Research statement

My mathematical research can be approximately split into four different areas: algebraic varieties with an action of a torus, actions (including representations) of semisimple algebraic groups, combinatorics of polytopes, and combinatorics on words.

All algebraic varieties in this research statement are considered over $\mathbb{C}$.

One of the areas where I did my research was algebraic varieties with an action of an algebraic torus, more specifically, equivariant deformation theory of such varieties. Deformation theory is (or can be used as) a tool to study singularities of varieties. Informally speaking, to deform a (singular) variety X is to include X in a family of varieties parametrized by some parameter space S. So, one gets varieties, which are "close" or "similar" to X and (usually) "less singular" than X. And if an algebraic group, for example a torus, acts on X, equivariant deformations show how a variety can be changed if we want to preserve the action of the group on each "deformed version" of X. In my research, I computed the space of so-called infintesimal equivariant deformations of a certain class of three-dimensional affine varieties with an action of a torus (see [13, Theorem 4.32]) and found so-called formally versal equivariant deformations of these varieties (see [13, Theorem 6.54]), i. e. deformations, which, in a certain sense, contain information about all possible equivariant deformations of these varieties. For more details about my research and my plans in this area, see Section 1.

In the area of semisimple group actions and representations, my research mostly dealt with so-called generalized flag varieties. In one part of my research I studied actions of various groups on generalized flag varieties with an open orbit. More precisely, I have continued the research started by [37] and classified all triples (G, P, n), where G is a simple algebraic group not of type A, $P \subseteq G$ is a nonmaximal parabolic subgroup, $n \in \mathbb{N}$, and G acts on $(G/P)^n$ with an open orbit (see [10, Theorem 1.2]); such a classification for maximal parabolic subgroups P was known before. I have also extended the previously known (see [30], [31]) classification of all triples (G, P, n), where G is a simple algebraic group of type A or C, $P \subseteq G$ is a parabolic subgroup, $n \in \mathbb{N}$, and G acts on $(G/P)^n$ with finitely many orbits to the cases when G is a simple algebraic group of any other type (and all other conditions are true), see [10, Corollary 1.4]. Later I continued the research of commutative unipotent group actions started in [21] for actions on arbitrary varieties and in [4] on generalized flag varieties. Namely, a classification of actions of commutative unipotent groups with an open orbit on several classes of varieties, including projective spaces, was obtained in [21]. Also, all generalized flag varieties X such that at least one action of the commutative unipotent group of dimension $\dim X$ on X with an open orbit exists were previously found in [4]. I extended these results by classifying the actual actions on these generalized flag varieties X, see [11, Theorem 25]. My results and plans in this area can be found in Section 2.

In another part of my research I studied cohomology theory of generalized flag varieties, so-called Schubert calculus. I particularly studied generalized flag varieties of groups of type A, D, and E, and I found a criterion when a Schubert cell occurs in the decomposition of a product of several divisors on a generalized flag variety into a linear combination of Schubert cells with coefficient 1, see [15, Theorem 8.1]. Using this criterion, I found the maximal possible codimension of such a Schubert cell, see [15, Lemma 11.1, Proposition 11.4, and Theorem 11.5]. This result is available on my personal website and will appear on arxiv shortly. It can be used for computation of the *canonical dimension* of such generalized flag varieties, introduced in [27]. My results and plans in this area can be found in Section 3.

I also participated in two joint research projects. The first one was aimed to compute the K-theory of certain *twisted flag varieties*, i. e. varieties over non-algebraically-closed fields that become generalized flag varieties after scalar expansion to the algebraic closure of the field, and of *cohomological invariants* (defined in [19]) of such twisted flag varieties. These results were published in [5]. My input consisted of finding explicit descriptions (finite lists of generators and relations) of certain commutative algebras, see [5, Theorem 3.4]. These algebras were known to be equal to the K_0 ring of some of the twisted flag varieties. Also, some of the relations from these lists were used to find indecomposable degree 3 cohomological invariants (the definition can be found in [32]) of twisted flag varieties (see proofs of

Propositions 7.1 and 8.1 in [5]). A more detailed description of my results in this project and their applications can be found in Section 4.

The second joint project studies *equivariant cohomology* (see [23] for a general reference on equivariant cohomology) of products of flag varieties. This work is still in progress, and the current version of results is available on arxiv as [14]. My input is a classification of the orbits of a certain Weyl group action, and a result of the whole project is an explicit presentation (i. e. an explicit list of generators and relations) of the subgroup of W-invariants in the T-equivariant cohomology group of $G/P \times G/Q$, where G is a *classical* simple algebraic group (i. e., an algebraic group of types A, B, C, D), and P and Q are parabolic subgroups. More details about this project can be found in Section 5.

The third area I studied was combinatorics of polytopes. I found a series of examples of so-called neighborly polytopes, which are well-known and classical objects in combinatorics of polytopes. These results are contained in [9]. See Section 6 for definitions and more details.

The fourth area of my research interests is combinatorics on words. A general reference on combinatorics on words is, for example, [29]. In this area I studied and I am going to study objects related to so-called morphic sequences. I proved that the subword complexity of a morphic sequence is either asymptotically equivalent to one of explicitly written functions, or is $O(n \log n)$, where n is the length of the subword (see [12, Theorem 1.1]). For definitions, motivation, precise formulation of my results in this area and future plans, see Section 7.

1. Deformation of varieties with an action of a torus

1.1. **Overview.**

1.1.1. *T-varieties.* A *toric variety* is a normal algebraic variety with an action of an algebraic torus with an open orbit. Toric varieties provide easy examples of different kinds of varieties and operations on them, for example, Fano varieties, Gorenstein varieties, quotient singularities, blow-ups, etc. There is a classification of toric varieties by some combinatorial data, namely so-called polyhedral fans. There is a well-developed theory of toric varieties, which explains, for example, what kind of singularities they can have, what line bundles exist on them, etc. A general reference on toric varieties is, e. g., [16].

A natural generalization of toric varieties is the class of so-called *T-varieties*, which are by definition normal varieties with an action of an algebraic torus of smaller dimension. A survey on T-varieties can be found in [2] and [3]. The geometry of T-varieties is studied in many recent papers, see, for example, [24], [25], [41]. One of possible motivations to study T-varieties is that, unlike toric varieties, T-varieties form continuous families, so they can be used as a source of examples of deformations of singularities, see below. Like toric varieties, affine T-varieties are also parametrized by some combinatorial and geometric data, namely so-called proper polyhedral divisors. This classification is widely used in studies of T-varieties.

More specifically, a ($\mathbb{Q}$-Cartier) polyhedral divisor on a variety Y is a formal linear combination $\mathfrak{D} = \sum \Delta_i Z_i$, where Z_i are $\mathbb{Q}$-Cartier divisors on Y, and Δ_i are convex polyhedra ("possibly unbounded polytopes") in the same vector space with the same *tail cone.* (A *tail cone* of a convex polyhedron Δ in a vector space V is the set of all vectors $v \in V$ such that $\Delta + v \subseteq \Delta$.)

