Multiplicity-free products of Schubert divisors

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Abstract

Let G/B be a flag variety over $\mathbb{C}$. We say that the product of classes of Schubert divisors in the Chow ring is *multiplicity free* if it is possible to multiply it by a Schubert class (not necessarily of a divisor) and get the class of a point. In the present paper we find the maximal possible degree (in the Chow ring) of a multiplicity free product of classes of Schubert divisors.

1 Introduction

Let G be a simple algebraic group over $\mathbb{C}$ with a simply laced Dynkin diagram. Consider the *generalized flag variety* G/B, where $B \subseteq G$ is a Borel subgroup. We are going to study the Chow ring of G/B.

The Chow ring of G/B is generated (as a $\mathbb{Z}$-algebra) by the classes of *Schubert divisors* in G/B (actually, to define the Schubert divisors canonically, we need to first fix a maximal torus in B, which canonically defines the root system, the Weyl group, and its action on G/B, so we assume that a maximal torus in B is fixed until the end of the paper, although we will not need it explicitly). Denote the classes of Schubert divisors by $D_1, \ldots, D_r$, where $r = \operatorname{rk} G$. We will be particularly interested in monomials in classes D_i. Let us say that a monomial $D_1^{r_1} \ldots D_r^{n_r}$ is *multiplicity free* if there exists a Schubert class X (this is the class of a Schubert variety, not necessarily of a Schubert divisor) such that $D_1^{r_1} \ldots D_r^{n_r} X = [\mathrm{pt}]$. Our goal is to answer the following question: *What is the maximal degree (in the Chow ring) of a multiplicity-free monomial in $D_1, \ldots, D_r$ (i. e., what is the maximal value of the sum $n_1 + \ldots + n_r$ over all n-tuples $n_1, \ldots, n_r$ of nonnegative integers such that $D_1^{n_1} \ldots D_r^{n_r}$ is a multiplicity-free monomial?)* This question is particularly interesting in the case when G is of type E_8, because the answer may be used to compute upper bounds on the *canonical dimension* (see definition in [6]) of G/B over non-algebraically-closed fields, similarly to the arguments of [5, Section 5].

The answer to this question for E_8 is 34, see Theorem 11.5. More generally, we will answer this question for any simple group G with simply-laced Dynkin diagram. In particular, we also get an answer for the "classical" variety of complete flags, i. e. for the case when $G = SL_{r+1}$. Namely, for a group of type A_r (e. g. $G = SL_{r+1}$, the Weyl group in this case is the permutation group S_{r+1}) we get $r(r+1)/2$, see Lemma 11.1, for a group of type D_r ($r \geq 4$) we get $r(r+1)/2 - 1$, see Proposition 11.4, and for a group of type E_r ($6 \leq r \leq 8$) we get $r(r+1)/2 - 2$, see Theorem 11.5. The answer for type A_r agrees with the fact that the torsion index of SL_{r+1} is 1.

To explain how we are going to solve this question, let us introduce notation and terminology more carefully. Recall that we have fixed a maximal torus, so we have a canonically defined root system. Denote it by Δ. Also, denote the simple roots by $\alpha_1, \ldots, \alpha_r$, denote the Weyl group by W, and denote the reflection corresponding to a root $\alpha \in \Delta$ by σ_α. Also, set $d = \dim(G/B)$. In G/B, one can associate a so-called *Schubert subvariety* to any $w \in W$. There are many different ways to establish such a correspondence, we choose the following one: for each $w \in W$, denote $Z_w = [\overline{B\dot{w}w^{-1}B/B}]$, where $\dot{w}$ is the longest element of the Weyl group. Then $\operatorname{codim} Z_w = \ell(w)$, where $\ell(w)$ is the length of an element $w \in W$. In other words, Z_w belongs to the $\ell(w)$th graded component of the Chow ring. This notation corresponds to the notation for classes D_i we introduced before as follows: $Z_{\sigma_{\alpha_i}} = D_i$.

The classes of Z_w in Chow ring for all $w \in W$ form a basis of the Chow ring as of a linear space. The highest possible, the dth degree, of the Chow group is $\mathbb{Z}$-generated by $Z_{\dot{w}} = [\mathrm{pt}]$.

It is known (see, for example, [2, Proposition 1.3.6]) that all products of Schubert classes are linear combinations of Schubert classes with nonnegative coefficients. In particular, if $w_1, \ldots, w_k \in W$ and $\ell(w_1) + \ldots + \ell(w_k) = d$, then $Z_{w_1} \ldots Z_{w_k}$ is a nonnegative integer multiple of [pt].

If we have several Schubert classes such that the sum of their dimensions is d, we say that their product is *multiplicity-free* if it equals [pt], the class of a point, in the Chow ring. More generally, we say that a product of Schubert classes $Z_{w_1}, \ldots, Z_{w_k}$ ($w_i \in W$) is *multiplicity-free* if there exists a Schubert class Z_w ($w \in W$) such that $Z_{w_1} \ldots Z_{w_k} Z_w = [\mathrm{pt}]$. This agrees with and generalizes the definition of a multiplicity-free monomial in $D_1, \ldots, D_r$ we introduced above.

