Fano classification and mirror duality:

a progress report

V. Przyjalkowski, Weak Landau–Ginzburg models for smooth Fano threefolds.

S.Galkin, Toric del Pezzo surfaces and pencils of elliptic curves with low ramification.

V.G. Spectra and strains, hep-th.

Missing: a singularity theory *in globo* ?

Typical setup of the singularity theory:

study R local ring, either alone, more frequently with $f \in R$ a (germ of a) function.

May study complex R's or R's such as $\mathbb{C}[x_i]$ and complex f's.

It is usual to study single and local objects and phenomena.

Envision a singularity theory in globo:

a mass of singularities inscribed in a global object and defining it by imposing so many conditions as to render the existence “almost impossible” = essentially combinatorial in nature

The classical Morse/Lefschetz theory:

A global function on a variety / structure of a pencil over $\mathbb{A}^1$,

critical points simple;

critical values distinct.

Anti–Morse/Lefschetz:

still a pencil $f : X \longrightarrow \mathbb{A}^1$, but

critical values come together and merge as much as possible.

Qualitatively: can be for two reasons;

either simple critical points at the same level;

or fewer more complex critical points.

Need to quantify contributions from both factors in the future.

An emblematic example:

consider a general semistable rational elliptic pencil $f : \mathcal{E} \longrightarrow \mathbb{A}^1$.

Known to have 12 bad fibers, each a nodal rational curve.

By passing to a codimension 1 subspace in its moduli, can make two of the critical values merge.

Continue until the moduli are exhausted.

Beauville's theorem:

Beauville classified [1982] completely the stable families of elliptic curves over P^1 having exactly four singular fibres. The minimal number of singular fibres is 4. There are 6 such families.

$$X^3 + Y^3 + Z^3 + tXYZ = 0$$

$$X(X^2 + Z^2 + 2ZY) + tZ(X^2 - Y^2) = 0$$

$$X(X - Z)(Y - Z) + tZY(X - Y) = 0$$

$$(X + Y)(Y + Z)(Z + X) + tXYZ = 0$$

$$(X + Y)(XY - Z^2) + tXYZ = 0$$

$$X^2Y + Y^2Z + Z^2X + tXYZ = 0$$

One would think of the Beauville surfaces as instances of the anti–Lefschetz behaviour.

Remark. Theory of dessins d'enfants must be viewed is in this framework. Object becoming essentially combinatorial in nature once the singularities merge extremally.

With a little bit of stretching, we are looking for generalizations of dessins to morphisms of higher relative dimension.

Classifying Fano varieties

A Fano variety F is a complex smooth projective variety with $-K_F > 0$. Fanos are completely classified in dimensions $1, 2$ and 3.

Birational geometers say they do not have ideas that are good for higher dimensions.

Frobenius manifolds.

A Frobenius manifold is associated with a Fano F.

Recall that a Frobenius manifold is a complex differential geometric structure:

Definition. The Frobenius manifold attached to the variety V is the following system of objects:

1) the linear space $M = H^\bullet(V)$

2) the potential Φ on it, given by the formula

$$\Phi(\Gamma) = \sum_{n\geqslant 3} \sum_{\beta} \frac{q^\beta}{n!} < I^V_{0,n,\beta} > (\Gamma^{\otimes n}).$$

More precisely, considering a homogeneous basis $\Delta_a (a = 0, \ldots, D)$ of the space $H^\bullet(V)$, one has, at a point $\Gamma = \sum_a x^a \Delta_a$,

$$\Phi(\Gamma) = \sum_{n=n_0+n_1+\ldots+n_D\geqslant 3} \frac{1}{n_0! \ldots n_D!} \left\langle I^V_{0,n,\beta} \right\rangle (\Delta_0^{\otimes n_0} \otimes \ldots \otimes \Delta_D^{\otimes n_D})(x^0)^{n_0} \ldots (x^D)^{n_D}.$$

3) the constant metric g_{ab}on the tangent bundle $\mathcal{T}_M$ that becomes just the Poincare pairing under identification $\mathcal{T}_M = M$,

4) the functional $A : S^3(\mathcal{T}_M) \to \mathcal{O}(M)$ such that for any triple of constant vector fields one has

$$A(X, Y, Z) = (XYZ)\Phi.$$

The periods conjecture of mirror symmetry:

The (essential subquotient of) the “second extended structural connection” in the lambda–direction at the divisorial parameter (or, more specifically, the parameter $-K \in H^2(F)$ of a Fano is a connection of Gauss–Manin type.