Here we don't describe the parametrization of T-varieties in full precision, but we say the following. Suppose that an affine variety X with an action of a torus T corresponds to a polyhedral divisor $\sum \Delta_i Z_i$ on a variety Y. It is known that all algebraic group homomorphisms $T \to \mathbb{C}^*$ with the operation of (pointwise) multiplication form a group isomorphic to $\mathbb{Z}^n$ (a lattice), this lattice is called the *character lattice* of T. Denote it by M, denote the dual lattice by N, and denote $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Then Y is, roughly speaking, a quotient of X modulo the torus action, and the polyhedra Δ_i are contained in $N_{\mathbb{Q}}$. There is a rational quotient map $\pi\colon X \to Y$, and its restriction to the regularity locus is a degenerate fibration. The divisors Z_i consist of points where the fibration degenerates, and the polyhedra, informally speaking, explain how exactly it degenerates.

1.1.2. *Deformation theory.* Let us give a precise definition of a deformation. Let X be a variety, S be a scheme, and $s \in S$ be a closed point. A *deformation* of X with *parameter space* S and *marked point* (or *basepoint*) s (or, briefly, a deformation of X *over* S) is a flat morphism $\xi\colon Y \to S$, together with an isomorphism ι between X and $\xi^{-1}(z)$. Y is called the *total space* of the deformation, and X is usually called the *special fiber*.

For a general reference on deformation theory one can use [20]. Deformation theory is, among other applications, a source of invariants of singularities. For example, if a variety X has one isolated singular point, then the dimension of the space of first-order deformations (see below) depends only on the formal infinitesimal neighborhood of the singular point, not on the whole X. Also, if we have a deformation with parameter space S, then, informally speaking, different kinds of singularities (for example, isolated singularities, nodal singularities) usually correspond to locally closed subsets of S, and "bad" singularities correspond so "small" subsets, so we get something like a topology on types of singularities.

Here we consider the *equivariant* version of deformation theory (in its simplest version). All definitions we give below have obvious non-equivariant analogs, but we will not need them. Let us start with the following definition. Let X be a variety with an action of an algebraic torus T. A deformation $(\xi\colon Y \to S, \iota\colon X \to \xi^{-1}(s))$ with a torus action $T : Y$ is called *T-equivariant* if ι is T-equivariant and ξ is T-equivariant for the trivial action of T on S. Informally speaking, equivariant deformations show how a variety can be changed if we want to preserve the action of T on each "deformed version" of X. One can define a morphism, an isomorphism, and a pull-back of T-equivariant deformations in the natural way, so that all morphisms involved in these definitions are T-equivariant and commute with the embeddings of X into total spaces.

The following particular cases of the notion of a deformation are important in deformation theory.

An equivariant deformation over the double point (i. e. over $\operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2)$) is called a *first-order* equivariant deformation, The set of isomorphism classes of first-order deformations of an affine variety X is denoted by $T^1(X)_0$. One can introduce a vector space structure on $T^1(X)_0$, and this structure has good category-theoretical properties.

More precisely, suppose that $\xi\colon Y \to S$ is a T-equivariant deformation of X with an arbitrary parameter space S, and $s \in S$ is the marked point. Each tangent vector v to S at s defines an embedding of the double point into S. If we pull back ξ via this embedding, we will get a T-equivariant deformation over the double point. So we get a map from the tangent space of S at s (denote it by T_sS) to $T^1(X)_0$. And there is a theorem that says that this map $T_sS \to T^1(X)_0$ (which is called the *Kodaira-Spencer* map) is linear. So, the space $T^1(X)_0$ gives some information about the tangent spaces of parameter spaces at the marked points.

Let us also mention the *obstruction space* for deformations of a T-variety X, which is denoted by $T^2(X)$. We omit the precise definition, but we say that this is a vector space that can be computed from a presentation of $\mathbb{C}[X]$ using generators and equations. There is also a map $\lambda\colon T^1(X)_0 \to T^2(X)$ (the so-called *quadratic form corresponding to the cup product*), which has, in particular, the following property. Let $\xi\colon Y \to S$ again be a T-equivariant deformation, let $s \in S$ be the marked point, and let T_sS be the tangent space of S at s. Denote the Kodaira-Spencer map by φ. Then, if $v \in T_sS$ belongs to the *tangent cone* at s, then $\lambda(\varphi(v)) = 0$. So, the obstruction space and the map λ give some information about the possible "shape" of S. S cannot be "too large" in the directions where λ yields a non-zero element of $T^2(X)$, and that's, informally speaking, why $T^2(X)$ is called the obstruction space.

In particular, the spaces of first-order equivariant deformations and of obstructions provide information about the "shape" of so-called versal and formally versal deformation spaces. A T-equivariant deformation $\xi\colon Y \to S$ of a variety X is called *versal* if any other T-equivariant deformation $\xi'\colon Y' \to S'$ can be obtained from ξ by pull-back via a map $f\colon S' \to S$. Unlike universal objects, this definition does not require the uniqueness of such f. However, it turns out that versality is a very strong property of a deformation, for many kinds of varieties it is not known whether a versal deformation space exists at all. A weaker notion is a formally versal T-equivariant deformation. A T-equivariant deformation $\xi\colon Y \to S$

of X with base point $s \in S$ is called *formally versal* if any other *formal* T-equivariant deformation of X (or, in other words, the restriction to the formal neighborhood of the special fiber of any other T-equivariant deformation of X) can be obtained from ξ via pullback (maybe, in a non-unique way).

In fact, formally versal equivariant deformations and first-order equivariant deformations have even more connections. First, a formally versal equivariant deformation exists if and only if $T^1(X)_0$ is a finite dimensional vector space. Next, it is also easy to see that the Kodaira-Spencer map of a formally versal T-equivariant deformation is always surjective. Moreover, if a formally versal deformation space exists, then there also exists a so-called *formally miniversal* T-equivariant deformation, which is a formally versal equivariant deformation such that the Kodaira-Spencer map is an isomorphism. Also, if $T^1(X)_0$ is a finite-dimensional vector space, the parameter space of a versal T-equivariant deformation can be constructed as an algebraic subscheme of the vector space $T^1(X)_0$.

1.2. **Past research.** I learned about open problems in this area from my PhD thesis advisor in Berlin, Klaus Altmann. In my research (see [13]), I studied deformations of T-varieties of a particular form. Namely, I consider affine rational three-dimensional T-varieties with an action of a two-dimensional torus such that the Chow quotient modulo this action equals $\mathbf{P}^1$, and all finite stabilizers are trivial.

These T-varieties are defined exactly by the polyhedral divisors with the following properties.

(1) The polyhedral divisor is on $\mathbf{P}^1$.
(2) The polyhedral divisor is of the form $\mathfrak{D} = \sum \Delta_i p_i$, where $p_i \in \mathbf{P}^1$ (divisors on $\mathbf{P}^1$ are just points), and each Δ_i is a polyhedron with all its vertices in lattice points.
(3) The tail cone σ of each Δ_i (recall that the tail cones of all polyhedra Δ_i are the same by definition) is full-dimensional (i. e. it is not contained in any proper subspace of $N_{\mathbb{Q}}$) and is pointed (i. e. it contains no lines).
(4) In terms of the notation from the previous conditions, the Minkowski sum of all polyhedra Δ_i is strictly contained in σ.

