from the square $(x_0, 0)$ to the square (x, y), starting *and ending* with an upwards-right move. The length square of the vector is the *probability to find a right electron at* (x, y)*, emitted by a source of wave length* λ *and impulse width* Δ. It is denoted by $P(x, y, \lambda, \Delta, +)$. Define $\vec{a}(x, y, \lambda, \Delta, -)$ and $P(x, y, \lambda, \Delta, -)$ analogously. For instance, $P(x, y, \lambda, 1, +) = P(x, y, +)$ for each λ, and $\vec{a}(x + 1, 1, \lambda, \Delta, +) = \frac{1}{\sqrt{\Delta}}\left(-\sin\frac{2\pi x}{\lambda}, \cos\frac{2\pi x}{\lambda}\right)$ for even $|x| < \Delta$.

16. Let $\Delta = 3$, $\lambda = 4$. Find the vector $\vec{a}(x, y, 4, 3, +)$ and the probability $P(x, y, 4, 3, +)$ for $y = 1, 2, 3$ and each x. What is $\sum_{x\in\mathbb{Z}}(P(x, y, 4, 3, +) + P(x, y, 4, 3, -))$ for fixed $y = 1, 2, 3$? When $P(x, 3, 4, 3, +) = 0$?

17. Probability/charge conservation. Solve analogues of Problems 5 and 6 for $\vec{a}(x, y, \lambda, \Delta, -)$, $\vec{a}(x, y, \lambda, \Delta, +)$, and $P(x, y, \lambda, \Delta, +) + P(x, y, \lambda, \Delta, -)$ instead of $a_1(x, y)$, $a_2(x, y)$, and $P(x, y)$.

18. Causality. Both $\sqrt{\Delta}\vec{a}(x, y, \lambda, \Delta, +)$ and $\Delta P(x, y, \lambda, \Delta, +)$ do not depend on Δ for $\Delta > y + |x|$.

Denote $\vec{a}(x, y, \lambda, \pm) = \sqrt{\Delta}\vec{a}(x, y, \lambda, \Delta, \pm)$ and $P(x, y, \lambda, \pm) = \Delta P(x, y, \lambda, \Delta, \pm)$ for any $\Delta > y + |x|$.

19. Wave. How to find $\vec{a}(x, 100, \lambda, +)$ for all even x, if we know it for just one even x?

20. Dispersion relation. Find all compex-valued functions of the form $a_1(x, y) = v_1 e^{2\pi i(x/\lambda + y/T)}$ and $a_2(x, y) = v_2 e^{2\pi i(x/\lambda + y/T)}$ satisfying the Dirac equation from the solution of Problem 5. Here v_1 and v_2 are complex constants. Which relation emerges between λ and T (*wavelength* and *period*)?

21.* Wave propagation. For each x, y, λ find $\vec{a}(x, y, \lambda, -)$, $P(x, y, \lambda, -)$, $P(x, y, \lambda, -) + P(x, y, \lambda, +)$.

22.* Fourier integral. Prove that $a_1(x, y) = \frac{1}{2\pi\sqrt{2}}\int_{-\pi}^{\pi}\frac{\sin(\omega_p(y-1))}{\sin\omega_p}e^{-ipx}\,dp$ for each $y > 0$ and each x, where $\omega_p := \arccos(\cos p/\sqrt{2})$. Find an analogous integral formula for $a_2(x, y)$ (puzzle).

5 Medium

Question: which part of light of given color reflects from a glass plate of given width?
Assumptions: right angle of incidence, no polarization nor dispersion of light; the mass now depends on x.
Results: thin-film reflection.

Our model can be applied also to describe propagation of light in transparent media such as glass[9]. Light propagates as if it had some nonzero mass inside the media, while the mass remains zero outside[10]. The wavelength determines the color of light.

23. (Puzzle) Define an analogue of $\vec{a}(s, m\varepsilon)$ in the case when the mass $m = m(x)$ depends on x so that analogues of Problems 5 and 6 remain true.