Definition. The Euler vector field w.r.t. the chosen basis Δ_a is the field

$$E = \sum_a \left(1 - \frac{|\Delta_a|}{2}\right) x^a \partial_a + \sum_{b_i \Delta_b = 2} r^b \partial_b,$$

where r^b are the coefficients of the decomposition of $-K_V$ w.r.t. the chosen basis:

$$-K_V = \sum_{b | \Delta_b = 2} r^b \Delta_b.$$

Definition. Extended first structural connection on the extended Frobenius manifold $\hat{M} = M \times \mathbb{P}^1$ is the connection $\hat{\nabla}$ on $\hat{\mathcal{T}} = pr_M^*(\mathcal{T}_M)$:

$$\widehat{\nabla}_X(Y) = \lambda X \circ Y$$

$$\hat{\nabla}_{\frac{\partial}{\partial \lambda}}(Y) = E \circ Y - \frac{1}{\lambda}\,[E, Y]\,.$$

Definition. The second structural connection is the Fourier transform of the first connection in the t–direction.

We usually use t for the coordinate on the base of the second connection.

The periods conjecture of mirror symmetry:

The (essential subquotient of) the second extended structural connection in the lambda–direction at the divisorial parameter is a Gauss–Manin connection.

Strategy of classification modulo the periods mirror conjecture:

Classify Fano varieties by classifying their Frobenius manifolds.

In turn, classify Frobenius manifolds by classifying the respective Gauss–Manin connection or pencils that give rise to them.

It is more natural to start with rank 1 Fanos: Pic $F = \mathbb{Z}$. We recall that the index of such a Fano is defined to be the integer d such that $-K = d \cdot 1 \in \mathbb{Z} = \text{Pic } F$.

An argument in favor of the strategy is a theorem proved, case by case, by Przyjalkowski and myself.

The Eta Theorem. The second structural connection of the Frobenius manifold of a rank 1 Fano 3-fold with invariants $(N = (-K)^3/d^2, d)$ is of Picard–Fuchs type and can be obtained in the following uniform way. Let $T = T(q)$ be the inverse of the suitable Hauptmodul on $X_0(N)$ (that is, the one with the "right" constant term). Consider

$$\Phi = (q^{1/24} \prod (1 - q^n) q^{N/24} \prod (1 - q^{Nn}))^2 T^{-\frac{N+1}{12}}.$$

Then the second connection is isomorphic to the Picard–Fuchs differential equation satisfied by Φ with respect to $t = T^{\frac{1}{d}}$.

This theorem sends contradictory signals:

One can succeed with the strategy in principle;

however, dimension 3 is misleading: starting with dimension 4, the Lie algebra of the differential Galois group of a general second connection is NOT $sl(2)$ but typically so/sp of higher rank.

Modularity not at the root. A by–product of something deeper.

In order to break the ‘modularity barrier’ and pass to higher dimensions, we operate on the heuristics that the second connections arise from anti–Lefschetz pencils (or,anti–Lefschetz Landau–Ginzburg potentials).

We define extremality of a pencil in cohomological terms.

Geometric ramification. Let $U \subset \mathbb{P}^1/\mathbb{C}$ be a Zariski open subset, $S = \mathbb{P}^1 - U$. Let L be a rank r non-trivial irreducible polarized local system over U, i.e. a representation $\varphi : \pi_1(U^{an}) \longrightarrow O(r)/Sp(r)$, and let L_x be its generic fiber. Its ramification is

$$R(L) = \sum_{s \in S(\mathbb{C})} \dim L_x / L_x^{I_s},$$

the sum over all bad points of the codimensions of the invariants of the respective local monodromy.

Low ramified local systems and their conductors. A local system L as above is said to be low ramified if

$$R(L) = 2\,\mathrm{rk}\,L.$$

Its geometric conductor is the respective closed subscheme of $\mathbb{P}^1\mid_{\mathbb{C}}$: the union of points $s \in S(\mathbb{C})$ each taken with multiplicity $\dim L_x/L_x^{I_s}$.

In other words, a local system is low ramified if the Euler characteristic of its $*$-extension as a constructible sheaf is 0. A pencil $f : \mathcal{E} \longrightarrow \mathbb{A}^1$ is low ramified if the (essential constituent of) the local system of Rf^{n-1} is low ramified.

The key problem:

Establish a link between the low ramified pencils and Frobenius manifolds associated to Fano varieties/stacks.

$$\Rightarrow? \Leftarrow?$$

A version of the anti–Morse: the special Laurent polynomials zoo.

Let M be the standard lattice in $\mathbb{R}^N$, and let P be a lattice polytope, i.e. a convex hull of a finite number of lattice points. Inasmuch as it is allowed by P, a generic non–zero polynomial π 'tends' to be Morse(–Lefschetz), i.e.have simple singularities and critical values.

Stratify the vector space X of the Laurent polynomials supported on P according to the ramification; call the the geometric points in the highest codimension strata <u>special Laurent polynomials</u>.

Weak Landau–Ginzburg model

Consider a torus $T_{LG} = \mathbb{G}_{\mathrm{m}}^n = \prod_{i=1}^n \operatorname{spec} \mathbb{C}[x_i^{\pm 1}]$ and a function f on it. This function is represented by Laurent polynomial: $f = f(x_1, x_1^{-1} \ldots, x_n, x_n^{-1})$. Let $\varphi_f(i)$ be the constant term (i. e. the coefficient at $x_1^0 \cdot \ldots \cdot x_n^0$) of f^i. Put

$$\Phi_f = \sum_{i=0}^{\infty} \varphi_f(i) \cdot t^i \in \mathbb{C}[[t]].$$

Definition The series $\Phi_f = \sum_{i=0}^{\infty} \varphi_f(i) \cdot t^i$ is called *the constant terms series* of f.