The paper [3] contains another identification of elements of the Weyl group and subvarieties of G/B, namely (see [3, §3.2]), $X_w = [\overline{BwB/B}]$. These notations are related as follows: $Z_w = X_{\dot{w}w^{-1}}$. In particular, $X_{\mathrm{id}} = Z_{\dot{w}} = [\mathrm{pt}]$.

In terms of this notation, the product $X_w X_{w'}$, where $w, w' \in W$, can be computed as follows, see [3, §3.3, Proposition 1a]: $X_w X_{w'} = X_{\mathrm{id}} = [\mathrm{pt}]$ if and only if $w = \dot{w}w'$. Otherwise, $X_w X_{w'} = 0$. In terms of the notation Z, this can be rewritten as follows: $Z_w Z_{w'} = [\mathrm{pt}]$ if and only if $w = w'\dot{w}$. Otherwise, $Z_w Z_{w'} = 0$.

The classes Z_w in Chow group for all $w \in W$ form a basis of the Chow group as of a linear space. In particular, every monomial in classes D_i equals a linear combination of some classes Z_w:

$$D_1^{n_1} D_2^{n_2} \ldots D_r^{n_r} = \sum C_{w,n_1,\ldots,n_r} Z_w.$$

We fix the notation $C_{w,n_1,\ldots,n_r}$ in the whole paper. It follows from the multiplication formulas mentioned above that $D_1^{n_1} D_2^{n_2} \ldots D_r^{n_r}$ is multiplicity-free if and only if there exists $w \in W$ such that $C_{w,n_1,\ldots,n_r} = 1$. (In more details, if we multiply the above equality by $Z_{w\dot{w}}$, then $C_{w,n_1,\ldots,n_r} Z_w$ will become $C_{w,n_1,\ldots,n_r}[\mathrm{pt}]$, and all other summands on the right-hand side will vanish.)

So, in fact we are trying to answer the following question: What is the maximal degree of a monomial of the form $D_1^{n_1} D_2^{n_2} \ldots D_r^{n_r}$ such that at least one coefficient $C_{w,n_1,\ldots,n_r}$ equals 1?

It also seems natural to ask when, for given numbers $n_1, \ldots, n_r$, all coefficients $C_{w,n_1,\ldots,n_r}$ for all $w \in W$ equal either 0 or 1. But this happens quite rarely, and we are not trying to answer this question here. We will return to this question in a subsequent paper.

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2 Preliminaries

We denote the subset of positive roots by Δ^+, and the set of simple roots by Π. Denote the fundamental weight corresponding to a simple root α_i by ϖ_i.

We choose the scalar multiplication on Δ so that the scalar square of each simple root is 2. The scalar product of two roots α and β is denoted by (α, β). Note that with this choice of scalar multiplication, we can use a simple formula for reflection: usually, we write

$$\sigma_\alpha \beta = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \alpha.$$

But with our choice of scalar product, we can write

$$\sigma_\alpha \beta = \beta - (\alpha, \beta)\alpha.$$

We enumerate simple roots as in [1].

We use the following Pieri formula:

Proposition 2.1 ([3, §4.4, Corollary 2])**.** *Let $\alpha_i \in \Pi$, and let $w \in W$. Then*

$$D_i Z_w = \sum_{\substack{\alpha \in \Delta^+ \\ \ell(\sigma_\alpha w) = \ell(w)+1}} \varpi_i(\alpha) Z_{\sigma_\alpha w}.$$

Proof. In fact, Corollary 2 in [3, §4.4] is formulated in terms of X_w (and also, in another form, in terms of other classes Y_w, but we don't need those), and looks as follows:

$$X_{\dot{w}\sigma_{\alpha_i}} X_{w'} = \sum_{\substack{\alpha\in\Delta^+\\ \ell(w'\sigma_\alpha)=\ell(w')-1}} \varpi_i(\alpha) X_{w'\sigma_\alpha}.$$

If we substitute $\dot{w}w^{-1}$ instead of w', we will get:

$$X_{\dot{w}\sigma_{\alpha_i}} X_{\dot{w}w^{-1}} = \sum_{\substack{\alpha\in\Delta^+\\ \ell(\dot{w}w^{-1}\sigma_\alpha)=\ell(\dot{w}w^{-1})-1}} \varpi_i(\alpha) X_{\dot{w}w^{-1}\sigma_\alpha}.$$