Given a *mass* $m = m(x)$, define $\vec{a}(x, y, m(x), \lambda, \pm)$ and $P(x, y, m(x), \lambda, \pm)$ analogously to $\vec{a}(x, y, \lambda, \pm)$ and $P(x, y, \lambda, \pm)$, only replace $\vec{a}(s)$ by $\vec{a}(s, m\varepsilon)$ in the definition. So far the sidelength $\varepsilon = 1$.

24. One-surface reflection. Find $P(x, y, m(x), \lambda, +)$ and $P(x, y, m(x), \lambda, -)$ for

$$m(x) = m_0(x) \equiv 0 \qquad \text{and} \qquad m(x) = m_1(x) = \begin{cases} 0.2, & \text{for } x = 0; \\ 0, & \text{for } x \neq 0. \end{cases}$$

Now fix odd $L > 1$ called the *width of a glass plate.* First assume for simplicity that light is reflected only by the two surfaces of the plate and thus take[11]

$$m_2(x) = \begin{cases} -0.2, & \text{for } x = 1; \\ +0.2, & \text{for } x = L; \\ 0, & \text{otherwise.} \end{cases}$$

The *reflection/transmission probabilities*[12] *of light of wavelength* λ *by glass plate of width* L are respectively

$$P(\lambda, L, -) = \lim_{\substack{y\to+\infty \\ y \text{ even}}} P(0, y, m_2(x), \lambda, -);$$

$$P(\lambda, L, +) = \lim_{\substack{y\to+\infty \\ y \text{ even}}} P(L + 1, y, m_2(x), \lambda, +).$$

[9]Beware: in general Feynman checkerboard is inappropriate to describe light; partial reflection is a remarkable exception.
[10]The mass is proportional to $(n - 1)/2\sqrt{n}$, where n is the refractive index; for glass $n \approx 1.5$ and $(n - 1)/2\sqrt{n} \approx 0.2$.
[11]This simplifying assumption requires negative mass for the left surface; the origin of that becomes clear after solving 27.
[12]It is more conceptual to define $P(\lambda, L, -) = \lim_{\varepsilon\to 0}\lim_{\Delta\to+\infty}\lim_{y\to+\infty, y\in\varepsilon\mathbb{Z}}\sum_{x\in\varepsilon\mathbb{Z}} P(x, y, m_3(x), \varepsilon, \lambda, \Delta, -)$ but we do not.

25. Plot the graph of the function $f(L) = P(0, 2L, m_2(x), 16, -)$.

26.* (Second main problem) **Two-surface reflection.** Find $P(\lambda, L, -)$, $P(\lambda, L, +)$, $P(\lambda, L, -)+P(\lambda, L, +)$, and $\max_L P(\lambda, L, -)$.

In fact the light is reflected *inside* the plate; this should be taken into account for a more accurate computation of the reflection probability (there is no hope for an exact solution for complicated matter such as glass). For this purpose we need to modify the model essentially.

First, the square side $\varepsilon > 0$ is now arbitrary. Fix $m > 0$ and let $m_3(x) = m$ for $0 < x \leq L$ and $m_3(x) = 0$ otherwise. Second, for each move starting in a square inside the glass, our vector is now *additionally rotated* through the angle $\arctan m\varepsilon$ clockwise (independently on if the checker does or does not turn in the square)[13]. In other words, set

$$\vec{a}(x, y, m(x), \varepsilon, \lambda, \text{modified}, -) := \sum_{x_0 : x_0/\varepsilon \in \mathbb{Z}} \sum_s R^{2\pi x_0/\lambda} R^{-k(s)\arctan m\varepsilon}\, \vec{a}(s, m(x)\varepsilon),$$

where the second sum is over all checker paths s from the square with the center $(x_0, 0)$ to the square with the center (x, y), starting with an upwards-right move and ending with an upwards-right move, and $k(s)$ is the number of moves starting inside the strip $0 < x \leq L$ in the path s. Set $P(x, y, m(x), \varepsilon, \lambda, \text{modified}, -) := |\vec{a}(x, y, m(x), \varepsilon, \lambda, \text{modified}, -)|^2$.

