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Generically Transitive Actions on Multiple Flag Varieties

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Let G be a semisimple algebraic group whose decomposition into the product of simple components does not contain simple groups of type A, and $P \subseteq G$ be a parabolic subgroup. Extending the results of Popov [7], we enumerate all triples (G, P, n) such that (a) there exists an open G-orbit on the *multiple flag variety* $G/P \times G/P \times \ldots \times G/P$ (n factors), (b) the number of G-orbits on the multiple flag variety is finite.

1 Introduction

Let G be a semisimple connected algebraic group over an algebraically closed field of characteristic zero, and $P \subseteq G$ be a parabolic subgroup. One easily checks that the G-orbits on $G/P \times G/P$ are in bijection with P-orbits on G/P. The Bruhat decomposition of G implies that the number of P-orbits on G/P is finite and that these orbits are enumerated by a subset in the Weyl group W corresponding to G. In particular, there is an open G-orbit on $G/P \times G/P$. So we come to the following questions: for which G, P and $n \geq 3$ is there an open G-orbit on the *multiple flag variety* $(G/P)^n := G/P \times G/P \times \ldots \times G/P$? For which G, P and n is the number of orbits finite?

Notice that if G is locally isomorphic to $G^{(1)} \times \ldots \times G^{(k)}$, where $G^{(i)}$ are simple, then there exist parabolic subgroups $P^{(i)} \subseteq G^{(i)}$ such that $G/P \cong G^{(1)}/P^{(1)} \times \ldots \times G^{(n)}/P^{(n)}$. Hence in the sequel we may assume that G is simple. Moreover, let $\pi\colon \widetilde{G} \to G$ be a simply connected cover. Then π induces a bijection between parabolic subgroups $P \subseteq G$ and $\widetilde{P} \subseteq \widetilde{G}$, namely $\widetilde{P} = \pi^{-1}(P)$, and an isomorphism $\widetilde{G}/\widetilde{P} \to G/P$. Also, $\widetilde{G}/\widetilde{P}$ may be considered as G-variety since $\operatorname{Ker}\pi$ acts trivially on it. In this sense the isomorphism is G-equivariant. Therefore we may consider only one simple group of each type.

The classification of multiple flag varieties with an open G-orbit for maximal subgroups P was given by Popov in [7]. We need some notation to formulate his result. Fix a maximal torus in G and an associated simple

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root system $\{\alpha_1, \dots, \alpha_l\}$ of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. We enumerate simple roots as in [3]. Let $P_i \subset G$ be the maximal parabolic subgroup corresponding to the simple root α_i.

Theorem 1.1. [7, Theorem 3] Let G be a simple algebraic group. The diagonal G-action on the multiple flag variety $(G/P_i)^n$ is generically transitive if and only if $n \le 2$ or (G, n, i) is an entry in Table 1:

Table 1. Generically transitive actions for maximal parabolic subgroups

Type of G	(n, i)
A_l	$n < \frac{(l+1)^2}{i(l+1-i)}$
$B_l, l \ge 3$	$n = 3, i = 1, l$
$C_l, l \ge 2$	$n = 3, i = 1, l$
$D_l, l \ge 4$	$n = 3, i = 1, l-1, l$
E_6	$n = 3, 4, i = 1, 6$
E_7	$n = 3, i = 7$

$\square$

In [7], the following question was posed: for which non-maximal parabolic subgroups $P \subset G$ is there an open G-orbit in $(G/P)^n$? We solve this problem for all simple groups except for those of type A_l.

Denote the intersection $P_{i_1} \cap \ldots \cap P_{i_s}$ by $P_{i_1,\dots,i_s}$. It is easy to see that $P_{i_1,\dots,i_s}$ is a parabolic subgroup and that every parabolic subgroup is conjugated to some $P_{i_1,\dots,i_s}$.

Theorem 1.2. Let G be a simple algebraic group which is not locally isomorphic to SL_{l+1}, $P \subset G$ be a non-maximal parabolic subgroup and $n \ge 3$. Then the diagonal G-action on the multiple flag variety $(G/P_i)^n$ is generically transitive if and only if $n = 3$ and (G, P) is one of the pairs in Table 2:

Table 2. Generically transitive actions for non-maximal parabolic subgroups

Type of G	P
$D_l, l \ge 5$ is odd	$P_{1,l-1}, P_{1,l}$
$D_l, l \ge 4$ is even	$P_{1,l-1}, P_{1,l}, P_{l-1,l}$

$\square$

Now let us consider actions with a finite number of orbits. Recall that a G-variety X is called *spherical* if a Borel subgroup $B \subseteq G$ acts on X with an open orbit. It is well-known that the number of B-orbits on a spherical variety is finite, see [1, 9]. Equivalently, the number of G-orbits on $G/B \times X$ is finite if X is spherical. Therefore, if $P \subseteq G$ is a parabolic subgroup and X is a spherical G-variety, then the number of G-orbits on $G/P \times X$ is finite. The classification of all pairs of parabolic subgroups (P, Q) such that $G/P \times G/Q$ is spherical is given in [4, 8]. According to this classification, if (G, P_i) is an entry in Table 1, then $G/P_i \times G/P_i$ is spherical and hence the number of G-orbits on $G/P_i \times G/P_i \times G/P_i$ is finite. In the last section we prove that the number of G-obits on $(G/P)^n$ is infinite if $n \ge 4$. We also check directly that if (G, P) is an entry in Table 2, then the number of G-orbits on $G/P \times G/P \times G/P$ is infinite. Thus we come to the following result.

Theorem 1.3. Let G be a simple algebraic group, $P \subset G$ be a parabolic subgroup and $n \geq 3$. The following properties are equivalent.

1. The number of G-orbits on $(G/P)^n$ is finite.
2. $n = 3$, P is maximal, and there is an open G-orbit on $G/P \times G/P \times G/P$.
3. $n = 3$, and $G/P \times G/P$ is spherical.

