Homologically minimal fourfolds.

joint work with

Anton Mellit

(IPMU 11-0100)

<u>Def.</u> $E \in \mathcal{D}^b_{coh}(X)$ is exceptional if $\mathrm{Hom}(E, E) = \mathbb{C}$

$$\mathrm{Hom}(E, E[k]) = 0$$

for $k \neq 0$

E_1, E_2 is exc. pair

if $\operatorname{Hom}(E_2, E_1[k]) = 0$

$E_1, \dots, E_R$ is exc. collect.

$\operatorname{Hom}(E_k, E_\ell[u]) = 0$

$k > \ell$

$E C$ is <u>full</u>

if

$$\langle E_1, \ldots, E_R \rangle = D^b_{coh}(X)$$

Thm (Beilinson)

$\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$

is FEC on

$\mathbb{P}^n$

Def X is homologically minimal if it has FEC of $(\dim X + 1)$ objects

Examples

odd-dim quadrics (Kapranov)

V_5 (Orlov)

V_{22} (Kuznetsov)

G_2/P, 5-fold sections of $OGr(5,10)$ & $LGr(3,6)$

„Russian conjecture"
(unpublished)

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even-dimensional HM variety is $\mathbb{P}^{2n}$.

Not known

even in $\dim = 2$.

Easy for X-Fano

$$HH_k(X) = \bigoplus_{p-q=k} H^{p,q}(X)$$

$$\parallel$$

$$\bigoplus HH_k(\langle E_i \rangle)$$

$$HH_0(\langle E \rangle) = \mathbb{C}$$

$$HH_{k\neq 0}(\langle E \rangle) = 0$$

Cor. $h^{p,q} = 0 \quad p \neq q$

$$R = \sum h^{p,p} \geq (\dim X + 1)$$

If $R = \dim X + 1$

$\Downarrow$

Same Hodge # as $\mathbb{P}^d$

Thm (Libgober-Wood)

$[c_1 c_{d-1}]$

depends only on Chern numbers

Cor for HM X

$$c_1 c_{d-1} = \frac{d(d+1)^2}{2} \neq 0$$

Cor X is projective

1) either Fano

2) or K_X is ample

Thm (Bondal–Polishchuk)

E_i are sheaves

$\Updownarrow$

coll is strict

$\Downarrow$

X is Fano

Thm (Positselski)

$$\frac{\begin{array}{c} E_i\text{-sheaves} \\ \cup \\ E_i\text{-vector bundles}\end{array}}{}$$

(up to common sheaves)

[A]

K-theory $K^0(X)$

has bilinear

Euler pairing

$$\chi([E],[F]) = \sum (-1)^n \dim \operatorname{Hom}(E, F[n])$$

X has FEC

$\Updownarrow$

$K^0(X) \simeq \mathbb{Z}^R$

$[E_1], \ldots, [E_R]$

is uni-upper Δ $\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \\ & & & 1 \end{pmatrix}$

Warm-up

$\dim X = 2$

X - HM

$$\Downarrow$$

$$c_1^2 = 9, \quad c_2 = 3$$

X-Fano $\Longrightarrow$ $X \simeq \mathbb{P}^2$ ✓

$-K_X$ ample $\Longrightarrow$ $X = \mathrm{Ball}/\Gamma$

$\boxed{\text{torsion in } H^2}$ (100 cases)

Thm X - Fano 4-fold

$E_1, \dots, E_s \in D^b_{coh}(X)$

is FEC

$$\Downarrow$$

$$X \simeq \mathbb{P}^4$$

Proof in 2 steps

1) Chern numbers

2) K-theory

Thm (Wilson 1986)

If $X \not\cong \mathbb{P}^4$

then $\boxed{\text{Wilson 4-folds}}$

$$\left(\begin{array}{c} c_1^4 = 225,\ c_1^2 c_2 = 150, \\ c_2^2 = 100,\ c_1 c_3 = 50 \\ c_4 = 5 \end{array} \right)$$

Also Yeung 2000, Orlov (un)

Mysterious:

same Pontrjagin #
as $\mathbb{P}^4$

L-genus - clear

$\hat{A}$-genus - why?

Chern numbers

y

Hilbert polynomial

$$P(n) = \chi(\mathcal{O}(n)) =$$

$$1 + \frac{25}{8} n(n+1)\left(3n(n+1) + 2\right)$$

$$\Lambda = \langle [\mathcal{O}], [\mathcal{O}(1)], [\mathcal{O}(2)], [\mathcal{O}(3)], [\mathcal{O}(4)] \rangle$$

is sublattice in $K^0(X)$ of

index $(c_1(X)^4)^5 = 225^5$

$\boxed{\text{odd \#}} \longrightarrow$

index odd

$$\Downarrow$$

$$K^0(X) / 2K^0(X)$$

$$\uparrow ?$$

$$\Lambda / 2\Lambda$$

$\Rightarrow$ no $\underline{1}-\Delta$

basis in

$V = \Lambda / 2\Lambda$

$\Downarrow$

no FEC in X

V is 5-dim
vector space
$/\mathbb{F}_2$
with bilinear
form

$$A_{i,j} = P(j-i)$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$q(x) = \langle x, x \rangle$$

is a quadric

of corank 1

$$q = \sum_i x_i^2 + x_3(x_1 + x_5) + x_2 x_4$$

$$[Q] = (1 + [A^1])^2 \cdot [\cdots$$

$$\Downarrow$$

$$\frac{19 \text{ points on } Q}{12 \text{ points } \notin Q}$$

(roots) $\to$ 12 exc. obj.

mod 2.

Serre functor

$$E \to E \otimes \omega_X[4]$$
$$\parallel$$
$$E \otimes \omega(1)$$

is isometry:

$$\langle a, b \rangle = \langle b, Sa \rangle = \langle Sa, Sb \rangle$$

$$S = A^{-1} A^{T}$$

$P(\ell) \bmod 2$ depends on $\ell \bmod 8$

$$S^{8} = \mathrm{Id} \pmod 2$$

There are

2 orbits of S

8 „line bundles"

4 „rank 2 bundles"

Braid group

acts on set of FECs

V can be i-th

$\Updownarrow$

V can be 1st

Vectors from small orbit have only 4 right orthogonals

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$ — not in FEC

„line bundles" have only right orthogonal „line bundles"

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{- basis}$$

doesn't exist

Problem

$P(x)$ — poly of $\deg = d$

$A_{i,j} = P(j-i)$

$(d+1)\times(d+1)$ — matrix

$D = \det A$

When A is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ bil. form

$/\mathbb{Z}[\frac{1}{D}]$?

Thm (Bondal) non-degenerate bilinear form / $\mathbb{C}$ is $\begin{pmatrix}1 & & * \\ & 1 & \\ 0 & & 1\end{pmatrix}$-able

$$\Uparrow$$

it is not skew-symmetric!