Prokhorov-Hacking degenerations in the mirror

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① $W \in$ ~~k[x,y]~~ Lorent polynomial in d variables:

$w: (\mathbb{C}^*)^d \longrightarrow \mathbb{C}$

Consider also form $\omega := \prod \frac{dx_i}{x_i}$ and "j-function"

$$J_w(t) = \frac{1}{(2\pi i)^d} \int_{|x_1|=\dots=|x_d|=\varepsilon} \frac{1}{1-tw}\,\omega$$

Examples: $w_{II} = x + y + \frac{1}{xy}$

$w_{III} = x + y + z + \frac{1}{xyz}$

Dutch Trick: $J_w(t)$ is an analytic solution of the Picard-Fuks equation for pencil $\{1-tw=0\}$

Take $f \in Cr_d$ s.t. $J_{f^*w}(t) = J_w(t)$

Consider "special subgroup" $SCr_d \subset Cr_d$ s.t. $f^*\omega = \omega$ $\forall f \in SCr_d$ and f^*w is again a Lorent polynomial

($d=2$, $SCr_2 = Symp$ is called symplectic (Usnich)). Let us study this group:

② • $SCr_{d=2}$ contains maximal torus, since $\omega = \prod_i d\log x_i$ $T := (k^*)^2$;

• For normalizer of T, we have $N(k^{*2})/_{k^{*2}} = Sl_2(\mathbb{Z})$;

• $p: (x,y) \mapsto (y, \frac{1+y}{x})$ also belongs to SCr_2.

Conjecture 1 (Usnich). The above generate whole SCr_2

Take $H \subset SCr_2$

$H = \langle p, Sl_2(\mathbb{Z}) \rangle$

$Sl_2(\mathbb{Z}) = \{ C = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, C^3 = I^4 = [C, I^2] = 1 \}$ (*)

$p^5 = 1$ $\quad pCp = I$

Conjecture 2 (Usnich). (*) are all gen relations for H.

Tropicalization

Set $x = t^a$ $y = t^b$ this gives a homomorphism μ: Symp $\to \mathbb{T}$ Thompson's group of piecewise-linear transformations of $\mathbb{Z}^2$

$SL_2(\mathbb{Z})$ acts linearly on $\mathbb{Z}^2$

$\mu(P)$

$(a,b) \mapsto (b, \min\{0, b^3 - a)$

Th. (Usnich) $\left\langle L, C, I \;\middle|\; \begin{array}{l} I = LCL \\ C^3 = I^4 = L^5 = 1 \\ IC^3 = CI \\ (CIL)^7 = 1 \end{array} \right\rangle = \mathbb{T}$.

Comming back to (1):

(3) Newton $(w) = \left\langle \begin{array}{l} m \in \mathbb{Z}^d \\ a_m \neq 0 \end{array} \right\rangle$, $w = \sum a_m x^m$

Examples: 1) $w = x + y + z + \frac{1}{xyz}$ $w_i = m_i w$, $i = 1,2,3$ s.t. some ~~linear~~ affine transformations

$$w_1 = z(x+y) + z^{-1}\left(1 + \frac{1}{xy}\right),$$
$$w_2 = z(x+1) + y + \frac{1}{xyz^2},$$
$$w_3 = z(x+y+1) + \frac{1}{xyz^3}.$$

Apply more interesting transformations:

$$f_1 : (x,y,z) \mapsto \left(x, y, \frac{z}{x+y}\right),$$
$$f_2 : (x,y,z) \mapsto \left(x, y, \frac{z}{1+x}\right),$$
$$f_3 : (x,y,z) \mapsto \left(x, y, \frac{z}{1+x+y}\right)$$

$$\tilde{w}_1 = z + z^{-1}(x+y)\left(1 + \frac{1}{xy}\right),$$
$$\tilde{w}_2 = z + y + \frac{(1+x)^2}{xyz^2},$$
$$\tilde{w}_3 = z + \frac{(1+x+y)^3}{xyz^3}$$

Take Newton polytopes of $\tilde{w}_i$ and corresponding Toric varieties:

$\tilde{w}_1 \rightsquigarrow T_1 = \mathrm{Cone}\,(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-K))$ $\tilde{w}_3 \rightsquigarrow \mathbb{P}(1,1,1,9)/\mu_3 = T_3$

$\tilde{w}_2 \rightsquigarrow T_2 = \mathbb{P}(1,1,2,4)$

Claim. T_i are degenerations of $\mathbb{P}^3$

Proof. e.g. $X_2 \subset \mathbb{P}(1,1,1,1,2)$

④ Degenerations of $\mathbb{P}^2$:

Take $w = x + y + \frac{1}{xy}$,

Apply $y \rightsquigarrow yx \quad x \rightsquigarrow x$, obtain something

— // — for $w = x(1+y) + \frac{1}{x^3 y}$ obtain:

$$x \rightsquigarrow \frac{x}{1+y}, \quad y \rightsquigarrow y$$

$$x + \frac{(1+y)^2}{xy} \quad (**)$$

Newton $(**)$ = Polytope corresponding to $\mathbb{P}(1,1,4)$,

— // — $\mathbb{P}(1,4,25)$ for the first case

take $w = \sum x^i f_i(y) \rightsquigarrow \tilde{w} = \sum x^i \frac{f_i(y)}{(1+y)^2}$

$\tilde{w}$ is Lorent iff $f_i(y) \vdots (1+y) \ \forall i > 0$.

This gives a „map" between $\mathbb{P}(1,1,4)$ and $\mathbb{P}(1,4,25)$

~~Ex~~. ~~Are~~ 1, 2, 5, 23 ... are Markov numbers.

Q. Can we produce more Lorent polynomials?

Answ.: Yes for all solution of Markov eq. there is a polynomial.

Markov (1881): $x^2 + y^2 + z^2 = 3xyz$

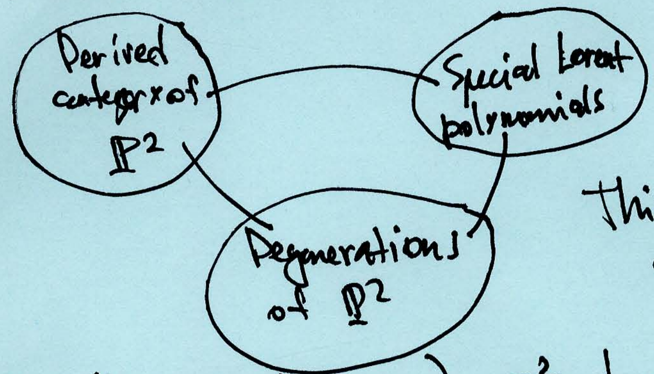
(1, 1, 1) is a solution, and any other positive solution can be obtain from (1, 1, 1) by a sequence of Markov's mutations (Markov): $(x, y, z) \longmapsto (x, y, z')$

$$z \cdot z' = x^2 + y^2$$

(claster mutation (Fomin - Zelevinsky for claster algebra))

The above ~~equa~~, actually, is an example of St_2 embedding of $Sl_2(\mathbb{Z})$ in Cr_2: $(x,y,z)\mapsto(xz : yz : x^2+y^2)$

Markov triples arise in



This is a mirror symmetry

<u>Th</u>. (Hacking-Prokhorov) • $\mathbb{P}^2$ has $\mathbb{Q}$-Gorenstein deformations to $\mathbb{P}(x^2, y^2, z^2)$, where (x,y,z) is a Markov triple;

• all other deformations are ~~dominated~~ ($\mathbb{Q}$-Gorenstein with cyclic quotient singularities) are dominated by above.