27.** **Thin-film reflection.** Find $\lim_{\varepsilon\to 0+} \lim_{\substack{y\to+\infty \\ y/\varepsilon \text{ even}}} P(0, y, m_3(x), \varepsilon, \lambda, \text{modified}, -)$. For which m the maximum of the expression over L equals $\max_L P(\lambda, L, -)$? It is allowed to use the existence of the limit $\lim_{\substack{y\to+\infty \\ (x+y)/\varepsilon \text{ even}}} R^{2\pi y/\lambda} \vec{a}(x, y, m_3(x), \varepsilon, \lambda, \text{modified}, -)$ for each x without proof.

6 External field

Question: what is the probability to find a right electron at (x, y), if it moves in a given electromagnetic field u?
Assumptions: the electromagnetic field vanishes outside the xy-plane, it is not affected by the electron.
Results: deflection of electron and spin 'precession' in a magnetic field, charge conservation.

An external electromagnetic field changes the motion as follows[14].

A common point of 4 squares of the checkerboard is called a *vertex*. An *electromagnetic field*[15] is a fixed assignment u of numbers $+1$ and -1 to all the vertices. For instance, in Figure 6, the field is -1 at the top-right vertex of each square (x, y) with both x and y even. Modify the definition of the vector $\vec{a}(s)$ by reversing the direction each time when the checker passes through a vertex with the field -1. Denote by $\vec{a}(s, u)$ the resulting vector. Formally, put $\vec{a}(s, u) = \vec{a}(s)u(C_1)u(C_2)\dots u(C_y)$, where $C_1, C_2, \dots, C_y$ are all the vertices passed by s. Define $\vec{a}(x, y, u, \pm)$ and $P(x, y, u, \pm)$ analogously to $\vec{a}(x, y, \pm)$ and $P(x, y, \pm)$ replacing $\vec{a}(s)$ by $\vec{a}(s, u)$ in the definition. For instance, if $u(C) = +1$ identically, then $P(x, y, u) = P(x, y)$.

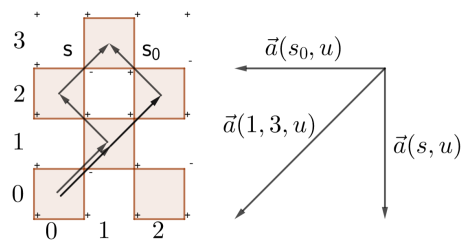

Figure 6: Paths in a field

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[13]This additional rotation is explained as follows. The light can be scattered in each square inside the glass several times. Each individual scattering gives a factor of $-im\varepsilon$ to our vector (viewed as a complex number) and may or may not change the movement direction. Assume that $m\varepsilon < 1$. Thus a move without changing the direction contributes a factor of

$$1 \text{ (no scattering)} - im\varepsilon \text{ (1 scattering)} + (-im\varepsilon)^2 \text{ (2 scatterings)} + \dots = \frac{1}{1 + im\varepsilon}.$$

Without a scattering, the checker moves straightly. Thus a turn in a particular square inside the glass contributes a factor

$$-im\varepsilon \text{ (1 scattering)} + (-im\varepsilon)^2 \text{ (2 scatterings)} + (-im\varepsilon)^3 \text{ (3 scatterings)} + \dots = \frac{-im\varepsilon}{1 + im\varepsilon}.$$

These are the same factors as in the model from §3 but additionally rotated through the angle $\arctan m\varepsilon$ clockwise.

[14]Beware that this method of adding the magnetic field, although well-known, is very different from the one from [Feynman].

[15]Or more precisely *rotation by the angle equal to a component of the electromagnetic vector-potential* (in the direction of the checker move). For simplicity we allow only two angles 0° and 180°; usually arbitrary angles are considered. The field u is interpreted as *magnetic* or *electric* depending on if the y-coordinate is interpreted as *position* or *time*.