An example of a polyhedron satisfying conditions 2 and 3 is shown in Fig. 1.

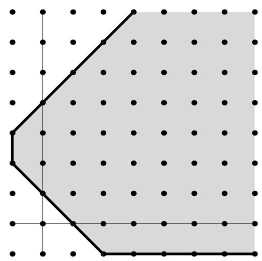

FIGURE 1. An example of a polyhedron.

The T-variety constructed from such a polyhedral divisor can have either one- or zero-dimensional singular locus, or it can be even smooth.

In my research, I have computed the space of first-order equivariant deformations for such T-varieties and found a versal deformation space. To formulate these results, let us fix some notation. Let X be the T-variety obtained from a polyhedral divisor $\mathfrak{D} = \sum_{i=1}^{r} \Delta_i p_i$ satisfying the above conditions.

For each of the polyhedra Δ_i, the lattice points split its boundary into minimal segments (which themselves do not contain lattice points inside anymore). Denote the number of these segments by k_i. In other words, k_i is the "integral length" of the part of $\partial\Delta_i$ that consists of segments (i. e. $\partial\Delta_i$ without the rays). Also, denote by q the number of points p_i such that $k_i > 0$ (in other words, $\Delta_i \neq \sigma + v$ for any $v \in N_{\mathbb{Q}}$). The following theorem is a combination of [13, Theorems 4.32 and 6.54] proved by myself.

Theorem 1. *In terms of the above notation,* $\dim T^1(X)_0 = \max(0, q-3) + \sum_{i=1}^{r} \max(0, k_i - 1)$.

There exists a formally versal equivariant deformation of X with smooth parameter space of the same dimension (in other words, the deformation is unobstructed).

1.3. **Future research.** In my subsequent research I plan to do the following. First, I am going to make my proofs in [13] more "conceptual" (for example, to use more coordinate-free language in linear-algebraic arguments) and to write papers based on my thesis to submit them to journals.

Second, I plan to relax some of the conditions I imposed on the polyhedral divisor. For example, I am going to consider the whole class of rational three-dimensional varieties with an action of a two-dimensional torus. The next step could be to increase the dimension of X and of T so that $\dim X - \dim T$ (the *complexity of the action*) still equals one.

Finally, there is a much more complicated direction where this research can be continued, namely, I will try to consider actions of complexity more than one. Already actions of one-dimensional torus on three-dimensional varieties look quite interesting. If the complexity equals two, then the polyhedral divisor describing this action is located on a two-dimensional variety, so it is formed not by points anymore, like in the cases I considered before, but by of one-dimensional varieties. So, a priori this situation looks like there is much more freedom to deform such a variety.

In all these cases, my goals will be to find the first-order equivariant deformation space and the obstruction space. If the first-order deformation space turns out to be finite, then my next goal will be to find and a formally versal equivariant deformation.

2. Semisimple algebraic group actions and representations

2.1. **Overview.** Let us give a precise standard definition of a generalized flag variety. A maximal connected solvable algebraic subgroup of a semisimple algebraic group G is called a *Borel subgroup* of G. If G is a semisimple algebraic group, $B \subseteq G$ is a Borel subgroup, and $P \subseteq G$ is another algebraic subgroup containing B, then P is called a *parabolic* subgroup of G.

Given an affine algebraic group G and an algebraic subgroup $H \subseteq G$, there always exists a *quotient variety* G/H (also called a *homogeneous space*), i. e. a variety G/H with a transitive action of G and with a point $x \in G/H$ (usually denoted by eH) such that the stabilizer of x equals H. If G is a semisimple affine algebraic group, and $P \subseteq G$ is a parabolic subgroup, then G/P is called a *generalized flag variety.*

It is known that all generalized flag varieties are projective. Moreover, each projective variety with a transitive action of an affine semisimple algebraic group G is isomorphic to a generalized flag variety, and this isomorphism preserves the group action. Some examples of generalized flag varieties are: projective spaces, Grassmannians, Lagrangian Grassmannians, classical (full and partial) flag varieties, etc.

2.2. **Past research.** I started to study actions and representations of algebraic groups at Ernest Vinberg's seminar at Moscow State University, when I was a student there. I learned about some open problems from Ernest Vinberg and from Ivan Arzhantsev, who also helped to organize this seminar, and I started to work on these problems under their supervision. When I became a PhD student, I continued this research under the supervision of Sergey Loktev, my PhD thesis advisor in Moscow, at Mathematics Department of Higher School of Economics.

The first problem I studied was the following. Given a semisimple affine algebraic group G, a parabolic subgroup $P \subseteq G$ and a number n, one has to determine whether the diagonal action of G on $(G/P)^n$ has an open orbit and whether the number of orbits is finite. Both questions can be easily reduced to the case when G is simple. The question about open orbits was previously solved in [37] for maximal P, and in the same paper it was formulated in the whole generality. The question about finiteness of the number of orbits was previously solved for $G = SL_k$ in [30] and for $G = Sp_{2k}$ in [31] (these papers actually contain more general results for these particular groups, see the "Future research" section below). In [10, Theorem 1.2], I extended classification of [37] to non-maximal parabolic subgroups in all types except for type A, and in [10, Corollary 1.4], I extended the known results on the finiteness of the number of orbit of the action $G : (G/P)^n$ to all simple algebraic groups and all parabolic subgroups. The precise formulations of these generalized results can be found below.

To formulate these results, we will need to recall some more facts about simple algebraic groups. First, there is a known classification of simple algebraic groups up to quotients over central finite subgroups. It is known that every parabolic subgroup contains the center (and therefore all finite central subgroups), so the classification up to quotients over central finite subgroups is suitable for our needs.

Second, for each simple group G, fix a Borel subgroup B. Then it is known that each parabolic subgroup of G is conjugate to a parabolic subgroup containing B, and each parabolic subgroup containing B equals the intersection of some of the maximal parabolic subgroups containing B. The maximal parabolic subgroups containing B are enumerated by so-called simple roots. We don't give a precise definition here, we just mention that there are finitely many simple roots for each simple algebraic group, and that there is a standard way to enumerate them, which can be found, for example, in [6]. The number of simple roots is called the *rank* of G and is denoted by $\operatorname{rk} G$. Denote the maximal parabolic group containing B and corresponding to the ith simple root ($1 \le i \le \operatorname{rk} G$) by P_i.

Using this notation, we can say that [10, Theorem 1.2] (proved by myself) and [37, Theorem 3] yield the following classification:

Theorem 2. *Let G be a simple algebraic group, which is not of type A (i. e. it is not a quotient of SL_k modulo a finite subgroup for any k), $P \subset G$ be a non-maximal parabolic subgroup, and let $n \in \mathbb{N}$. If $n \le 2$, then the diagonal G-action on $(G/P)^n$ always has an open orbit. If $n \ge 3$, then the diagonal G-action on $(G/P)^n$ has an open orbit if and only if (G, P, n) is an entry in the following table.*

Type of G	*P (up to conjugation)*	n
$B_l, l \ge 3$	P_1, P_l	3
$C_l, l \ge 2$	P_1, P_l	3
$D_l, l \ge 4$ *is even*	$P_1, P_{l-1}, P_l, P_1 \cap P_{l-1}, P_1 \cap P_l, P_{l-1} \cap P_l$	3
$D_l, l \ge 5$ *is odd*	$P_1, P_{l-1}, P_l, P_1 \cap P_{l-1}, P_1 \cap P_l$	3
E_6	P_1, P_6	$n = 3, 4$
E_7	P_7	$n = 3$

The case $G = SL_k$ remains open for now.