Now, using the facts that $\ell(\dot{w}w'') = \ell(\dot{w}) - \ell(w'')$ for any $w'' \in W$, that $\sigma_\alpha^{-1} = \sigma_\alpha$, and that $\ell(w''^{-1}) = \ell(w'')$, we can rewrite this:

$$X_{\dot{w}\sigma_{\alpha_i}} X_{\dot{w}w^{-1}} = \sum_{\substack{\alpha\in\Delta^+\\ \ell(\dot{w})-\ell(w^{-1}\sigma_\alpha)=\ell(\dot{w})-\ell(w^{-1})-1}} \varpi_i(\alpha) X_{\dot{w}(\sigma_\alpha w)^{-1}}.$$

$$X_{\dot{w}\sigma_{\alpha_i}} X_{\dot{w}w^{-1}} = \sum_{\substack{\alpha\in\Delta^+\\ \ell(\sigma_\alpha w)=\ell(w)+1}} \varpi_i(\alpha) X_{\dot{w}(\sigma_\alpha w)^{-1}}.$$

Now, using the notation Z:

$$Z_{\sigma_{\alpha_i}} Z_w = \sum_{\substack{\alpha\in\Delta^+\\ \ell(\sigma_\alpha w)=\ell(w)+1}} \varpi_i(\alpha) Z_{\sigma_\alpha w}.$$

Recall that $Z_{\sigma_i} = D_i$:

$$D_i Z_w = \sum_{\substack{\alpha\in\Delta^+\\ \ell(\sigma_\alpha w)=\ell(w)+1}} \varpi_i(\alpha) Z_{\sigma_\alpha w}.$$

□

Note that $\varpi_i(\alpha)$ is precisely the coefficient at α_i in the decomposition of α into a linear combination of simple roots.

We will use the following well-known combinatorial Hall representative lemma and its generalization.

Lemma 2.2 (Hall representative lemma)**.** *Let $A_1, \dots, A_n$ be several finite sets. Suppose that for each subset $I \subseteq \{1, \dots, n\}$ one has $|\cup_{i\in I} A_i| \geq |I|$. Then one can choose $a_i \in A_i$ for all i ($1 \leq i \leq n$) so that all elements a_i are different.*

Lemma 2.3 (Generalized Hall representative lemma)**.** *Let $A_1, \dots, A_r$ be several finite sets, and let $k_1, \dots, k_r \in \mathbb{N}$. Suppose that for each subset $I \subseteq \{1, \dots, n\}$ one has*

$$|\cup_{i\in I} A_i| \geq \sum_{i\in I} k_i$$

Then one can choose $a_i \in A_i$ for all i ($1 \leq i \leq n$) so that all elements a_i are different.

Proof. Consider the following collection of sets S_{ij}: $S_{ij} = A_i$, $1 \leq i \leq r$, $1 \leq j \leq k_i$. Let J be a subset of double indices. Let m_i ($1 \leq i \leq r$) be the number of double indices in J that begin with i. Then $m_i \leq k_i$. Also denote the projection of J onto the first coordinate by I. Then $\cup_{(i,j)\in J} S_{ij} = \cup_{i\in I} A_i$, and

$$|\cup_{(i,j)\in J} S_{ij}| = |\cup_{i\in I} A_i| \geq \sum_{i\in I} k_i \geq \sum_{i\in I} m_i = |J|.$$

So, the collection $\{S_{ij}\}$ satisfies the hypothesis of Lemma 2.2. □

The following facts about root systems and Weyl groups are well-known and can e found, for example, in [4].

Lemma 2.4. *Let $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, $\alpha \neq -\beta$. Then all possible values of (α, β) are 0, 1, and -1.*

Lemma 2.5. *Let $\alpha, \beta \in \Delta$. Then:*

1. *$\alpha + \beta \in \Delta$ if and only if $(\alpha, \beta) = -1$.*
2. *$\alpha - \beta \in \Delta$ if and only if $(\alpha, \beta) = 1$.*

Corollary 2.6. *For each $\alpha \in \Delta$, the reflection σ_α has the following orbits on Δ:*

1. *$\{\alpha, -\alpha\}$*
2. *$\{\beta\}$ (a fixed point) for each $\beta \in \Delta$, $(\alpha, \beta) = 0$.*
3. *$\{\beta, \gamma\}$ for $\beta, \gamma \in \Delta$, $(\alpha, \beta) = 1$, $(\alpha, \gamma) = -1$, and $\beta = \alpha + \gamma$.*

Lemma 2.7. *If $\alpha, \beta, \gamma \in \Delta$ and $(\alpha, \beta) = 1$, $(\beta, \gamma) = 1$, $(\alpha, \gamma) = 0$,*
then $\delta = \alpha + \gamma - \beta \in \Delta$, and $(\alpha, \delta) = 1$, $(\delta, \gamma) = 1$, $(\delta, \beta) = 0$

Proof. Direct computation of scalar products.
$\alpha - \beta \in \Delta$ by Lemma 2.5.
$(\alpha - \beta, \gamma) = 0 - 1 = -1$
$\delta = \alpha - \beta + \gamma \in \Delta$ by Lemma 2.5.
$(\delta . \alpha) = 2 - 1 + 0 = 1$.
$(\delta, \beta) = 1 - 2 + 1 = 0$.
$(\delta, \gamma) = 0 - 1 + 2 = 1$. □

Lemma 2.8. *If $\alpha, \beta, \gamma \in \Delta$ and $(\alpha, \beta) = 1$, $(\beta, \gamma) = 1$, $(\alpha, \gamma) = 0$, and there exists a simple root α_i that appears in the decompositions of all three roots α, β, and γ into linear combinations of simple roots with coefficient 1,*
then α_i appears in the decomposition of $\delta = \alpha - \beta + \gamma$ into a linear combination of simple roots also with coefficient 1, and $\delta \in \Delta^+$.