28. Homogeneous field. Let $u(C) = -1$, if C is the top-right vertex of a square (x, y) with both x and y even, and $u(C) = +1$ otherwise. Find the vector $\vec{a}(x, y, u, +)$ and the probability $P(x, y, u, +)$ for $y = 1, 2, 3, 4$ and each integer x. What is $\sum_{x\in\mathbb{Z}}(P(x, y, u, +) + P(x, y, u, -))$ for fixed $y = 1, 2, 3$, or 4?

29. Spin 'precession' in a magnetic field. Plot the graph of the function $P(y) = \sum_{x\in\mathbb{Z}} P(x, y, u, -)$ for the field u from the previous problem using a computer.

For a given field u, a white square is *negative*, if u equals -1 at 1 or 3 vertices of the square.

30. Gauge transformations. Changing the signs of the values of u at the 4 vertices of one black square simultaneously does not change $P(x, y, u, +)$.

31. Curvature. One can make u to be identically $+1$ in a rectangle formed by checkerboard squares using the transformations from Problem 30, if and only if there are no negative white squares in the rectangle.

32. Homology. The field u equals $+1$ on the boundary of a rectangle $m \times n$ formed by checkerboard squares. Which can be the number of negative white squares in the rectangle?

33. Probability/charge conservation. Solve analogues of Problems 5, 6 for u not being identically $+1$.

7 Identical particles

Question: what is the probability to find electrons (or electron+positron) at F and F', emitted from A and A'?
Assumptions: the same as in the basic model; y-coordinate is interpreted as time.
Results: exclusion principle.

The motion of several electrons is described by a similar model as follows.

To each pair of paths s, s' of the checker, consisting of y moves each, assign a vector $\vec{a}(s, s')$ as follows. Start with the vector $(0, 1)$. Move the checker consecutively along both paths, and rotate the vector according to the same rule as in §1: each time when the checker changes the direction, the vector is rotated through 90° *clockwise*. (Thus the vector is rotated totally $t(s) + t(s')$ times, where $t(S)$ is the number of turns in a path S.) In addition, at the very end the vector is divided by 2^{y-1}. The final position of the vector is denoted by $\vec{a}(s, s')$. For instance, in Figure 2 we have $\vec{a}(s, s_0) = (-1/4, 0)$.

Fix squares $A = (0, 0)$, $A' = (x_0, 0)$, $F = (x, y)$, $F' = (x', y)$ and their diagonal neighbors $B = (1, 1)$, $B' = (x_0 + 1, 1)$, $E = (x - 1, y - 1)$, $E' = (x' - 1, y - 1)$, where $x_0 \neq 0, x' \geq x$. Denote[16]

$$\vec{a}(AB, A'B' \to EF, E'F') := \sum_{\substack{s:AB\to EF \\ s':A'B'\to E'F'}} \vec{a}(s, s') - \sum_{\substack{s:AB\to E'F' \\ s':A'B'\to EF}} \vec{a}(s, s'),$$

where the first sum is over all pairs consisting of a checker path s starting with the move AB and ending with the move EF, and a path s' starting with the move $A'B'$ and ending with the move $E'F'$, whereas in the second sum the final moves are interchanged.

The length square $P(AB, A'B' \to EF, E'F') = |\vec{a}(AB, A'B' \to EF, E'F')|^2$ is called the *probability*[17] *to find right electrons at F and F', if they are emitted from A and A'*. In particular, $P(AB, A'B' \to EF, EF) = 0$, i.e., two right electrons cannot be found at the same point; this is called *exclusion principle.*

34. Independence. For $x_0 \geq 2y$ and $x' > x$ express $P(AB, A'B' \to EF, E'F')$ through $P(x, y, +)$ and $P(x' - x_0, y, +)$.