□

Corollary 1.4. Let $n \geq 3$. The number of G-orbits on $(G/P)^n$ is finite if and only if $n = 3$ and (G, P) is one of the pairs listed in Table 3:

Table 3. Actions with finite numbers of orbits

Type of G	P
A_l	any maximal
$B_l, l \geq 2$	P_1, P_l
$C_l, l \geq 3$	P_1, P_l
$D_l, l \geq 4$	P_1, P_{l-1}, P_l
E_6	P_1, P_6
E_7	P_7

□

Let us mention a more general result for classical groups. Let $Q_{(1)}, \ldots, Q_{(n)}$ be parabolic subgroups in G. We call the variety $G/Q_{(1)} \times \ldots \times G/Q_{(n)}$ a *generalized multiple flag variety*. The classification of all generalized multiple flag varieties with a finite number of G-orbits is given in [5] for $G = SL_{l+1}$ and in [6] for $G = Sp_{2l}$.

Proofs of Theorems 2 and 3 use methods developed in [7]. The results concerning existence of an open orbit in a linear representation space in Section 3 may be of independent interest. In several cases for $G = SO_{2l}$ the existence of an open orbit on a multiple flag variety is checked directly.

2 Preliminaries

Let G be a connected simple algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero and $\mathfrak{g} = \operatorname{Lie} G$. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. These data determine a root system Φ of $\mathfrak{g}$, a positive root subsystem Φ^+ and a system of simple roots $\Delta \subseteq \Phi^+$, $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Choose a corresponding Chevalley basis $\{x_i, y_i, h_i\}$ of $\mathfrak{g}$. We have $[h, x_i] = \alpha(h)x_i$, $[h, y_i] = -\alpha(h)y_i$ for all $h \in \mathfrak{t} = \operatorname{Lie} T$ and $h_i = [x_i, y_i]$.

Let $I = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subseteq \Delta$ be a subset. The Lie algebra of the parabolic subgroup $P_I := P_{i_1,\ldots,i_s}$ is

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha,$$

where $\mathfrak{b} = \operatorname{Lie} B$ and $\Phi_I \subseteq \Phi^-$ denotes the set of the negative roots such that their decomposition into the sum of simple roots *does not* contain the roots α_i, $i \in I$. For example, $P_\Delta = B$ and $P_\varnothing = G$. It is known that

[2, Theorem 30.1] if a parabolic group P contains B, then $P = P_I$ for some $I \subseteq \Delta$. Therefore any parabolic subgroup $P \subseteq G$ is conjugate to some P_I. If $P = P_I$ for some $I \subseteq \Delta$, we denote by P^- the parabolic subgroup whose Lie algebra is

$$\mathfrak{p}^- = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\Phi_I \cup \Phi^-} \mathfrak{g}_\alpha.$$

Denote the weight lattice of T by $\mathfrak{X}(T)$. Let $\mathfrak{X}^+(T)$ be the subsemigroup of dominant weights with respect to B. Assume first that G is simply connected. Then $\mathfrak{X}^+$ is generated by the fundamental weights $\pi_1, \ldots, \pi_l$. Given a dominant weight λ, denote the simple G-module with the highest weight λ by $V(\lambda)$. If G is not simply connected, we may consider a simply connected cover $p\colon \widetilde{G} \to G$, the dominant weight lattice $\mathfrak{X}^+(p^{-1}(T))$ and the highest weight $\widetilde{G}$-module $V(\lambda)$.

Let G be a simple group and $P = P_{i_1,\ldots,i_s}$ be a parabolic subgroup. Notice that if there is an open G-orbit on $(G/P)^n$, then there exists an open G-orbit on $(G/P_i)^n$ for all $i \in \{i_1, \ldots, i_s\}$. Indeed, since $P \subseteq P_i$, one has the surjective G-equivariant map $G/P \to G/P_i$, $gP \mapsto gP_i$. It induces the surjective G-equivariant map $\varphi\colon (G/P)^n \to (G/P_i)^n$, and the image of an open G-orbit on $(G/P)^n$ under φ is an open G-orbit on $(G/P_i)^n$. Similarly, if G acts on $(G/P)^n$ with an open orbit and $m < n$, then G acts on $(G/P)^m$ with an open orbit.

Theorem 1 leaves us very few cases of non-maximal parabolic groups to consider. Namely, if $n > 3$ and G is of type B_l, C_l or D_l, then G never acts on $(G/P)^n$ with an open orbit. If $n = 3$ and G is of type B_l or C_l, it suffices to consider $P = P_{1,l}$, and we show that there is no open orbit in this case. If $n = 3$ and G is of type D_l, an open orbit may exist only if $P = P_I$ where $I \subseteq \{\alpha_1, \alpha_{l-1}, \alpha_l\}$. So there are four cases to consider. We reduce the case $P_{1,l-1}$ to the case $P_{1,l}$. If G is of type E_6, the only parabolic group we should consider is $P = P_{1,6}$. We show that there is no open orbit for $n = 3$. If G is of type E_7, if there existed an open G-orbit on $(G/P)^n$ for $n \geq 3$, then the only maximal parabolic subgroup containing P would be P_7, but in this case P should be maximal itself. If G is of type E_8, F_4 or G_2, an open orbit exists for no maximal parabolic subgroups for $n \geq 3$, so there are no cases to consider.

Given a group G acting on an irreducible variety X with an open orbit, according to [7] we denote the maximal n such that there is an open G-orbit on X^n by $\operatorname{gtd}(G : X)$. If G acts on X^n with an open orbit, we say that the action $G : X$ is *generically n-transitive.*

We make use of the following fact proved by Popov.