Proof. Direct calculation. □

Lemma 2.9. *If $w \in W$, then $\ell(w) = |\Delta^+ \cap w\Delta^-|$. Moreover, the set $|\Delta^+ \cap w\Delta^-|$ determines w uniquely.*

We will have several examples involving permutation groups. More precisely, there permutation groups will appear as the Weyl groups of groups of type A_r. The Weyl group of a group of type A_r is S_{r+1}. For brevity, we will write $(s_1, s_2, \ldots, s_{r+1})$ instead of

$$\begin{pmatrix} 1 & 2 & \ldots & r+1 \\ s_1 & s_2 & \ldots & s_{r+1} \end{pmatrix}.$$

The transposition interchanging the ith and the jth positions will be denoted by $(i \leftrightarrow j)$.

Example 2.10. The length of an element $(s_1, \ldots, s_{r+1}) \in W$ is the number of inversions, i. e. the number of pairs (i, j) with $i < j$ and $s_i > s_j$.

We use the following terminology to compute products of several divisors using Pieri formula.

Definition 2.11. Let $\alpha \in \Delta^+$ and let $w \in W$. We say that the reflection σ_α is:

1. A *sorting reflection* for w if $\ell(\sigma_\alpha w) < \ell(w)$;
2. A *desorting reflection* for w if $\ell(\sigma_\alpha w) > \ell(w)$;
3. An *admissible sorting reflection* for w if $\ell(\sigma_\alpha w) = \ell(w) - 1$;

4. An *admissible desorting reflection* for w if $\ell(\sigma_\alpha w) = \ell(w) + 1$;

5. An *antisimple sorting reflection* for w if $\ell(\sigma_\alpha w) = \ell(w) - 1$ and $w^{-1}\alpha \in -\Pi$.

6. An *antisimple desorting reflection* for w if $\ell(\sigma_\alpha w) = \ell(w) + 1$ and $w^{-1}\alpha \in \Pi$.

Example 2.12. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \ldots, s_{r+1})$, then the sorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ with $i < j$ and $s_i > s_j$, and the desorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ with $i < j$ and $s_i < s_j$. This example motivates the usage of the words "sorting" and "desorting".

We will also need to consider two different kinds of orders on Δ. First, there is the standard order $\prec$ on Δ: we say that $\alpha \prec \beta$ if $\beta - \alpha$ is a sum of positive roots. Additionally, for each $w \in W$ we will need an order we will denote by $\prec_w$: we say that $\alpha \prec_w \beta$ if $w^{-1}\alpha \prec w^{-1}\beta$.

Remark 2.13. *If $\alpha, \beta \in \Delta$ and $(\alpha, \beta) = 1$, then, by Lemma 2.5, α and β are comparable for $\prec$ and for the orders $\prec_w$ for all $w \in W$.*

Definition 2.14. Let v be a linear combination of roots, $v = \sum a_i \alpha_i$. The set of simple roots α_i such that $a_i \neq 0$ is called the *support* of v (notation: $\operatorname{supp} v$).

Lemma 2.15. *Let $w \in W$.*
If $\alpha, \beta, \gamma \in w\Delta^-$ and $(\alpha, \beta) = 1$, $(\beta, \gamma) = 1$, $(\alpha, \gamma) = 0$, and ($\alpha \prec_w \beta$ or $\gamma \prec_w \beta$), then $\delta = \alpha - \beta + \gamma \in w\Delta^-$

Proof. Without loss of generality, $\alpha \prec_w \beta$.
By Lemma 2.5, $\alpha - \beta \in \Delta$. $\alpha \prec_w \beta$, so $\alpha - \beta \in w\Delta^-$.
By Lemma 2.7, $\delta = \alpha - \beta + \gamma \in \Delta$. $\alpha - \beta \in w\Delta^-$ and $\gamma \in w\Delta^-$, so $\delta \in w\Delta^-$. □

3 Sorting

Lemma 3.1. *Let $\alpha \in \Delta^+$, and $\beta \in \Delta$. Suppose that $(\alpha, \beta) = 1$. σ_α interchanges β with another simple root, which we denote by γ.*
Then there are exactly three possibilities:

(i) $\beta, \gamma \in \Delta^+$.

(ii) $\beta \in \Delta^+$, $\gamma \in \Delta^-$.

(iii) $\beta, \gamma \in \Delta^-$.