35. Exclusion principle (for intermediate states). Prove that $\vec{a}(AB, A'B' \to EF, E'F')$ is not changed, if the sums in the definition are over only those pairs of paths s, s' which have *no common moves.*

Define $P(AB, A'B' \to EF, E'F')$ analogously also for $E = (x \pm 1, y - 1)$, $E' = (x' \pm 1, y - 1)$. Here we require $x' \geq x$, if both signs are the same, and allow arbitrary x' and x, otherwise.

36. Probability conservation. For each fixed $y \geq 1$ we have $\sum_{E,E',F,F'} P(AB, A'B' \to EF, E'F') = 1$, where the sum is over all quadruples $F = (x, y)$, $F' = (x', y)$, $E = (x \pm 1, y - 1)$, $E' = (x' \pm 1, y - 1)$, such that $x' \geq x$, if the latter two signs are the same.

[16] Here it is essential that s and s' are paths of particles of *the same sort*, e.g., two electrons. Otherwise the 2nd sum is omitted. The sign before the 2nd sum is changed to plus for some other sorts of particles, e.g., *photons* (particles of light).

[17] One should think of y as fixed, and the unordered quadruples $\{F, F', E, E'\}$ as the possible outcomes of an experiment.

8 Antiparticles

Question: the same as in §7 "Identical particles".
Assumptions: electron-positron pairs now created and annihilated, no interaction, encircling by reflecting walls.
Results: Feynman propagator in the continuum limit.

Finally we have reached an unexplored area: **we state an almost 40-years-old open problem.** Start with mentioning what is *not done* in this section:

- we do *not* give a definition of the new upgrade (it is unknown so far);
- the upgrade (even if defined) would *not* explain any new experimental results.

But

- we *do* give a precise statement of the problem: which exactly properties of the upgrade are requested;
- the upgrade is an important ingredient of further ones fantastically agreeing with experiment.

Informally, our plan is as follows. Checker paths turning downwards or upwards or forming cycles mean creation and annihilation of electron-positron pairs. Even if we start with just one electron, we might end up with many electrons and positrons. To each possible configuration of the resulting particles, we want to assign a complex number so that its length square is the probability of the configuration in a sense. The number itself is the sum over all possible transitions from the initial configuration to the final one, that is, all possible paths configurations joining them. To make the sum finite, we put reflecting walls around.

Fix a rectangle R formed by all the squares (x, y) such that $x_{\min} \le x \le x_{\max}$ and $0 \le y \le y_{\max}$ (the lines $x = x_{\min}$ and $x = x_{\max}$ are called *reflecting walls*). Fix $m, \varepsilon > 0$ called *mass* and *lattice step*.

An *initial configuration* is any assignment[18] of one of the signs "+" or "−" to some vertices inside R lying between the lines $y = 0$ and $y = 1$. Physically the signs mean the initial positions of positrons and electrons respectively (and points without a sign are vacant). For our game, this means that the checkers pass the vertices with the "−" sign upwards-left or -right, and the vertices with the "+" sign — downwards-left or -right (and do not pass through vertices without a sign). Analogously, a *final configuration* is an assignment to vertices between the lines $y = y_{\max}$ and $y = y_{\max} - 1$. An *intermediate configuration* is any assignment of one of the signs "+" or "−" to some vertices inside R such that the difference between the number of "+" and "−" signs on the 2 top vertices of each black square in the strip $1 \le y \le y_{\max} - 1$ equals the difference on the 2 bottom vertices. For our game, this means that the checkers start and finish motion in the lines $y = 0$ or $y = y_{\max}$ only. In other words, the signs at the vertices of each black square are in one of the 19 positions[19] shown in Figure 7 to the left. These 19 positions are called *basic configurations*.