Proposition 2.1. [7, Corollary 1 (ii) of Proposition 2] Let G be a simple algebraic group, P be a parabolic subgroup, P^- be an opposite parabolic subgroup, $L = P \cap P^-$ be the corresponding Levi subgroup and $\mathfrak{u}^-$ be the Lie algebra of the unipotent radical of P^-. If P is conjugate to P^-, then $\operatorname{gtd}(G : G/P) = 2 + \operatorname{gtd}(L : \mathfrak{u}^-)$. □

We suppose that the group SO_l acts in the l-dimensional space and preserves the bilinear form whose matrix with respect to a standard basis is

$$Q = \begin{pmatrix} & & & & 1 \\ & 0 & & \iddots & \\ & & 1 & & \\ & & 1 & & \\ & \iddots & & 0 & \\ 1 & & & & \end{pmatrix}.$$

We denote the l-dimensional projective space by $\mathbf{P}^l$ and the Grassmannian of k-dimensional subspaces in $\mathbb{K}^l$ by $\mathrm{Gr}(k,l)$.

3 Existence of an Open Orbit

3.1 Groups of type B_l

By Theorem 1.1, it is sufficient to consider the case $P = P_{1,l}$. The Dynkin diagram B_l has no automorphisms, hence P is conjugate to P^-. So we may apply Proposition 2.1, and it suffices to check that $\mathrm{gtd}(L : \mathfrak{u}^-) = 0$, i. e. L acts on $\mathfrak{u}^-$ with no open orbit.

Let $G = SO_{2l+1}$. Then $L = \mathbb{K}^* \times GL_{l-1}$ and the L-module $\mathfrak{u}^-$ can be decomposed into the direct sum $V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$. Here V_1 is a GL_{l-1}-module $(\mathbb{K}^{l-1})^*$ dual to the tautological one and its $\mathbb{K}^*$-weight is 1, V_2 is a trivial one-dimensional GL_{l-1}-module of weight 1, V_3 is a tautological GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 0, V_4 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_5 is a GL_{l-1}-module $\Lambda^2\mathbb{K}^{l-1}$ of weight 0. According to this decomposition, we denote components of a vector $u \in \mathfrak{u}^-$ by u_1, u_2, u_3, u_4, u_5.

Notice that there exists a GL_{l-1}-invariant pairing between V_1 and V_3. Its $\mathbb{K}^*$-weight is 1. Also there exists a GL_{l-1}-invariant pairing between V_1 and V_4, whose $\mathbb{K}^*$-weight is 2. Therefore the rational function

$$\frac{(u_1, u_3)^2}{(u_1, u_4)}$$

is a non-constant invariant for $L : \mathfrak{u}^-$, and the action of G on G/P is not generically 3-transitive.

3.2 Groups of type C_l

This case is completely similar to the previous one, and again the only thing we should do is to prove that there is no open L-orbit on $\mathfrak{u}^-$, where $L = P \cap P^-$ is a Levi subgroup of $P = P_{1,l}$ and $\mathfrak{u}^-$ is the Lie algebra of the unipotent radical of P^-.

Let $G = Sp_{2l}$. Then $L = \mathbb{K}^* \times GL_{l-1}$ and the L-module $\mathfrak{u}^-$ can be written as $V_1 \oplus V_2 \oplus V_3 \oplus V_4$. Here V_1 is a GL_{l-1}-module $(\mathbb{K}^{l-1})^*$ and its $\mathbb{K}^*$-weight is 1, V_2 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_3 is a GL_{l-1}-module

$S^2\mathbb{K}^{l-1}$ of weight 0, V_4 is a trivial GL_{l-1}-module of weight 2. According to this decomposition, we denote components of a vector $u \in \mathfrak{u}^-$ by u_1, u_2, u_3, u_4.

We see that there exists a GL_{l-1}-invariant pairing between V_1 and V_2 with $\mathbb{K}^*$-weight is 2. Therefore we have the rational invariant

$$\frac{(u_1, u_2)}{u_4}$$

for $L : \mathfrak{u}^-$, and the action of G on G/P is not generically 3-transitive.

3.3 Groups of type D_l

This time we should consider the following four cases of parabolic subgroups: $P = P_{1,l-1}, P_{1,l}, P_{l-1,l}, P_{1,l-1,l}$. One easily checks that P and P^- are conjugate except for the cases $P = P_{1,l}$, l odd, and $P = P_{1,l-1}$, l odd.

Let $G = SO_{2l}$. There exists a diagram automorphism of G that interchanges α_{l-1} and α_l. It preserves the maximal torus and the Borel subgroup and interchanges $P_{1,l-1}$ and $P_{1,l}$. Therefore, the actions $G : G/P_{1,l-1}$ and $G : G/P_{1,l}$ are either generically 3-transitive or not generically 3-transitive simultaneously.

3.3.1 $P = P_{l-1,l}$

In this case, P and P^- are conjugate, and we have to find $\mathrm{gtd}(L : \mathfrak{u}^-)$.

The Levi subgroup L is isomorphic to $\mathbb{K}^* \times GL_{l-1}$ and the L-module $\mathfrak{u}^-$ is isomorphic to $V_1 \oplus V_2 \oplus V_3$, where V_1 is a GL_{l-1}-module $\Lambda^2\mathbb{K}^{l-1}$ and its $\mathbb{K}^*$-weight is 0, V_2 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_3 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight -1. We denote components of a vector $u \in \mathfrak{u}^-$ by u_1, u_2, u_3.

Let l be odd. Then a generic element $u_1 \in V_1$ gives rise to a non-degenerate skew-symmetric bilinear form on the GL_{l-1}-module $(\mathbb{K}^{l-1})^*$. Furthermore, one can consider the corresponding skew-symmetric form on the tautological GL_{l-1}-module. This form is obtained by matrix inversion and we denote it by u_1^{-1}. The following function is a rational L-invariant:

$$u_1^{-1}(u_2, u_3).$$

Thus the action of G on G/P is not generically 3-transitive.