Proof. The only remaining case is $\beta \in \Delta^-$, $\gamma \in \Delta^+$. Let us check that this is impossible. Note that $\beta = \alpha + \gamma$. So, if $\alpha \in \Delta^+$, $\gamma \in \Delta^+$, then $\beta = \alpha + \gamma \in \Delta^+$, a contradiction. □

Lemma 3.2. *Let $w \in W$, $\alpha \in \Delta^+$, and $\beta \in \Delta$. Suppose that $(\alpha, \beta) = 1$. σ_α interchanges β with another simple root, which we denote by γ.*
Then there are exactly three possibilities:

1. $\alpha \in w\Delta^-$, $\beta \in \Delta^+$, $\gamma \in \Delta^-$, $\beta \in w\Delta^-$, $\gamma \in w\Delta^+$.
   Then $\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, *and* $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| > |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

2. $\alpha \in w\Delta^+$, $\beta \in \Delta^+$, $\gamma \in \Delta^-$, $\beta \in w\Delta^+$, $\gamma \in w\Delta^-$.
   Then $\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-) = \varnothing$, $\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$, *and* $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| < |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

3. *Otherwise,* $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| = |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. *More precisely:*

(a) *If* $\alpha \in w\Delta^-$, $\beta \in \Delta^+$, $\gamma \in \Delta^+$, $\beta \in w\Delta^-$, *and* $\gamma \in w\Delta^+$, *then* $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\gamma\}$,

(b) *If* $\alpha \in w\Delta^+$, $\beta \in \Delta^+$, $\gamma \in \Delta^+$, $\beta \in w\Delta^+$, *and* $\gamma \in w\Delta^-$, *then* $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\gamma\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$,

(c) *Otherwise,* $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)$.

Proof. Note that $(\alpha,\gamma) = -1$, and $\beta = \alpha + \gamma$.

Note also that $\beta \in w\Delta^-$ if and only if $\gamma \in \sigma_\alpha w\Delta^-$, and $\gamma \in w\Delta^-$ if and only if $\beta \in \sigma_\alpha w\Delta^-$.

Let us consider the 3 cases from Lemma 3.1:

(i) $\beta,\gamma \in \Delta^+$.

Then $\beta \in \Delta^+ \cap w\Delta^-$ if and only if $\gamma \in \Delta^+ \cap \sigma_\alpha w\Delta^-$, and $\gamma \in \Delta^+ \cap w\Delta^-$ if and only if $\beta \in \Delta^+ \cap \sigma_\alpha w\Delta^-$. Therefore, $|\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-)| = |\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

If $\beta,\gamma \in w\Delta^-$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta,\gamma\}$, and 3c is true.

If $\beta,\gamma \in w\Delta^+$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 3c is true.

If $\beta \in w\Delta^+$ and $\gamma \in w\Delta^-$, then α must be in $w\Delta^+$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^-$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\gamma\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$, and 3b is true.

If $\beta \in w\Delta^-$ and $\gamma \in w\Delta^+$, then α must be in $w\Delta^-$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^+$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\gamma\}$, and 3a is true.

(ii) $\beta \in \Delta^+$, $\gamma \in \Delta^-$.

If $\beta,\gamma \in w\Delta^-$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$, and 3c is true.

If $\beta,\gamma \in w\Delta^+$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 3c is true.

If $\beta \in w\Delta^+$ and $\gamma \in w\Delta^-$, then α must be in $w\Delta^+$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^-$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \varnothing$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\gamma\}$, and 2 is true.

If $\beta \in w\Delta^-$ and $\gamma \in w\Delta^+$, then α must be in $w\Delta^-$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^+$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 1 is true.

(iii) $\beta,\gamma \in \Delta^-$.

Then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 3c is true.

□

Lemma 3.3. *Let $w \in W$ and let $\alpha \in \Delta^+$. Then:*

σ_α is a sorting reflection for w if and only if $\alpha \in \Delta^+ \cap w\Delta^-$. Otherwise, σ_α is a desorting reflection for w.

Proof. The reflection σ_α acting on Δ has some fixed points (they are precisely the roots orthogonal to α), and the other roots can be split into pairs (β,γ) such that σ_α interchanges β and γ ($(\alpha,-\alpha)$ is one of such pairs).

Consider a pair (β,γ) such that σ_α interchanges β and γ. Suppose also that $\beta \neq \pm\alpha$. Then, since the Dynkin diagram is simply laced, $(\alpha,\beta) = \pm 1$. Without loss of generality, let us assume that $(\alpha,\beta) = 1$. Then $(\alpha,\gamma) = -1$, and $\beta = \alpha + \gamma$.

Suppose first that $\alpha \in w\Delta^-$. Then, in the classification of Lemma 3.2, case 2 is impossible, since it requires $\alpha \in \Delta^+$. And in both of the other cases, we have $|\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-)| \geq |\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

END Suppose first that $\alpha \in w\Delta^-$.