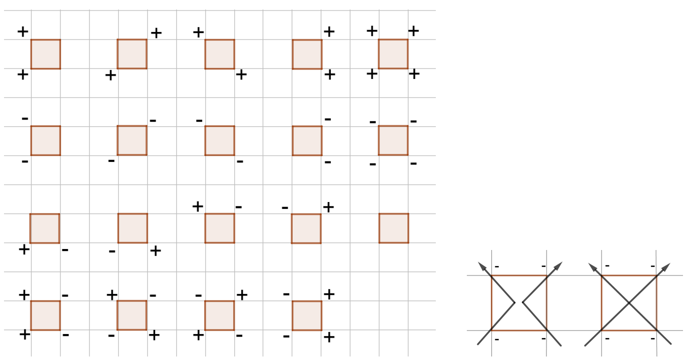

Figure 7: Basic configurations

Suppose that one has assigned a complex number (depending on $m\varepsilon$) to each of the 19 basic configurations. A "right" choice of the numbers is unknown; think of them as fixed parameters of our model.

Then take any intermediate configuration. In each black square inside R not having common points with the boundary, write the complex number assigned to the basic configuration in the black square. Assign the product of all the written numbers to the intermediate configuration.

[18]The assignment has nothing to do with the magnetic field from §6.
[19]We consider positions of signs, but *not* possible checker paths in a particular black square like in Figure 7 to the right.

Now to any pair of initial and final configurations, sum the complex numbers assigned to all possible intermediate configurations between them. Assign the resulting complex number to the pair.

37. (Puzzle) Restrict to configurations without "+" signs. Let $A, A', B, B', E, E', F, F'$ be the black squares defined in §7. Take any $x_{\min} < -y$ and $x_{\max} > x_0 + y$. Fix the initial and final configurations with exactly two "−" signs, located at the top-right vertices of the squares A, A', and E, E' respectively. Assign complex numbers to the 6 basic configurations without "+" signs so that the sum of the numbers assigned to all intermediate configurations *without "+" signs* equals to the vector $\vec{a}(AB, A'B' \to EF, E'F')$ from §7.

A case of particular interest is when the initial configuration consists of just one "−" sign at the top-right vertex of the square $(0,0)$ and the final configuration consists of just one "−" sign at the bottom-right or bottom-left vertex of a black square $(x, y_{\max})$. The complex numbers assigned to the pairs in question are denoted by $\vec{a}(x, y_{\max}, m\varepsilon, x_{\min}, x_{\max}, -)$ and $\vec{a}(x, y_{\max}, m\varepsilon, x_{\min}, x_{\max}, +)$ respectively.

The desired continuum limit of these complex numbers involves the following *modified Bessel functions and Hankel functions*:

$$K_0(x) = \int_1^\infty \frac{e^{-xt}}{\sqrt{t^2-1}}\,dt \qquad H_0^{(1)}(x) = \frac{2}{i\pi}\int_1^\infty \frac{e^{ixt}}{\sqrt{t^2-1}}\,dt$$

$$K_1(x) = x\int_1^\infty e^{-xt}\sqrt{t^2-1}\,dt \qquad H_1^{(1)}(x) = -\frac{2x}{i\pi}\int_1^\infty e^{ixt}\sqrt{t^2-1}\,dt$$

38.*** **Continuum limit.** Assign complex numbers to the 19 basic configurations so that for each $|y| < |x|$

$$\lim_{n\to\infty}\lim_{x_{\max}\to\infty} n\,\vec{a}\left(2\left\lfloor\frac{nx}{2}\right\rfloor, 2\left\lfloor\frac{ny}{2}\right\rfloor, \frac{m}{n}, -x_{\max}, x_{\max}, -\right) = \frac{m}{2\pi}K_0(m\sqrt{x^2-y^2});$$

$$\lim_{n\to\infty}\lim_{x_{\max}\to\infty} n\,\vec{a}\left(2\left\lfloor\frac{nx}{2}\right\rfloor, 2\left\lfloor\frac{ny}{2}\right\rfloor, \frac{m}{n}, -x_{\max}, x_{\max}, +\right) = -i\frac{m}{2\pi}\cdot\frac{x+y}{\sqrt{x^2-y^2}}K_1(m\sqrt{x^2-y^2});$$