Let l be even. We prove that there is an open L-orbit on $\mathfrak{u}^-$.

Consider the GL_{l-1}-module $V' = V_1 \oplus V_2$, where V_2 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ and $V_1 = \Lambda^2\mathbb{K}^{l-1}$.

Since $l-1$ is odd, the rank of a generic element $w \in V_1$ is $l-2$. Denote the set of all $w \in V_1$ such that $\mathrm{rk}\, w = l-2$ by Z. Any element $w \in Z$ gives rise to a (degenerate) skew-symmetric form on V_2^*, and $\dim \mathrm{Ker}\, w = 1$. Consider the subspace $(\mathrm{Ker}\, w)^\perp \subset V_2$ where all the functions from the kernel vanish. Denote $V_2 \setminus (\mathrm{Ker}\, w)^\perp$ by X_w. Clearly, $W_1 = \cup_{w \in Z}(w \times X_w)$ is an open GL_{l-1}-invariant subset of V'.

Let us prove that GL_{l-1} acts transitively on W_1. First, given an element $u = (u_1, u_2) \in W_1$, one can apply an element of GL_{l-1} such that the matrix of the bilinear form u_1 in the corresponding basis is

$$R = \begin{pmatrix} 0 & & & & & \\ & 0 & 1 & & \text{\Large 0} & \\ & -1 & 0 & & & \\ & & & \ddots & & \\ & \text{\Large 0} & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

The first coordinate of u_2 in the new basis is non-zero since $u_2 \in X_{u_1}$. Denote the i-th coordinate of u_2 by $(u_2)_i$. The following element of GL_{l-1} preserves the bilinear form with matrix R:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ -(u_2)_3/(u_2)_1 & & & \\ \vdots & & I_{l-2} & \\ -(u_2)_{l-1}/(u_2)_1 & & & \end{pmatrix}.$$

When we apply it to u_2, all its coordinates will be zero except for the first one.

So any element of W_1 can be transformed by GL_{l-1}-action to an element of the form $u_1 = R$, $u_2 = ((u_2)_1, 0, \dots, 0)^T$, where $(u_2)_1 \neq 0$. Clearly, all these elements belong to the same GL_{l-1}-orbit. Call such an element of V' *canonical*, i. e. call an element $(u_1, u_2) \in V'$ *canonical* if $u_1 = R$, $u_2 = ((u_2)_1, 0, \dots, 0)^T$, where $(u_2)_1 \neq 0$. The stabilizer of $(u_1, \langle u_2 \rangle)$ consists of direct sums of a non-zero 1×1 matrix and a symplectic $(l-2) \times (l-2)$ matrix. Such an element fixes u_2 as well if and only if the first 1×1 matrix is 1.

Now we are ready to consider the L-action on $\mathfrak{u}^-$. Maintain the above notation. Since V_2 and V_3 are isomorphic as GL_{l-1}-modules, for each $w \in Z \subset V_1$ we can similarly consider the open subset $V_3 \setminus (\operatorname{Ker} w)^\perp$. Denote it by Y_w. Define the subsets $W_2 = \cup_{w \in Z}(w \times X_w \times Y_w)$ and $W = \{u \in W_2 : u_2 \text{ is not a multiple of } u_3\}$. Let us prove that L acts on W transitively.

We may suppose that u_1 and u_2 are canonical in the sense stated above. Applying a diagonal matrix from $(GL_{l-1})_{u_1, \langle u_2 \rangle}$, we may assume that $(u_2)_1 (u_3)_1 = 1$ since V_2 and V_3 are both tautological GL_{l-1}-modules. Since u_3 is not a multiple of u_2, the vector $v = ((u_3)_2, (u_3)_3, \dots, (u_3)_{l-1})$ is not zero. Since Sp_{l-2} acts transitively on $\mathbb{K}^{l-2} \setminus 0$, there exists an element $g \in (GL_{l-1})_{u_1, u_2}$, $g = g_1 \oplus g_2$, $g_1 = 1$, $g_2 \in Sp_{l-2}$ such that $g_2 v = ((u_3)_1, 0, \dots, 0)^T$. In other words, we may suppose that u_1 and u_2 are canonical, the only non-zero coordinates of u_3 are the first one and the second one, they are equal, and $(u_2)_1 (u_3)_1 = 1$.

Now recall that $L = GL_{l-1} \times \mathbb{K}^*$, the $\mathbb{K}^*$-weights of V_1, V_2 and V_3 are 0, 1 and -1, respectively. Therefore, after applying a suitable element of $\mathbb{K}^*$, we have $u_1 = S$, $u_2 = (1, 0, \dots, 0)^T$ and $u_3 = (1, 1, 0, \dots, 0)^T$, and the G-action on G/P is generically 3-transitive.

3.3.2 $P = P_{1,l}$

In this case, Proposition 2.1 applies if and only if l is even.

Let l be even. It is sufficient to prove that there is an open L-orbit on $\mathfrak{u}^-$.

Again $L = \mathbb{K}^* \times GL_{l-1}$, and the L-module V can be decomposed into three summands, $V = V_1 \oplus V_2 \oplus V_3$, but this time V_1 is a GL_{l-1}-module $\Lambda^2\mathbb{K}^{l-1}$ and its $\mathbb{K}^*$-weight is 0, V_2 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_3 is a GL_{l-1}-module $(\mathbb{K}^{l-1})^*$ of weight 1. We denote components of an element $u \in \mathfrak{u}^-$ by u_1, u_2, u_3.