In [10, Corollary 1.4] I proved that the following classification is complete. The fact that in all these cases the number of orbits is finite (the "if" part) follows very easily from previously known resluts. It was also known before that in some (but not all) of the other cases the number of orbits is infinte, and my contribution was ruling out all of the remaining possibilities.

Theorem 3. *Let G be a simple algebraic group. Let $P \subset G$ be a non-maximal parabolic subgroup, and let $n \in \mathbb{N}$. If $n \le 2$, then the diagonal G-action on $(G/P)^n$ always has finitely many orbits.*

If $n \ge 3$, then the number of G-orbits on $(G/P)^n$ is finite if and only if $n = 3$ and (G, P) is one of the pairs in the following table.

Type of G	*P (up to conjugation)*
A_l	*any maximal parabolic*
$B_l (l \ge 3)$	P_1 *or* P_l
$C_l (l \ge 2)$	P_1 *or* P_l
$D_l (l \ge 4)$	P_1 *or* P_{l-1} *or* P_l
E_6	P_1 *or* P_6
E_7	P_7

The second problem deals with so-called *commutative unipotent group*, which is just a vector space considered as an algebraic group with the operation of addition. For each $m \in \mathbb{N}$, denote the m-dimensional commutative unipotent group by $\mathbf{G}_a^m$. In 1999, B. Hassett and Yu. Tschinkel formulated

[21] the following general question. Given $m \in \mathbb{N}$, one has to classify all possible actions with an open orbit of $\mathbf{G}_a^m$ on projective varieties. The idea is to "develop a theory" for varieties with an action of $\mathbf{G}_a^m$ in a similar way to theory of toric varieties.

This general question is far from a complete solution now, but the following more particular question is solved in some cases. Namely, one fixes an m-dimensional projective variety X and wants to classify actions of $\mathbf{G}_a^m$ on this particular variety X with an open orbit. For example, if X is a projective space, the answer was found in the original paper [21]. In [39] it was proved that if X is a projective quadric, then such an action is unique up to conjugation by automorphisms of X and up to automorphisms of $\mathbf{G}_a^m$. In [4], the following question was answered: given a generalized flag variety X, does there exist at least one $\mathbf{G}_a^m$-action ($m = \dim X$) on X with an open orbit.

My goal was to study the case when X is a generalized flag variety. I have classified the $\mathbf{G}_a^m$-actions on any generalized flag variety completely, this classification is contained in [11]. This classification problem can also be reduced to the case when $X = G/P$, where G is a *simple* algebraic group and P is a parabolic subgroup. If G is simple, then [11, Theorem 25] proved by myself, [4, Theorem 1] and the correspondence between so-called exceptional and non-exceptional pairs of a simple group and a parabolic subgroup (for example, see [8, Section 2]) yield the following answer:

Theorem 4. *Let G be a simple algebraic group, let P be a parabolic subgroup, $P \neq G$, and let $m = \dim(G/P)$.*

If G/P is a projective space, then the actions $(\mathbf{G}_a)^m : G/P$ with an open orbit, considered up to conjugation by the automorphisms of the variety G/P and up to the automorphisms of the group $(\mathbf{G}_a)^m$, are parametrized by the isomorphism classes of commutative associative m-dimensional algebras with nilpotent multiplication operators.

Otherwise, if an action $(\mathbf{G}_a)^m : G/P$ with an open orbit exists, then it is unique up to conjugation by the automorphisms of the variety G/P and up to the automorphisms of the group $(\mathbf{G}_a)^m$. The pairs (type of G, P) such that an action $(\mathbf{G}_a)^m : G/P$ with an open orbit exists (but G/P is not a projective space) are listed in the following table.

Type of G	*P (up to conjugation)*
A_l	$P_i (1 < i < l)$
$B_l (l \geq 3)$	P_1 *or* P_l
$C_l (l \geq 2)$	P_l
$D_l (l \geq 4)$	P_1 *or* P_{l-1} *or* P_l
E_6	P_1 *or* P_6
E_7	P_7
G_2	P_1

The proof of this theorem is independent of the results of [21]. As an intermediate result, which can be of independent interest, the proof of this theorem also contains a solution of the following problem in representation theory of Lie algebras. Namely, let $\mathfrak{g}$ be a reductive Lie algebra, and V be a faithful finite dimensional representation of V. One has to find bilinear maps ("multiplications") $V \times V \to V$ that have the following properties: the multiplication has to be associative, commutative, all multiplication operators (i. e., operators of the form $v \mapsto wv$ for a fixed $w \in V$) have to be nilpotent, and for each $v \in V$ there has to exist $x \in \mathfrak{g}$ such that the operator of multiplication by v equals the operator of action of x. Let us call a multiplication satisfying these four properties *a $\mathfrak{g}$-compatible multiplication.*

This problem can be reduced to the case when $\mathfrak{g}$ is a simple Lie algebra, and V is an irreducible representation of $\mathfrak{g}$. In this case, the answer is given by the following theorem proved by myself as [11, Theorem 21].

Theorem 5. *Let $\mathfrak{g}$ be a simple Lie algebra, V be an irreducible representation of $\mathfrak{g}$. Suppose that there exists a $\mathfrak{g}$-compatible multiplication on V, which is nonzero (i. e. $V \cdot V \neq 0$). Then there are exactly two possibilities:*

(1) *$\mathfrak{g} = \mathfrak{sl}_k$, V is the tautological representation or the dual one. Then any commutative associative multiplication such that all multiplication operators are nilpotent is $\mathfrak{g}$-compatible.*
(2) *$\mathfrak{g} = \mathfrak{sp}_{2k}$ ($k \geq 2$), V is the tautological representation. Let ω be a skew-symmetric nondegenerate form on V such that $\mathfrak{g}$ consists of the linear operators on V skew-symmetric with respect to ω. Then the trilinear forms dual to the $\mathfrak{g}$-compatible multiplications on V are exactly the totally symmetric trilinear forms c on V such that $\ker c$ contains a Lagrangian subspace of V (i. e. such that there exists a Lagrangian subspace $V_1 \subset V$ such that $c(V_1, V, V) = 0$).*

2.3. **Future research.** This research has the following possible continuation.

First, I am going to finish the investigation started by Theorem 2 and find out when an open orbit exists if $G = SL_k$. This case is expected to be more complicated than the others. Even for maximal parabolic subgroups, depending on k and the parabolic subgroup, an open G-orbit on $(G/P)^n$ may exist for arbitrarily large n (see [37]), while if G is not of type A, and an open G-orbit on $(G/P)^n$ exists, then, as it follows from Theorem 2, $n \leq 4$. I hope that quiver representation theory can be a useful tool for this problem, because it was already helpful in [37] to consider the case when G is of type A and P is maximal.