and for each $|y| > |x|$ we have

$$\lim_{n\to\infty}\lim_{x_{\max}\to\infty} n\,\vec{a}\left(2\left\lfloor\frac{nx}{2}\right\rfloor, 2\left\lfloor\frac{ny}{2}\right\rfloor, \frac{m}{n}, -x_{\max}, x_{\max}, -\right) = i\frac{m}{4}H_0^{(1)}(m\sqrt{y^2-x^2});$$

$$\lim_{n\to\infty}\lim_{x_{\max}\to\infty} n\,\vec{a}\left(2\left\lfloor\frac{nx}{2}\right\rfloor, 2\left\lfloor\frac{ny}{2}\right\rfloor, \frac{m}{n}, -x_{\max}, x_{\max}, +\right) = \frac{m}{4}\cdot\frac{x+y}{\sqrt{y^2-x^2}}H_1^{(1)}(m\sqrt{y^2-x^2}).$$

Let us discuss the physical meaning of the upgrade. For $y \gg |x|$ twice the absolute values of the right-hand sides of the latter two equations are very close to the right-hand sides of the answer to Problem 15. Thus one may wish to interpret the upgrade as a more accurate approximation for the probability to find the electron in a square (x, y). But this faces serious objections.

First, it is *in principle* impossible to measure the coordinates of an electron exactly [20]. Such a priori uncertainty has the same order of magnitude as the correction introduced by the upgrade. Thus the upgrade does not actually add anything to description of the electron motion.

Second, for fixed initial configuration, the squares of the absolute values of the numbers assigned to all the possible final configurations do *not* sum up to 1 (even in the continuum limit). The reason is that distinct configurations are not mutually exclusive: there is a positive probability to detect an electron at two distinct points during a measurement. There is nothing mysterious about that: Any measurement necessarily influences the electron. This influence might be enough to create an electron-positron pair from the vacuum. Thus one can detect a newborn electron in addition to the initial one; and there is no way to distinguish one from another.

To summarize, the upgrade lacks a direct physical interpretation, and should only be considered as an ingredient for further upgrades.

[20] This should not be confused with *uncertainty principle*, which does not allow simultaneous measurement of the coordinates and momentum.

9 Interaction

Question: the same as in §7 "Identical particles".
Assumptions: the electrons now generate an electromagnetic field affecting the motion.
Results: experimental: repulsion of like charges and attraction of opposite charges is expected.

Construction of the required model is a widely open problem as well because in particular it requires the missing *Minkowskian* lattice gauge theory.

10 $(1+1)$-dimensional quantum electrodynamics

Question: what is the probability to find electrons (or electron+positron) with momenta q and q' in the far future, if they were emitted with momenta p and p' in the far past?
Assumptions: interaction now switched on; all simplifying assumptions removed except the default ones: no nuclear forces, no gravitation, electron moves only along the x-axis (and y-coordinate is interpreted as time).
Results: experimental: quantum corrections.

Unifying the (so far unknown) upgrades discussed in the previous two sections would give an elementary definition of $(1+1)$-dimensional QED.

Epilogue (underwater rocks)

We hope that at least some of our readers have become interested in quantum theory and want to learn more about it. As an epilogue, let us give a few warnings to such readers.

In popular science, this theory is usually oversimplified. This sequence of problems is not an exception. The toy models introduced here are very rough and should be considered with a grain of salt. Simplicity is their only advantage; if taken too seriously, the models could even give a wrong physical intuition. Real understanding of quantum theory requires excellent knowledge of both physics and mathematics.

We should also remark that quantum filed theory still has not been constructed on a mathematical level of rigor, and in its lattice models, there are almost no mathematical results; what we have is usually just a numeric simulation. Finally, there are "*theories of New Physics*" which are developed without any objective truth criterion: such theories are supported by neither experimental nor mathematical proofs (and some of them have experimental disproofs).

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