Recall the notation we have introduced for the GL_{l-1}-module V'. Also this time denote $Y_w = V_3 \setminus \operatorname{Ker} w$. Define $W_2 = \cup_{w \in Z}(w \times X_w \times Y_w)$ and $W = \{u \in W_2 : \langle u_2, u_3\rangle \neq 0\}$. Here $\langle \cdot, \cdot\rangle$ denotes the GL_{l-1}-invariant pairing between V_2 and V_3. Its $\mathbb{K}^*$-weight is 2, but the condition $\langle u_2, u_3\rangle \neq 0$ is not affected by $\mathbb{K}^*$-action, so W is L-invariant. We are going to prove that L acts transitively on W.

Again we may suppose that $u_1 = R$ and the only non-zero coordinate of u_2 is the first one. Notice that $(u_3)_1 \neq 0$ since $\langle u_2, u_3\rangle \neq 0$. This time V_2 and V_3 are dual GL_{l-1}-modules, so by applying a suitable element of $(GL_{l-1})_{u_1,\langle u_2\rangle}$ the coordinates $(u_2)_1$ and $(u_3)_1$ can be made equal.

Consider the vector $v = ((u_3)_2, (u_3)_3, \ldots, (u_3)_{l-1})$. It cannot be zero since $u_3 \notin \operatorname{Ker} u_1$. Since Sp_{l-2} acts transitively on $(\mathbb{K}^{l-2})^* \setminus 0$, there exists an element $g \in (GL_{l-1})_{u_1,u_2}$, $g = g_1 \oplus g_2$, $g_1 = 1$, $g_2 \in Sp_{l-2}$ such that $g_2 v = ((u_3)_1, 0, \ldots, 0)^T$. In other words, we may suppose that u_1 and u_2 are canonical, the only non-zero coordinates of u_3 are the first one and the second one, and $(u_2)_1 = (u_3)_1 = (u_3)_2$.

This time the $\mathbb{K}^*$-weights of V_2 and V_3 are both 1, so with the help of the $\mathbb{K}^*$-action we can satisfy the equality $(u_2)_1 = (u_3)_1 = (u_3)_2 = 1$. Thus, L acts transitively on W, and the G-action on G/P is generically 3-transitive.

Let l be odd. Proposition 2.1 does not apply, and we have to find $\operatorname{gtd}(G : G/P)$ directly.

Consider the tautological SO_{2l}-module $\mathbb{K}^{2l}$, and let $e_1, \ldots, e_{2l}$ be its standard basis. Let $X' \subset \operatorname{Gr}(l, 2l)$ be the set of all isotropic subspaces of dimension l in $\mathbb{K}^{2l}$. One easily checks that X' is a disjoint union of two SO_{2l}-orbits, and the group O_{2l} interchanges them. If two subspaces belong to the same SO_{2l}-orbit, then their intersection is non-zero.

Denote the orbit $SO_{2l}\,\langle e_1, \ldots, e_l\rangle \subset X'$ by X. Then X is an irreducible subvariety in $\operatorname{Gr}(l, 2l)$.

For each $s \in X$ let $Y_s \subset \mathbf{P}^{2l-1}$ be the set of all lines contained in s. Clearly, $W = \cup_{s \in X}(s \times Y_s)$ is a closed G-invariant subset in $\operatorname{Gr}(l, 2l) \times \mathbf{P}^{2l-1}$. One easily checks that $G/P = W$.

Let us prove that there exists an open G-orbit on $W \times W \times W$. We impose some conditions on the point $(s_1, a_1, s_2, a_2, s_3, a_3) \in W \times W \times W$ and so define an open subset $Y \subseteq W \times W \times W$. Then we define a point $p \in \operatorname{Gr}(l, 2l) \times \mathbf{P}^{2l-1} \times \operatorname{Gr}(l, 2l) \times \mathbf{P}^{2l-1} \times \operatorname{Gr}(l, 2l) \times \mathbf{P}^{2l-1}$ and prove that (a) each point $y \in Y$ belongs to the same G-orbit that p does, and (b) p belongs to Y. Condition (b) guarantees that Y is not empty.

Let $Y \subseteq W \times W \times W$ be the set of all tuples $(s_1, a_1, s_2, a_2, s_3, a_3)$ such that:

1. $s_1 \cap s_2 \cap s_3 = 0$.
2. $s_1 + s_2 + s_3 = \mathbb{K}^{2l}$.

3. $\dim s_1 \cap s_2 = \dim s_2 \cap s_3 = \dim s_1 \cap s_3 = 1$.
4. $\dim(a_1 + a_2 + a_3) = 3$.
5. The intersection of the subspaces $s = (s_1 \cap s_2) + (s_2 \cap s_3) + (s_1 \cap s_3)$ and $a = a_1 + a_2 + a_3$ is zero.
6. $a_i + s_j + s_k = \mathbb{K}^{2l}$, where $i = 1, 2, 3, j \neq i, k \neq i, j < k$.
7. The lines a_i and a_j are not orthogonal for all $i \neq j$.

Notice that if conditions (1)–(3) hold, the sum of subspaces $s_1 \cap s_2$, $s_2 \cap s_3$ and $s_1 \cap s_3$ is direct.

Let us prove that G acts transitively on Y. Choose vectors f_1, f_2, f_3 such that $\langle f_i \rangle = a_i$, and vectors f_4, f_5, f_6 such that $\langle f_4 \rangle = s_2 \cap s_3$, $\langle f_5 \rangle = s_1 \cap s_3$, $\langle f_6 \rangle = s_1 \cap s_2$. The restriction of the bilinear form to the subspace $S = \langle f_1, \ldots, f_6 \rangle$ is defined by the following matrix:

$$\begin{pmatrix} 0 & b_1 & b_2 & b_4 & & \\ b_1 & 0 & b_3 & & b_5 & \\ b_2 & b_3 & 0 & & & b_6 \\ b_4 & & & & & \\ & b_5 & & & & \\ & & b_6 & & & \end{pmatrix}.$$

Conditions (6) and (7) imply that $b_i \neq 0$ for all i. Clearly, this matrix is non-degenerate.