Next, this problem allows further generalization. Given a semisimple algebraic group G and n of its parabolic subgroups $P^{(1)}, \ldots, P^{(n)}$, one has to determine whether the action of G on $G/P^{(1)} \times \ldots \times G/P^{(n)}$ has an open orbit and whether it has finitely many orbits. In this generality the problem is far from being solved, but there are some more known results about it. Namely, the question about finitely many orbits for $G = SL_k$ and for $G = Sp_k$ is fully solved, see [30] and [31]. I will try to solve it at least for some other particular cases. For example, the paper [31] says that the question of finiteness of the number of orbits for $G = SO_k$ can be solved by the same techniques as used in the paper for the case $G = Sp_k$, but the actual solution is not given in the paper, because, as the author says, it is too complicated.

Speaking of the classification of $\mathbf{G}_a^m$-actions with an open orbit, in the future I will try to classify the actions of commutative unipotent groups with an open orbit on different classes of varieties. For example, I am going to consider the case of complete toric surfaces, because the case of Hirzebruch surfaces (which are a particular case of complete toric surfaces) was already considered in the original paper [21]. I hope that a useful tool for this problem is the classification of divisors on toric varieties and intersection theory for them. My first guess is that in most cases the set of isomorphism classes of such actions will be finite, because it is finite for Hirzebruch surfaces (see [21, Proposition 5.5]).

3. Schubert calculus

3.1. **Overview.** Again, let G be a semisimple algebraic group, $B \subseteq G$ be a Borel subgroup, and $T \subseteq B$ be a maximal torus. These data define a so-called *root system* Φ of G, which is a subset of a Euclidean space, the subset of positive roots $\Phi^+ \subseteq \Phi$, and the subset of simple roots $\Pi \subseteq \Phi^+$. Π is a basis of the ambient vector space of Φ. Denote also $r = \operatorname{rk} G = |\Pi|$.

Denote by W the *Weyl group* of G, $W = N_G(T)/T$. There exists a unique element $w_0 \in W$ such that $w_0^{-1} B w_0 \cap B = T$, it is called the *maximal* element of W. W acts faithfully on Φ, and it contains all reflections across the planes orthogonal to the roots. We will denote such a reflection by s_α for $\alpha \in \Phi$. Moreover, W is generated (this makes sense since the action $W : \Phi$ is faithful) by the reflections s_β, where $\beta \in \Pi$. If $w \in W$, the smallest number of reflections $s_{\beta_1}, \ldots, s_{\beta_l}$, where $\beta_i \in \Pi$ and $w = s_{\beta_1} \ldots s_{\beta_l}$, is called the *length* of w, and is denoted by $\ell(w)$: $\ell(w) = l$.

The Chow ring of a smooth algebraic variety X is an object in algebraic geometry similar to the cohomology ring in topology, see [17] for precise definitions.

The Chow ring of X is denoted by $A(X)$. For flag varieties over $\mathbb{C}$, it is isomorphic to the classical (topological) cohomology ring, and this isomorphism maps the classes of smooth algebraic subvarieties to the classes of the same submanifolds. Note also that it is a *graded* ring, and the degree of the class of a subvariety $Y \subseteq X$ is $\operatorname{codim}_X Y$.

There are two different ways to generate $A(G/B)$. First, for each $w \in W$, one can set $X_w = [\overline{BwB/B}]$. For our purposes, it will also be convenient to denote $Z_w = X_{w_0 w^{-1}}$. Then, the classes Z_w for all $w \in W$ freely generate $A(G/B)$ as an abelian group. These classes X_w are called *Schubert classes*.

Second, for each $\alpha \in \Pi$, Z_{s_α} is (the class of) a divisor. The classes Z_{s_α} generate $A(G/B)$ as a ring. For our purposes, it will also be convenient to denote $D_i = Z_{s_{\alpha_i}}$, where α_i is the ith simple root according to [6].

Schubert calculus studies this Chow ring, or, equivalently, the cohomology ring of G/B.

3.2. **Past research.** It is known that if the sum of codimensions of Schubert classes $Z_{w_1}, \ldots, Z_{w_k}$ equals $\dim(G/B)$ (in other words, $\ell(w_1) + \ldots + \ell(w_k) = \dim(G/B)$ for some $w_1, \ldots, w_k$), then the product $Z_{w_1} \ldots Z_{w_k}$ in the Chow ring is a nonnegative integer multiple of the class of a point, [pt]. This motivates the following terminology: let us call such a product *multiplicity-free* if $Z_{w_1} \ldots Z_{w_k} = [\mathrm{pt}]$. Moreover, if $w_1, \ldots, w_m \in W$ (but not necessarily $\ell(w_1) + \ldots + \ell(w_m) = \dim(G/B)$), then we call the product $Z_{w_1} \ldots Z_{w_m}$ *multiplicity-free* if there exists another Schubert class Z_w such that $Z_{w_1} \ldots Z_{w_m} Z_w = [\mathrm{pt}]$.

I was particluarly interested in products of Scubert classes of *divisors*, not of arbitrary Schubert classes, in other words, in monomials of the form $D_1^{n_1} \ldots D_r^{n_r}$. The *degree* of such a monomial in the Chow ring is $n_1 + \ldots + n_r$.

Since the classes Z_w freely generate $A(G/B)$ as an abelian group, for every r-tuple of numbers $n_1, \ldots, n_r$ one can write

$$D_1^{n_1} \ldots D_r^{n_r} = \sum_{w \in W} C_{w,n_1,\ldots,n_r} Z_w.$$

It is known that all coefficients $C_{w,n_1,\ldots,n_r}$ are nonnegative integers. These coefficients can also be viewed combinatorially as the numbers of path in the *Bruhat graph* of G with certain labels and counting certain multiplicities. Also, using known facts about Schubert calculus, one can easily prove that the monomial $D_1^{n_1} \ldots D_r^{n_r}$ is multiplicity-free if and only if there exists $w \in W$ such that $C_{w,n_1,\ldots,n_r} = 1$ My goal was to study these coefficients $C_{w,n_1,\ldots,n_r}$.

The folowing proposition is a direct corollary of [15, Proposition 4.4] proved by myself and [15, Lemma 3.26], which, in turn, follows easily from [7, §4.4, Corollary 2].

Proposition 6. *Assume, in addition, that G is a simple algebraic group of types A, D, or E. The following conditions are equivalent:*

(1) $C_{w,n_1,\ldots,n_r} \geq 1$.
(2) *There exists a function $f \colon \Phi^+ \setminus (w\Phi^+) \to \Pi$ such that:*
   (a) *For each $\alpha \in \Phi^+ \setminus (w\Phi^+)$, the simple root $f(\alpha)$ appears in the decomposition of α into a linear combination of simple roots with positive coefficient.*
   (b) *For each $\alpha_i \in \Pi$, f takes value α_i exactly n_i times.*

Under the same restrictions (G is a simple algebraic group of types A, D, or E), I have also found a criterion when $C_{w,n_1,\ldots,n_r} = 1$. This criterion is quite technical and complicated, but the following corollary can be formulated easily:

Proposition 7 (see [15, Lemma 11.1, Proposition 11.4, and Theorem 11.5])**.** *Again assume that that G is a simple algebraic group of types A, D, or E. The maximal degree of a multiplicity-free monomial in*

$D_1, \dots, D_r$ *is contained in the following table:*

Type of G	*maximal degree*
A_r	$r(r+1)/2$
$D_r, r \geq 4$	$r(r+1)/2 - 1$
$E_r, 6 \leq r \leq 8$	$r(r+1)/2 - 2$

These results may be used to compute upper bounds on the *canonical dimension* (see definition in [27]) of G/B over non-algebraically-closed fields, similarly to the arguments of [26, Section 5].