Now suppose that $\alpha \in w\Delta^+$. Then, in the classification of Lemma 3.2, case 1 is impossible, since it requires $\alpha \in \Delta^-$. And in both of the other cases, we have $|\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

END Now suppose that $\alpha \in w\Delta^+$.

END Consider a pair (β,γ)

So, we can conclude that if $\alpha \in w\Delta^-$, then for every pair (β, γ) such that σ_α interchanges β and γ, and $\beta \neq \pm\alpha$, we have $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \geq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. Also, if $\alpha \in w\Delta^-$, then $\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-) = \{\alpha\}$, $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and $|\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-)| > |\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. The summation over all orbits of σ_α in Δ gives us $|(\Delta^+ \cap w\Delta^-)| > |(\Delta^+ \cap \sigma_\alpha w\Delta^-)|$ if $\alpha \in w\Delta^-$.

And we can also conclude that if $\alpha \in w\Delta^+$, then for every pair (β, γ) such that σ_α interchanges β and γ, and $\beta \neq \pm\alpha$, we have $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. Also, if $\alpha \in w\Delta^+$, then $\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-) = \varnothing$, $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\alpha\}$, and $|\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-)| < |\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. The summation over all orbits of σ_α in Δ gives us $|(\Delta^+ \cap w\Delta^-)| < |(\Delta^+ \cap \sigma_\alpha w\Delta^-)|$ if $\alpha \in w\Delta^+$. □

Lemma 3.4. *Let $w \in W$ and $\alpha \in \Delta^+ \cap w\Delta^-$.*

Then σ_α is an admissible sorting reflection for w if and only if it is impossible to find roots $\beta, \delta \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \delta$.

Proof. Again, note that $\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-) = \{\alpha\}$, $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and $|\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-)| > |\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

Also note again that if (β, γ) is a pair such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$, then case 2 in Lemma 3.2 is not possible since it requires $\alpha \in w\Delta^+$, and $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

So, the summation over all orbits of σ_α on Δ tells us that $|(\Delta^+ \cap w\Delta^-)| = |(\Delta^+ \cap \sigma_\alpha w\Delta^-)| + 1$ if and only if all inequalities

$|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$ for all pairs (β, γ) such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$,

become equalities.

And all these inequalities become equalities if and only if case 1 does not occur for any pair (β, γ) such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$. In other words, $\ell(w) = \ell(\sigma_\alpha w) + 1$ if and only if there are no pairs (β, γ) such that

σ_α interchanges β and γ, $(\alpha, \beta) = 1$, $\beta \in \Delta^+$, $\gamma \in \Delta^-$, $\beta \in w\Delta^-$, $\gamma \in w\Delta^+$.

And if we denote $\delta = -\gamma$, then we see that the non-existence of such pairs is equivalent to the non-existence of pairs (β, δ) such that

$\alpha = \beta + \delta$, $(\alpha, \beta) = 1$, $\beta \in \Delta^+$, $\delta \in \Delta^+$, $\beta \in w\Delta^-$, $\delta \in w\Delta^-$.

Finally, note that by Lemma 2.5, if $\beta, \delta, \beta + \delta \in \Delta^+$, then automatically $(\beta, \delta) = -1$. □

Example 3.5. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \ldots, s_{r+1})$, then the admissible sorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ such that $i < j$, $s_i > s_j$, and there are no indices k such that $i < j < k$ and $s_i > s_k > s_j$.

Lemma 3.6. *Let $w \in W$ and $\alpha \in \Delta^+ \cap w\Delta^+$.*

Then σ_α is an admissible desorting reflection for w if and only if it is impossible to find roots $\beta, \delta \in \Delta^+ \cap w\Delta^+$ such that $\alpha = \beta + \delta$.

Proof. Again, note that $\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-) = \varnothing$, $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\alpha\}$, and $|\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-)| < |\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

Also note that if (β, γ) is a pair such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$, then case 1 in Lemma 3.2 is not possible since it requires $\alpha \in w\Delta^-$, so $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \geq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

So, the summation over all orbits of σ_α on Δ tells us that $|(\Delta^+ \cap w\Delta^-)| = |(\Delta^+ \cap \sigma_\alpha w\Delta^-)| + 1$ if and only if all inequalities

$|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \geq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$ for all pairs (β, γ) such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$, become equalities.

And all these inequalities become equalities if and only if case 2 does not occur for any pair (β,γ) such that σ_α interchanges β and γ and $\beta\neq\pm\alpha$. In other words, $\ell(w)=\ell(\sigma_\alpha w)+1$ if and only if there are no pairs (β,γ) such that

$$\sigma_\alpha \text{ interchanges } \beta \text{ and } \gamma,\ (\alpha,\beta)=1,\ \beta\in\Delta^+,\ \gamma\in\Delta^-,\ \beta\in w\Delta^+,\ \gamma\in w\Delta^-.$$

And if we denote $\delta=-\gamma$, then we see that the non-existence of such pairs is equivalent to the non-existence of pairs (β,δ) such that

$$\alpha=\beta+\delta,\ (\alpha,\beta)=1,\ \beta\in\Delta^+,\ \delta\in\Delta^+,\ \beta\in w\Delta^+,\ \delta\in w\Delta^+.$$