The above choice of the vectors f_i allows to multiply them by scalars. Up to scalar multiplication we may assume that all $b_i = 1$.

Notice that a cyclic permutation of f_1, f_2, f_3 and the same permutation of f_4, f_5, f_6 performed simultaneously define a linear operator on S that preserves the restriction of the bilinear form and whose determinant is 1.

Consider the following basis of S: $g_1 = f_1$, $g_2 = f_5$, $g_3 = f_6$, $g_4 = f_3 - f_4 - f_5$, $g_5 = f_2 - f_4$, $g_6 = f_4$. One checks directly that the matrix of the bilinear form with respect to this basis is Q. Obviously, there exists a matrix M such that $(f_1, \ldots, f_6) = (g_1, \ldots, g_6)M$ and whose elements do not depend on a_i and s_i.

The restriction of the bilinear form to S is non-degenerate, hence its restriction to $S^\perp$ is also non-degenerate. Since $s_i = s_i^\perp$, $\dim(s_i \cap S^\perp) = l - 3$ for all i.

Thus, $S^\perp$ is a subspace of even dimension equipped with a non-degenerate symmetric bilinear form. We have three isotropic subspaces of maximal dimension in $S^\perp$, and the intersection of any two of them is zero. Let us prove the following lemma.

Lemma 3.1. Let $(\cdot, \cdot)$ be a non-degenerate symmetric bilinear form in $\mathbb{K}^{2k}$, and U_1, U_2, U_3 be isotropic subspaces of dimension k with $U_i \cap U_j = 0$ for $i \neq j$. Then there exist matrices $M_1, M_2, M_3 \in Mat_{2k \times k}$ that do not depend on U_i and a basis $e_1, \ldots, e_{2k}$ of $\mathbb{K}^{2k}$ such that: (a) the matrix of the bilinear form is Q and (b) $(e_1, \ldots, e_{2k})M_i$ is a basis of U_i. $\square$

Proof. Consider the non-degenerate linear map $A\colon U_1 \to U_2$ whose graph is the subspace U_3. This is possible since $U_i \cap U_j \neq 0$ for $i \neq j$. In terms of the map, $U_3 = \{v + Av \mid v \in U_1\}$.

Consider the bilinear form $(v_1, v_2)_A = (v_1, Av_2)$ on U_1. Since U_3 is isotropic, we have $0 = (v_1 + Av_1, v_2 + Av_2) = (v_1, v_2) + (Av_1 + Av_2) + (v_1, Av_2) + (Av_1, v_2) = (v_1, Av_2) + (v_2, Av_1) = (v_1, v_2)_A + (v_2, v_1)_A$ for all $v_1, v_2 \in U_1$, hence the form $(\cdot,\cdot)_A$ is skew-symmetric. Assume that it is degenerate and $v \in U_1$ belongs to its kernel. Then $(v_1, v) = (v_1, Av) = 0$ for all $v_1 \in U_1$. Since the pairing between trivially intersecting isotropic subspaces U_1 and U_2 of maximal dimension is non-degenerate, $Av = 0$. Since $\operatorname{Ker} A = 0$, $v = 0$ and the form $(\cdot,\cdot)_A$ is non-degenerate.

Thus, we have a symplectic space U_1 with the skew-symmetric form $(\cdot,\cdot)_A$. Hence k is even. Choose a basis $\langle q_1, \dots, q_k \rangle$ of U_1 such that the matrix of the skew-symmetric form is

$$\begin{pmatrix} & & & & 1 \\ & \text{\Large 0} & & \iddots & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & \text{\Large 0} & \\ -1 & & & & \end{pmatrix}.$$

The vectors $q_1, \dots, q_k$ are linearly independent, so let them be the first k elements of a basis of $\mathbb{K}^{2l}$. Define the rest of the basis as follows: $q_{k+j} = -Aq_j$ if $j = 1, \dots, k/2$ and $q_{k+j} = Aq_j$ if $j = k/2+1, \dots, k$. The matrix of the bilinear form $(\cdot,\cdot)$ is Q. The subspaces U_i have the following bases:

$$U_1 = \langle q_1, \dots, q_k \rangle$$
$$U_2 = \langle q_{k+1}, \dots, q_{2k} \rangle$$
$$U_3 = \langle q_1 - q_{k+1}, \dots, q_{k/2} - q_{k+k/2}, q_{k/2+1} + q_{k+k/2+1}, \dots, q_k + q_{2k} \rangle.$$

This completes the proof of the lemma. ■

Consider the following basis of $\mathbb{K}^{2l}$: $g_1, g_2, g_3, q_1, \dots, q_{2l-6}, g_4, g_5, g_6$, where q_i are defined above in the proof of the lemma. Notice that the matrix of the bilinear form in this basis is Q. Define the operator $B\colon \mathbb{K}^{2l} \to \mathbb{K}^{2l}$ that maps this basis to the standard one. We know that the matrix of the bilinear form is Q in both bases, so $B \in O_{2l}$.

Let us check that $\det B = 1$. Assume that $\det B = -1$. Since $s_1 = \langle g_1, g_2, g_3, q_1, \dots, q_{l-3} \rangle$, $Bs_1 = \langle e_1, \dots, e_l \rangle \in X$. Since $s_1 \in X$, there exists an operator $C \in SO_{2l}$ such that $CBs_1 = s_1$. Thus, $CB \in (O_{2l})_{s_1}$, $\det CB = -1$, and the O_{2l}-orbit X' cannot be a union of two distinct SO_{2l}-orbits, a contradiction.