3.3. **Future research.** In the future, I am going to study other questions about the numbers $C_{w,n_1,\dots,n_r}$. For example, I am going to try to generalize my results to the cases of other type of simple groups (B, C, F, and G).

One more question one could ask is to describe all n-tuples $n_1, \dots, n_r$ such that each Z_w appears in the decomposition of $D_1^{n_1} \dots D_r^{n_r}$ into a linear combination of classes Z_w with coefficient 1 or 0 (in other words, $C_{w,n_1,\dots,n_r} = 1$ or 0 for all $w \in W$.)

One more question, which is also useful for the computation of canonical dimension, is the question of divisibility of the coefficients $C_{w,n_1,\dots,n_r}$ by various prime numbers.

4. K-theory of twisted flag varieties

Again consider a simple algebraic group G, choose a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We can associate several lattices to it, two of them are the *weight lattice*, which we denote by Λ, and the *character lattice of* T, which we denote by $\mathfrak{X}(T)$ and which is a sublattice of Λ. Consider the two corresponding Laurent polynomial algebras, $\mathbb{Z}[\Lambda]$ and $\mathbb{Z}[\mathfrak{X}(T)]$. (Note that of we took $\mathbb{C}[\mathfrak{X}(T)]$ instead of $\mathbb{Z}[\mathfrak{X}(T)]$, we would get the algebra of functions on T.) Since $\mathfrak{X}(T) \subseteq \Lambda$, we also have $\mathbb{Z}[\mathfrak{X}(T)] \subseteq \mathbb{Z}[\Lambda]$.

Again denote by $W = N_G(T)/T$ the Weyl group of G. Then W acts on Λ, on $\mathfrak{X}(T)$, on $\mathbb{Z}[\mathfrak{X}(T)]$, and on $\mathbb{Z}[\Lambda]$.

Denote by $I \subset \mathbb{Z}[\Lambda]$ the ideal generated by all Laurent polynomials f such that the sum of coefficients of f is 0 and f is invariant under the W-action. Then $I \cap \mathbb{Z}[\mathfrak{X}(T)]$ is an ideal in $\mathbb{Z}[\mathfrak{X}(T)]$.

My goal was to find an explicit list of generators of $I \cap \mathbb{Z}[\mathfrak{X}(T)]$. I found it for $G = SL_{2n}/\mu_2$ and for $G = Sp_{2n}/\mu_2$, where μ_2 is the subgroup (of order 2) generated by the minus identity matrix. This list of generators can be found in [5, Definition 3.2 and Theorem 3.4]. It is known (see [18, Corollary 5.5] and [7]) that under some additional conditions, if X is a variety over a non-algebraically-closed field that becomes a flag variety after the extension of scalars to the algebraic closure, then $K_0(X) = \mathbb{Z}[\mathfrak{X}(T)]/(I \cap \mathbb{Z}[\mathfrak{X}(T)])$.

An example of such an X can be constructed as follows (see [33, Section 5]). Let F be a non-algebraically-closed field of characteristic 0, and suppose we have a central simple division algebra D over F of dimension n^2 such that $D \otimes_F D = \mathrm{Mat}_{n^2 \times n^2}(F)$ (in other words, D is an element of order 2 of the *Brauer group*). An example of such D is the quaternion algebra over $\mathbb{R}$. Denote by Y the (classical) partial flag variety consisting of all flags of subspaces of dimensions $0, n, 2n, \dots, n(n-1), n^2$ of D as of a vector space. This is a projective variety over F. Then we can write homogeneous equations saying that all these subspaces are left ideals of D. These equations will have no solutions since D is a division algebra, but still they will define an algebraic variety over F (without rational points). This variety is the X we want to construct. It is called the Severi-Brauer flag variety of D.

5. Equivariant cohomology of products of flag varieties

Let G be a simple algebraic group. Again choose a Borel subgroup B and a maximal torus T, and denote by W the Weyl group $W = N_G(T)/T$. Also, fix two parabolc subgroups P and Q containing B. The goal of our joint research with M. Lanini and K. Zainoulline (this is still work in progress, but we have succeeded for the classical groups) is to find an explicit presentation for the subgroup of W-invariants in the T-*equivariant cohomology group* (see [23] for reference, the precise definition is [23, Definition 12]) of $G/P \times G/Q$. My personal goal is to solve the following problem.

Again denote the weight lattice by Λ, the root system by Φ, the set of positive roots by Φ^+, and consider the Weyl group action on Λ. For each parabolic subgroup $P' \subseteq G$ such that $B \subseteq P'$, it is also convenient to denote $W_{P'} = (P' \cap N_G(T))/T$. This is a subgroup of W. Choose a weight $\theta \in \Lambda$ whose stabilizer in W is W_P (one can check easily that the stabilizer of any dominant weight is of this form, so essentially, if we want to solve the orginal problem for any P, we have to consider an arbitrary dominant weight θ here).

Consider the orbit $W\theta$, and consider the graph whose vertices are points of this orbit, and two vertices μ and ν are connected with an edge labeled α, where α is a positive root, if $s_\alpha\mu = \nu$. Note that if $\alpha \in \Phi^+$, $\mu, \nu \in W\theta$, $w \in W$, and $s_\alpha\mu = \nu$, then $s_{w(\alpha)}(w\mu) = w\nu$. Therefore, we have a W-action on the edges of this graph, and it agrees with the original W-action on the vertices.

Let us restrict these actions to W_Q. One checks directly that for each pair of W_Q-orbits on the set of vertices $W_Q\mu$ and $W_Q\nu$, W_Q acts on (preserves) the set of edges between $W_Q\mu$ and $W_Q\nu$. My goal is to classify the W_Q-orbits on these sets of edges. I have done this for classical groups, it turns out that there is always only one orbit if G is of type A, at most two orbits of G is of type D, and at most three orbits if G is of types B and C.

Then, for the equivariant cohomology problem, we can roughly and briefly say that the answer looks as follows: we have a generator from a certain group for each W_Q-orbit in $W\theta$, and for each pair of such orbits $W_Q\mu$ and $W_Q\nu$ we have a relation for each W_Q-orbit on the set of edges connecting them. The precise form of the groups containing generators and the precise form of these relations can be found in [14, Theorem 9.1].

In the future, I plan to classify these W_Q-orbits for exceptional simple algebraic groups, and therefore answer the original equivariant cohomology question for these groups.

6. Combinatorics of polytopes

I started to study combinatorics of polytopes at Gaiane Panina's course at the "Contemporary Mathematics" summer school in Dubna, Russia in 2006. It was Gaiane Panina who told me about some open problems in this area.