Finally, note that by Lemma 2.5, if $\beta,\delta,\beta+\delta\in\Delta^+$, then automatically $(\beta,\delta)=-1$. ☐

Lemma 3.7. *Let $w\in W$ and $\alpha\in\Delta^+\cap w\Delta^-$. Suppose that σ_α is an admissible sorting reflection. Then the set $\Delta^+\cap\sigma_\alpha w\Delta^-$ can be obtained from the set $\Delta^+\cap w\Delta^-$ by the following procedure:*

For each $\beta\in\Delta^+\cap w\Delta^-$:

1. *If $\beta=\alpha$, don't put anything into $\Delta^+\cap\sigma_\alpha w\Delta^-$.*
2. *If $(\alpha,\beta)=1$, $\alpha\prec\beta$, and $\beta-\alpha\notin\Delta^+\cap w\Delta^-$, then put $\beta-\alpha$ into $\Delta^+\cap\sigma_\alpha w\Delta^-$.*
3. *Otherwise, put β into $\Delta^+\cap\sigma_\alpha w\Delta^-$.*

Note that this lemma in fact establishes a bijection between $(\Delta^+\cap w\Delta^-)\setminus\alpha$ and $\Delta^+\cap\sigma_\alpha w\Delta^-$.

Proof. Let us check that for every orbit of σ_α on Δ, the above procedure gives the correct intersection of this orbit with $\Delta^+\cap\sigma_\alpha w\Delta^-$. See Corollary 2.6 for the list of orbits.

If the orbit consists of one root, β, then $(\alpha,\beta)=0$. We apply case 3 of the procedure, and indeed, $\{\beta\}\cap(\Delta^+\cap w\Delta^-)=\{\beta\}\cap(\Delta^+\cap\sigma_\alpha w\Delta^-)$ since $\sigma_\alpha\beta=\beta$.

If the orbit is $\alpha,-\alpha$, then we apply case 1 of the procedure. And indeed, it is clear that $\{\alpha,-\alpha\}\cap(\Delta^+\cap\sigma_\alpha w\Delta^-)=\varnothing$.

Finally, consider an orbit $\{\beta,\gamma\}$, where $(\alpha,\beta)=1$, $(\alpha,\gamma)=-1$, and $\beta=\alpha+\gamma$. Lemma 3.2 gives us 5 possibilities in total, among them:

Case 1 is prohibited by Lemma 3.4 (if case 1 was true, then we would have $\beta\in\Delta^+\cap w\Delta^-$, $-\gamma\in\Delta^+\cap w\Delta^-$, and $\alpha=\beta+(-\gamma)$).

Case 2 is impossible since $\alpha\in w\Delta^-$.

If case 3a of Lemma 3.2 holds, then $\alpha,\beta\in\Delta^+\cap w\Delta^-$. Also, $\gamma\in\Delta^+$, $\gamma=\beta-\alpha$, so $\alpha\prec\beta$. Finally, $\gamma\notin w\Delta^-$, so the conditions of case 2 are satisfied. By Lemma 3.2, $\{\beta,\gamma\}\cap(\Delta^+\cap w\Delta^-)=\{\beta\}$, $\{\beta,\gamma\}\cap(\Delta^+\cap\sigma_\alpha w\Delta^-)=\{\gamma\}$, and indeed, case 2 tells us that we should put $\gamma=\beta-\alpha$ into $(\Delta^+\cap\sigma_\alpha w\Delta^-)$ instead of β.

Case 3b is impossible since $\alpha\in w\Delta^-$.

Finally, suppose that case 3c of Lemma 3.2 holds. Let us check that the conditions of case 2 of the procedure are not satisfied (and the procedure tells us that we should use case 3).

Clearly, the conditions of case 2 of the procedure are not satisfied for γ since $(\alpha,\gamma)=-1$

Assume the contrary, assume that the conditions of case 2 are satisfied for β. $\alpha\in\Delta^+$, $\alpha\in w\Delta^-$, $\beta\in\Delta^+$, $\beta\in w\Delta^-$. Since $\beta\prec\alpha$, $\gamma=\beta-\alpha\in\Delta^+$. Since $\beta-\alpha\notin\Delta^+\cap w\Delta^-$, $\gamma\in w\Delta^+$. So, case 3a of Lemma 3.2 holds, and we have assumed that case 3c of Lemma 3.2 holds. A contradiction.

END Assume the contrary.