Bases of the subspaces Ba_i and Bs_i can be written in terms of e_i using matrices that do not depend on a_i and s_i. Namely, they are the same matrices that we need to write bases of a_i and s_i using g_i and q_i, and the latter do not depend on a_i and s_i. Denote the 6-tuple $(Bs_1, Ba_1, Bs_2, Ba_2, Bs_3, Ba_3)$ by p. It suffices to prove that $p \in Y$. Conditions (1)–(7) hold by the construction of g_i and q_i, but we should check that $(Bs_1, Bs_2, Bs_3) \in X \times X \times X$. It is sufficient to find elements of SO_{2l} that map s_1 to s_2 and s_1 to s_3. Since $s_i = (s_i \cap S) \oplus (s_i \cap S^\perp)$, we find them as direct sums of elements of $SO(S)$ and $SO(S^\perp)$. The elements of $SO(S)$ are already found, they are cyclic permutations of f_1, f_2, f_3 and f_4, f_5, f_6. To interchange $s_1 \cap S^\perp$ and $s_2 \cap S^\perp$, consider the map that permutes all the pairs of vectors $g_i \leftrightarrow g_{2l+1-i}$, $i = 1, \dots l-3$. It is orthogonal and its determinant is 1 since $l-3$ is even. Finally, the operator with the following matrix in the basis g_i maps $s_1 \cap S^\perp$ to $s_3 \cap S^\perp$.

$$\begin{pmatrix} I_{l-3} & 0 \\ D & I_{l-3}, \end{pmatrix}$$

where

$$D = \begin{pmatrix} -1 & & & & & \\ & \ddots & & & \text{\Large 0} & \\ & & -1 & & & \\ & & & 1 & & \\ & \text{\Large 0} & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Therefore, SO_{2l} acts transitively on Y, and $\mathrm{gtd}(G : G/P) = 3$.

3.3.3 $P = P_{1,l-1,l}$

The subgroups P and P^- are conjugate for all l. It is sufficient to find $\mathrm{gtd}(L : \mathfrak{u}^-)$, where $L = (\mathbb{K}^*)^2 \times GL_{l-2}$ and the L-module $\mathfrak{u}^-$ is isomorphic to the direct sum of 7 simple modules that we denote by $V_1, \dots, V_7$. Namely, V_1 is a GL_{l-1}-module $(\mathbb{K}^{l-2})^*$ and its $(\mathbb{K}^*)^2$-weight is $(1, 0)$, V_2 is a trivial GL_{l-2}-module of weight $(1, 1)$, V_3 is a GL_{l-2}-module $\mathbb{K}^{l-2}$ of weight $(0, 1)$, V_4 is a trivial GL_{l-2}-module of weight $(1, -1)$, V_5 is a GL_{l-2}-module $\mathbb{K}^{l-2}$ of weight $(0, -1)$, V_6 is a GL_{l-2}-module $\mathbb{K}^{l-2}$ of weight $(1, 0)$, V_7 is a GL_{l-2}-module $\Lambda^2\mathbb{K}^{l-2}$ of weight $(0, 0)$. Denote the components of $u \in \mathfrak{u}^-$ by $u_1, \dots, u_7$.

There exists a GL_{l-2}-invariant pairing between V_1 and V_3 whose $(\mathbb{K}^*)^2$-weight is $(1, 1)$. The following function is a rational L-invariant:

$$\frac{(u_1, u_3)}{u_4}.$$

Thus, the G-action on G/P is not generically 3-transitive.

3.4 Groups of type E_6

The only parabolic subgroup to consider is $P = P_{1,6}$. The set $\{1, 6\}$ of Dynkin diagram vertices is invariant under all automorphisms of the Dynkin diagram. Hence the Weyl group element of the maximal length interchanges P and P^-. We have to find $\mathrm{gtd}(L : \mathfrak{u}^-)$.

The Levi subgroup L is locally isomorphic to $(\mathbb{K}^*)^2 \times SO_8$, and the L-module $\mathfrak{u}^-$ is isomorphic to $V_1 \oplus V_2 \oplus V_3$. Here V_1 is an SO_8-module with the lowest weight $-\pi_1$, i. e. a tautological SO_8-module, V_2 is an SO_8-module with the lowest weight $-\pi_3$, V_3 is an SO_8-module with the lowest weight $-\pi_4$. Denote the components of $u \in \mathfrak{u}^-$ by u_1, u_2, u_3.

Since V_1 is a tautological SO_8-module, there exists an SO_8-invariant symmetric bilinear form on it that we denote by (u_1, u_1). There exist diagram automorphisms of SO_8 that transform the tautological SO_8-module to SO_8-modules isomorphic to V_2 and V_3. So there exist an SO_8-invariant on V_2 that we denote by (u_2, u_2) and an SO_8-invariant on V_3 that we denote by (u_3, u_3). These bilinear forms are not necessarily $(\mathbb{K}^*)^2$-invariant, in general their $(\mathbb{K}^*)^2$-weights are three pairs of integers. There is a linear combination of these pairs that is equal to zero. Hence, there exists a non-trivial rational L-invariant of the form

$$(u_1, u_1)^a (u_2, u_2)^b (u_3, u_3)^c,$$

where $a, b, c \in \mathbb{Z}$, and the G-action on $G/P \times G/P \times G/P$ is not generically transitive.

4 Finite Number of Orbits

Proposition 4.1. Let G be a simple algebraic group and P be a proper parabolic subgroup. If $n \geq 4$, the number of G-orbits on $(G/P)^n$ is infinite. □

Proof. Let $P = P_{i_1,\ldots,i_s}$. Consider the dominant weight $\lambda = \pi_{i_1} + \ldots + \pi_{i_s}$. Then G/P is isomorphic to the projectivization of the orbit of the highest weight vector $v_\lambda \in V(\lambda)$. In the sequel we shortly write $i = i_1$. It is easy to check that $y_i^2 v_\lambda = 0$. Denote the unipotent subgroup $\exp(ty_i)$ by U_i. We see that $U_i v_\lambda$ is an affine line not containing zero. The closure of its image in the projectivization $\mathbf{P}(V(\lambda))$ is a projective line $\mathbf{P}^1 \subseteq G/P \subseteq \mathbf{P}(V(\lambda))$. Choose $n \geq 4$ points $(x_1, \ldots, x_n) \in \mathbf{P}^1 \times \ldots \times \mathbf{P}^1 \subseteq G/P \times \ldots \times G/P$. The double ratio of the first four of these points does not change under G-action. Hence, two n-tuples with different double ratios cannot belong to the same orbit, and the number of orbits is infinite. ■

Now we prove that in the cases $P = P_{1,l}$ and $P = P_{l-1,l}$ the number of orbits on $G/P \times G/P \times G/P$ is infinite.