I studied the following problem.

A convex polytope P in $\mathbb{R}^d$ is called *neighborly* if for each k, $1 \le k \le \lfloor d/2 \rfloor$, each set of k vertices of P forms a face of P.

The number $\lfloor d/2 \rfloor$ in this definition is essentially the largest possible one, if we replace it with $\lfloor d/2 \rfloor + 1$, the only polytope we obtain is a simplex.

Neighborly polytopes have various combinatorial properties. For example, let P be a neighborly polytope of dimension d with n vertices. Suppose, in addition, that P is simplicial, i. e. all of its facets are simplexes (if d is even, P is automatically simplicial). If Q is another (not necessarily neighborly) convex polytope of dimension d with n vertices, then for each k ($1 \le k \le d$), the amount of k-dimensional faces of P is bigger than or equal to the amount of k-dimensional faces of Q. (If $k \le d/2$, this is clear by the definition of a neighborly polytope. If $k > d/2$, then this is a corollary of McMullen upper bound theorem.)

Another combinatorial property of neighborly polytopes of *even* dimension is so-called *combinatorial rigidity*. Namely, for polytopes one can define two notions of combinatorial equivalence. First (this is usually called the combinatorial equivalence of polytopes themselves), one can say that two polytopes are combinatorially equivalent if their face lattices are isomorphic. Second (this is usually called the combinatorial equivalence of sets of vertices), one can define equivalence as follows. Let us define a *combinatorial covector* of a set of points in a real vector space (for example, of the set of vertices of a polytope). Namely, each hyperplane in the vector space splits the set of points into three subsets: the points contained in the hyperplane, the points lying at one side of the hyperplane, and the points lying at the other side of the hyperplane. Such a decomposition (constructed for an arbitrary hyperplane) is called a *combinatorial covector* of the set of vertices. And one says that the sets of vertices of two polytopes are combinatorially equivalent of their sets of combinatorial covectors coincide. Finally, combinatorial rigidity for even-dimensional neighborly polytopes means that these two notions of combinatorial equivalence coincide for even-dimensional neighborly polytopes.

My goal was to construct a series of examples of neighborly polytopes. I considered polytopes of an even dimension d with $d+4$ vertices. If we take less vertices (for example, $d+3$), the situation becomes almost trivial, with the only example being the (combinatorial class of) cyclic polytope or the (combinatorial class of) simplex. For odd dimensions the situation is also simpler, it was studied by Shemer in [40].

The main tool in my construction was Gale transformation. Given a set of n points in a d-dimensional affine space ($n \geq d+2$), which are not contained in any proper affine subspace, Gale transformation enables one to construct a so-called affine Gale diagram, which is a set of up to n points in a $(n-d-2)$-dimensional affine space plus some combinatorial data. Very briefly, this construction works as follows. Suppose we start with a d-dimensional affine space A and n points $a_1, \dots, a_n$ there. Embed A into a $(d+1)$-dimensional vector space V so that the last coordinate of each point in A is 1. Denote the images of $a_1, \dots, a_n$ by $v_1, \dots, v_n$. Consider also an n-dimensional space W with a prefixed basis $e_1, \dots, e_n$. We get a linear map $\varphi\colon W \to V$, $e_i \mapsto v_i$. The kernel of this map is $(n-d)$-dimensional. Consider another $(n-d)$-dimensional vector space X, and choose a map $\psi\colon X \to W$ that maps X isomorphically onto $\ker\varphi$. Then we have the dual map $\psi^*\colon W^* \to X^*$, and it maps $e_1^*, \dots, e_n^*$ to some vectors $x_1, \dots x_n$. Finally, choose a linear function l on X^* so that $l(x_i) \neq 0$ unless $x_i = 0$ and consider the affine space $B = \{l = 1\}$. If $x_i \neq 0$, choose λ_i so that $\lambda_i x_i \in B$. The affine Gale diagram of the sequence $a_1, \dots, a_n$ is the sequence $(\lambda_i x_i)_{1\leq i\leq n, x_i\neq 0}$ plus the following combinatorial data. First, we say that $\lambda_i x_i$ is a black (resp. white) point if $\lambda_i > 0$ (resp. $\lambda_i < 0$). Second, we remember the set of indices i such that $x_i = 0$. Various combinatorial properties of sets of points (and of polytopes, which have these sets of points as their sets of vertices) can be obtained from their affine Gale diagrams. For example, one can figure out whether a given subset of points is the set of all vertices of a face of a polytope. And if $n = d+4$, the affine Gale diagram is a two-dimensional object, which is relatively easy to study.

My series of examples is contained in [9]. It was constructed as follows. First, some restrictions on the affine Gale diagram of the polytope under construction were imposed. Then, among the diagrams satisfying these restrictions, all diagrams of neighborly polytopes were found. The following theorem was proved by myself as [9, Theorem 5.2].

Theorem 8. *For each even d, there exists a series of examples d-dimensional neighborly polytopes. The number of polytopes in this series is asymptotically (as $d \to \infty$)*

$$\Theta\left(\frac{1}{d^2}\binom{d}{d/2}\right) = \Theta\left(\frac{1}{d^{5/2}}2^d\right).$$

These results were later generalized and improved by Arnau Padrol in [34].

7. Combinatorics on words

7.1. Overview. Let us start with a precise definition of a morphism and of a pure morphic sequence.

Let Σ be a finite alphabet. The set of finite words in Σ is denoted by Σ^*. We use multiplicative notation for concatenation of words. By abuse of language we identify each letter with the word consisting of only this letter. Now let Ξ be another finite alphabet. A map $\varphi\colon \Sigma^* \to \Xi^*$ is called a *morphism* if $\forall u, v \in \Sigma^* \varphi(uv) = \varphi(u)\varphi(v)$. Clearly, a morphism can be defined by the images of all letters. Each morphism can be naturally extended to the set of infinite words. A morphism is called non-erasing (resp. coding) if the image of each letter is not the empty word (resp. is exactly one letter).

Let Σ be a finite alphabet, and let $\varphi\colon \Sigma^* \to \Sigma^*$ be a morphism. We denote the morphism obtained by n iterations of φ by φ^n. Suppose that there exists a letter $a \in \Sigma$ such that $\varphi(a)$ begins with a. Then

$a, \varphi(a), \varphi^2(a), \ldots, \varphi^n(a), \ldots$ is a sequence of finite words such that each word is a prefix of the next one. If the lengths of these words strictly increase (this is true, for example, if $\varphi(a) \neq a$ and φ is non-erasing), then all these words are prefixes of an infinite word (it is infinite to the right, not to the left), which is denoted by $\varphi^\infty(a)$ and is called a *pure morphic sequence.* If, in addition, Ξ is another alphabet and $\psi \colon \Sigma^* \to \Xi^*$ is a coding, then the infinite sequence $\psi(\varphi^\infty(a))$ is called a *morphic sequence.*

For a general reference on pure morphic and morphic sequences, one can use [1, Chapter 7]. Morphic sequences arise in different areas of combinatorics on words, for example, in so-called pattern avoidability problems [29, Chapter 3]. So-called Thue-Morse and Fibonacci words are also examples of morphic sequences. They also appear in other areas of mathematics, for example, in dynamical systems [38, Chapter 5].