Definition. Let $L_F I = 0$ be the second connection DE. (A unique) analytic solution of $L_F I = 0$ of type

$$I_{H^0}^F = 1 + a_1 t + a_2 t^2 + \ldots \in \mathbb{C}[[t]], \qquad a_i \in \mathbb{C},$$

is called *the fundamental term of the regularized I-series* of F.

Let $\mathbf{1}$ be the class in $H^0(F, \mathbb{Z})$ dual to the fundamental class of X. Then this series is of the form

$$I_{H^0}^X = 1 + \sum_{\beta > 0} \langle \tau_{-K_X \cdot \beta - 2} \mathbf{1} \rangle_\beta \cdot t^{-K_X \cdot \beta}.$$

Definition. Let F be a smooth Fano variety and $I^X_{H^0} \in \mathbb{C}[[t]]$ be its fundamental term of regularized I-series. The Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$ is called *a very weak Landau–Ginzburg model* for F if

$$\Phi_f(t) = I^X_{H^0}(t).$$

Definition. The Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$ is called *a weak Landau–Ginzburg model* for F if it is a very weak Landau–Ginzburg model for F and for almost all $\alpha \in \mathbb{C}$ the hypersurface $(1 - \alpha f = 0)$ is birational to a Calabi–Yau variety.

Theorem. [Przyjalkowski.] All 17 families of rank 1 Fano threefolds admit weak Landau–Ginzburg models that are special Laurent.

N.	Index	Degree	Description	Weak LG model
1	1	2	Sextic double solid V_2' (a double cover of $\mathbb{P}^3$ ramified over smooth sextic)	$\frac{(x+y+z+1)^6}{xyz}$
2	1	4	The general element of the family is quartic	$\frac{(x+y+z+1)^4}{xyz}$
3	1	6	A complete intersection of quadric and cubic	$\frac{(x+1)^2(y+z+1)^3}{xyz}$
4	1	8	A complete intersection of three quadrics	$\frac{(x+1)^2(y+1)^2(z+1)^2}{xyz}$
5	1	10	The general element is V_{10}, a section of $G(2,5)$ by 2 hyperplanes in Plücker embedding and quadric	$\frac{(x^2+x+z+zx+y+yx+yz)^2}{xyz}$
6	1	12	The variety V_{12}	$\frac{(x+z+1)(x+y+z+1)(z+1)(y+z)}{xyz}$
7	1	14	The variety V_{14}, a section of $G(2,6)$ by 5 hyperplanes in Plücker embedding	$\frac{(x+y+z+1)^2}{x} + \frac{(x+y+z+1)(y+z+1)(z+1)^2}{xyz}$
8	1	16	The variety V_{16}	$\frac{(x+y+z+1)(x+1)(y+1)(z+1)}{xyz}$
9	1	18	The variety V_{18}	$\frac{(x+y+z)(x+xz+xy+xyz+z+y+yz)}{xyz}$
10	1	22	The variety V_{22}	$\frac{xy}{z} + \frac{y}{z} + \frac{x}{z} + x + y + \frac{1}{z} + 4 + \frac{1}{x} + \frac{1}{y} + z + \frac{1}{xy} + \frac{z}{x} + \frac{z}{y} + \frac{z}{xy}$
11	2	$8 \cdot 1$	The general variety is double Veronese cone V_1 (a double cover of the cone over the Veronese surface branched in a smooth cubic)	$\frac{(x^2+y^2+z^2+1)^3}{xyz}$
12	2	$8 \cdot 2$	Quartic double solid V_2 (a double cover of $\mathbb{P}^3$ ramified over smooth quartic)	$\frac{(x+y+1)^4}{xyz} + z$
13	2	$8 \cdot 3$	A smooth cubic	$\frac{(x+y+1)^3}{xyz} + z$
14	2	$8 \cdot 4$	A smooth intersection of two quadrics	$\frac{(x+1)^2(y+1)^2}{xyz} + z$
15	2	$8 \cdot 5$	The variety V_5, a section of $G(2,5)$ by 3 hyperplanes in Plücker embedding	$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz$
16	3	$27 \cdot 2$	A smooth quadric	$\frac{(x+1)^2}{xyz} + y + z$
17	4	64	$\mathbb{P}^3$	$x + y + z + \frac{1}{xyz}$

Table 1: Weak Landau–Ginzburg models for Fano threefolds.

Theorem. [Galkin]. Del Pezzo surfaces of degrees $9, 6, 5, 4, 3, 2, 1$ admit weak Landau–Ginzburg models that are Laurent polynomials of low ramification. The del Pezzo surfaces of degrees $8, 7$ admit weak LG models that are of minimal ramification among the Laurent polynomials on the polytope on which they are supported.