We suppose that $G = SO_{2l}$. Let $\mathbb{K}^{2l}$ be the tautological SO_{2l}-module and let $e_1, \ldots, e_{2l}$ be the standard basis. Let $X' \subset \mathrm{Gr}(l, 2l)$ be the set of all isotropic subspaces of dimension l in $\mathbb{K}^{2l}$. It is known that G/P_l is isomorphic to a connected component of X'. In the sequel we suppose that $G/P_l = X \subseteq X'$. For each $s \in X$ let $Y_s \subset \mathbf{P}^{2l-1}$ be the set of all lines contained in s. One easily checks that the closed subset $Y = \cup_{s \in X} (s \times Y_s) \subset \mathrm{Gr}(l, 2l) \times \mathbf{P}^{2l-1}$ is isomorphic to $SO_{2l}/P_{1,l}$.

Similarly, if $s \in X$, denote by $Z_s \subset \mathrm{Gr}(l-1, 2l)$ the set of all subspaces of dimension $l-1$ in s. Let Z be the closed subset $\cup_{s \in X} (s \times Z_s) \subset \mathrm{Gr}(l, 2l) \times \mathrm{Gr}(l-1, 2l)$. One easily checks that it is isomorphic to $SO_{2l}/P_{l-1,l}$.

First, let $l=3$. Consider the following isotropic subspaces: $S_1=\langle e_1,e_2,e_4\rangle$, $S_2=\langle e_2,e_3,e_6\rangle$, $S_3=\langle e_1,e_3,e_5\rangle$. They belong to the same SO_6-orbit, so we may suppose that $S_1,S_2,S_3\in X$. Choose a line $T_1\subset S_1$ such that $T_1\subset\langle e_1,e_2\rangle$. Also choose lines $T_2\subset S_2$ and $T_3\subset S_3$ such that $T_2\subset\langle e_2,e_3\rangle$ and $T_3\subset\langle e_1,e_3\rangle$. Impose one more restriction, namely, the sum T_2+T_3 should be direct and should not be equal to $\langle e_1,e_2\rangle$. Consider the point $((S_1,T_1),(S_2,T_2),(S_3,T_3))\in G/P_{1,l}\times G/P_{1,l}\times G/P_{1,l}$. There are four subspaces of $\langle e_1,e_2\rangle$: $\langle e_1\rangle=S_1\cap S_3$, $\langle e_2\rangle=S_2\cap S_3$, T_1 and $T_4=(T_2\oplus T_3)\cap\langle e_1,e_2\rangle$. Thus, we have defined four lines in $\mathbb{K}^6$ in terms of intersections and sums of S_i and T_i. If we apply an element $g\in G$ to these four lines, we will obtain four lines defined in the same way using gS_i and gT_i instead of S_i and T_i. The double ratio of these four lines in their sum of dimension two is not changed under G-action. Since T_1 is chosen arbitrarily, this double ratio can be any number and the number of orbits is infinite.

Consider the same subspaces S_i and T_i and set $U_1=T_1\oplus\langle e_4\rangle$, $U_2=T_2\oplus\langle e_6\rangle$ and $U_3=T_3\oplus\langle e_5\rangle$. The point $((S_1,U_1),(S_2,U_2),(S_3,U_3))$ belongs to $Z\times Z\times Z$. Note that $\langle e_1,e_2,e_3\rangle=(S_1\cap S_2)\oplus(S_2\cap S_3)\oplus(S_1\cap S_3)$ and $T_i=U_i\cap\langle e_1,e_2,e_3\rangle$. Again we have a subspace of dimension two and four lines in it defined in terms of intersections and sums of S_i and U_i. The existence of SO_6-invariant double ratio in this case yields that the number of orbits is infinite.

Let $l>3$. Construct the subspaces S_i, T_i and U_i as above, using the last three basis vectors instead of e_4,e_5,e_6. Let $S_i'=S_i\oplus\langle e_4,\dots,e_l\rangle$ and $U_i'=U_i\oplus\langle e_4,\dots,e_l\rangle$. The points $((S_1',T_1),(S_2',T_2),(S_3',T_3))$ and $((S_1',U_1'),(S_2',U_2'),(S_3',U_3'))$ belong to $Y\times Y\times Y$ and $Z\times Z\times Z$, respectively. Consider also the subspace $V=(S_1'\cap S_2'\cap S_3')^\perp$. The restriction of the bilinear form to this subspace is degenerate, its kernel is $S_1'\cap S_2'\cap S_3'=\langle e_4,\dots,e_l\rangle$. The quotient is a space of dimension 6 with a bilinear form. The quotient morphism restricted to $\langle e_1,e_2,e_3,e_{2l-2},e_{2l-1},e_{2l}\rangle$ is an isomorphism, so we have subspaces S_i, T_i, U_i in the 6-dimensional space. This is exactly the same situation as we had above for the group SO_6, and it enables us to define double ratios for the points of $G/P_{1,l}\times G/P_{1,l}\times G/P_{1,l}$ and $G/P_{l-1,l}\times G/P_{l-1,l}\times G/P_{l-1,l}$ under consideration. Therefore the number of SO_{2l}-orbits on these multiple flag varieties is infinite. This finishes the proof of Theorem 1.3.

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