7.2. **Past research.** I first learned about open problems in combinatorics on words from Yuri Pritykin at the "Contemporary Mathematics" summer school in Dubna, Russia in 2007. Later I learned about more open problems in combinatorics on words from Alexander Bufetov.

My goal was to study one of the standard complexity characteristics of morphic sequences, namely so-called factor complexity. A finite word λ of length n is called a *factor* of an infinite word $(\alpha_i)_{i=1}^\infty$ if there exists an index i such that $\alpha_i \alpha_{i+1} \ldots \alpha_{i+n-1} = \lambda$. A factor of a finite word or of a bidirectional infinite word (i. e. a sequence $(\alpha_i)_{i\in\mathbb{Z}}$) is defined in a similar way. The *factor complexity* of a sequence α infinite to the right is the function $p_\alpha \colon \mathbb{N} \to \mathbb{N}$ such that $p_\alpha(n)$ is the amount of different factors of α of length n.

The factor complexity functions of different words were probably first introduced in [22] (where their values were called *permutation indices*). There are many interesting facts known about factor complexity. For example, if an infinite sequence γ containing k different letters is not eventually periodic, its factor complexity satisfies $p_\gamma(n) \geq n + k - 1 \geq n + 1$, see, for example, [29, Theorem 1.3.13]. (Clearly, if an infinite sequence is eventually periodic, its factor complexity is bounded.) Sturmian words, which are well known in combinatorics on words and have many interesting properties (see, for example, [29, Chapter 2]) and many equivalent definitions, can be defined as words in a two-letter alphabet with factor complexity $p(n) = n + 1$.

All possible asymptotic behaviors of *pure* morphic sequences were found by J.-J. Pansiot in [35]. In 1985, he formulated in [36] the question about possible asymptotic behavior of factor complexity of arbitrary morphic sequences.

In my research, I studied structure of pure morphic and morphic sequences and proved the following theorem, see [12, Theorem 1.1].

Theorem 9. *The factor complexity of a morphic sequence has the following asymptotic characteristics for $n \to \infty$: either it is $\Theta(n^{1+1/k})$ for some $k \in \mathbb{N}$, or it is $O(n \log n)$.*

7.3. **Future research.** In the future I can continue to study questions related with pure morphic sequences. First, I can try to figure out what happens if the factor complexity of a morphic sequence is $O(n \log n)$ (what exactly its asymptotic behavior can be).

Second, I am going to address the following problem about *substitutional and adic dynamical systems.* Let Σ be an alphabet, $\varphi \colon \Sigma \to \Sigma$ be a morphism, and $a \in \Sigma$ be a letter such that there exists a pure morphic sequence $\varphi^\infty(a)$. Then let us call a bidirectional infinite word *finitely morphic* if each of its factors is a factor of $\varphi^\infty(a)$. Suppose that φ and $\varphi^\infty(a)$ satisfy the following *unique admissible decomposition* condition:

> If $(\alpha_i)_{i\in\mathbb{Z}}$ is a finitely morphic bidirectional infinite word, then there exists a unique finitely morphic bidirectional infinite word $(\beta_i)_{i\in\mathbb{Z}}$ such that φ maps $(\alpha_i)_{i\in\mathbb{Z}}$ to $(\beta_i)_{i\in\mathbb{Z}}$ and α_0 is contained in the image of β_0.

All finitely morphic bidirectional infinite words with the operations of left and right shift form a discrete-time dynamical system. We will call it a *substitutional dynamical system.* For more information on such dynamical systems, see [28]. Let us introduce topology on substitutional dynamical systems. The base of this topology consists of the following sets $U_{-k,n}((\alpha_i)_{i=-\infty}^{\infty})$, where $k, n \in \mathbb{N}$ and $(\alpha_i)_{i=-\infty}^{\infty}$ is a finitely morphic bidirectional infinite word: $U_{-k,n}((\alpha_i)_{i=-\infty}^{\infty})$ consists of all finitely morphic bidirectional infinite words $(\beta_i)_{i=-\infty}^{\infty}$ such that $\alpha_i = \beta_i$ for $-k \leq i \leq n$. The problem is to determine whether two such substitutional dynamical systems coincide. In other words, if we only know the dynamical system as a topological space with the operation of shift, what can we say about the morphism φ, which was used to generate it?

A similar notion is a so called *adic dynamical system*. It is also constructed by an alphabet Σ and a morphism φ as follows. First, we say that a *path of letters* is an infinite sequence of alternating letters and numbers (for example, $a_0, k_0, a_1, k_1, a_2, k_2, \ldots$) such that for each i, a_i is the k_ith letter in $\varphi(a_{i+1})$. Denote the set of such paths by Ω. Denote the length of a finite word γ by $|\gamma|$. We can define the left and right shifts on Ω as follows. Given a path of letters $a_0, k_0, a_1, k_1, a_2, k_2, \ldots$, its *right shift* is the path $b_0, m_0, b_1, m_1, b_2, m_2, \ldots$ constructed as follows. First, we find the smallest i such that $k_i < |\varphi(a_{i+1})|$. Then we say that for $j > i$, $b_j = a_j$ and $m_j = k_j$, for $j = i$, $m_j = k_j + 1$ and b_j is the m_jth letter in $\varphi(b_{j+1})$, and for $j < i$, $m_j = 1$ and b_j is the first letter in $\varphi(b_{j+1})$. If such i does not exist (i. e. if $k_i = |\varphi(a_{i+1})|$ for all i), then we say that it is impossible to shift this path to the right. The left shift of a path is defined similarly. One checks easily that there are only finitely many orbits consisting of paths that cannot be shifted to the right.

It is also possible to introduce topology on an adic dynamical system. Let $A = a_0, k_0, a_1, k_1, a_2, k_2, \ldots$ be a path of letters, $n \in \mathbb{N}$. We say that $U_n(A)$ is the set of all paths of letters $b_0, m_0, b_1, m_1, b_2, m_2, \ldots$ such that $b_i = a_i$ for $i \leq n$ and $m_i = k_i$ for $i \leq n$. The sets $U_n(A)$ for all paths of letters A and for all $n \in \mathbb{N}$ form a base of a topology on Ω.

Suppose that φ satisfies the unique admissible decomposition condition, and there exists $n \in \mathbb{N}$ such that for each letter $a \in \Sigma$, the words $a, \varphi(a), \ldots, \varphi^n(a)$ together contain all letters of Σ. Then it is easy to see that if we remove some (finitely many) orbits from the substitutional dynamical system constructed by φ and all orbits in the adic dynamical system containing paths that cannot be shifted to the left or to the right (there are also only finitely many of such orbits), then the resulting dynamical systems will be homeomorphic (i. e. the homeomorphism preserves both the dynamical system structure and the topology). So, adic dynamical systems can be useful to deal with substitutional dynamical systems. I also hope that my studies of the structure of pure morphic sequences, which I preformed to solve the problem about factor complexity, will be helpful to study substitutional dynamical systems.

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