So, the procedure tells us that we should use case 3 and put all roots from $\{\beta,\gamma\}\cap(\Delta^+\cap w\Delta^-)$ into $\Delta^+\cap\sigma_\alpha w\Delta^-$. And this is correct since by case 3c of Lemma 3.2, $\{\beta,\gamma\}\cap(\Delta^+\cap w\Delta^-)=\{\beta,\gamma\}\cap(\Delta^+\cap\sigma_\alpha w\Delta^-)$. ☐

Lemma 3.8. *If $w\in W$, $\alpha\in\Delta^+\cap w\Delta^-$, and $w^{-1}\alpha\in-\Pi$, then σ_α is an antisimple sorting reflection.*

Proof. The only thing we have to check is that σ_α is an admissible sorting reflection. We use Lemma 3.4. Assume that there are roots $\beta,\gamma\in\Delta^+\cap w\Delta^-$ such that $\alpha=\beta+\gamma$. But then $-w^{-1}\alpha=(-w^{-1}\beta)+(-w^{-1}\gamma)$, $-w^{-1}\alpha\in\Pi$, and $-w^{-1}\beta,-w^{-1}\gamma\in\Delta^+$, a contradiction. ☐

Example 3.9. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \ldots, s_{r+1})$, then the antisimple sorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ such that $i < j$ and $s_i = s_j + 1$.

Lemma 3.10. *Let $w \in W$, $\alpha \in \Delta^+ \cap w\Delta^-$.*

The following conditions are equivalent:

1. $w^{-1}\alpha \in -\Pi$
2. *α is a maximal element of the set $\Delta^+ \cap w\Delta^-$ with respect to the order $\prec_w$.*
3. *It is impossible to find roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$ and It is impossible to find a root $\beta \in \Delta^+ \cap w\Delta^-$ such that: $\alpha \prec \beta$, $(\alpha, \beta) = 1$, $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$.*

Proof. $1 \Rightarrow 2$

Let $\alpha \in \Delta^+ \cap w\Delta^-$, $w^{-1}\alpha \in -\Pi$. Assume that $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec_w \beta$. Then, by the definition of $\prec_w$, $w^{-1}\alpha \prec w^{-1}\beta$. But $w^{-1}\beta \in \Delta^-$, $w^{-1}\alpha \in -\Pi$, a contradiction.

$2 \Rightarrow 3$

Let α be a maximal element of $\Delta^+ \cap w\Delta^-$ with respect to $\prec_w$.

If there exist $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$, then $-w^{-1}\gamma = w^{\beta} - w^{-1}\alpha \in \Delta^-$, so $\alpha \prec_w \beta$, a contradiction.

If there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that: $\alpha \prec \beta$, $(\alpha, \beta) = 1$, $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, then:

$\alpha \prec \beta$, $(\alpha, \beta) = 1$, so $\beta - \alpha \in \Delta^+$.

$\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so $\beta - \alpha \notin w\Delta^-$, $\beta - \alpha \in w\Delta^+$.

Again, $\alpha \prec_w \beta$, a contradiction with maximality of α.

$3 \Rightarrow 1$

Assume that $w^{-1}\alpha \notin -\Pi$. Then, since $w^{-1}\alpha \in \Delta^-$, it is possible to decompose $w^{-1}\alpha = \beta' + \gamma'$, where $\beta', \gamma' \in \Delta^-$. We have $w\beta' + w\gamma' = \alpha$. $w\beta'$ and $w\gamma'$ cannot be both negative, since their sum, α, is positive. At least one of the roots $w\beta'$ and $w\gamma'$ is positive, let us assume without loss of generality that $w\beta' \in \Delta^+$.

Set $\beta = w\beta'$, $\gamma = w\gamma'$.

If $\gamma = \alpha - \beta \in \Delta^-$, then $\beta \prec \alpha$ by definition, $(\beta, \alpha) = 1$ by Lemma 2.5, and $\beta - \alpha = -\gamma = w(-\gamma') \in w\Delta^+$, so $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$.

If $\gamma \in \Delta^+$, then $\beta, \gamma \in \Delta + \cap w\Delta^-$ and $\alpha = \beta + \gamma$. □

Corollary 3.11. *For every $w \in W$, $w \neq \mathrm{id}$, there exists at least one $\alpha \in \Delta^+ \cap w\Delta^-$ such that σ_α is an antisimple sorting reflection for w.* □

Corollary 3.12. *Let $w \in W$, $\alpha_i \in \Pi$. If there exists $\alpha \in \Delta^+ \cap w\Delta^-$ such that $\alpha_i \in \operatorname{supp} \alpha$, then there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha_i \in \operatorname{supp} \beta$ and σ_β is an antisimple sorting reflection.*

Proof. Consider the set A of all elements of $\Delta^+ \cap w\Delta^-$ whose support contains α_i. This set is nonempty since it contains α. Let β be a $\prec_w$-maximal element of A.

Assume that $w^{-1}\beta \notin -\Pi$. Then by Lemma 3.10, one of the two statements is true: Either there exists roots $\gamma, \delta \in \Delta^+ \cap w\Delta^-$ such that $\beta = \gamma + \delta$, or there exists $\gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta \prec \gamma$, $(\beta, \gamma) = 1$, and $\gamma - \beta \notin \Delta^+ \cap w\Delta^-$.

If there exists roots $\gamma, \delta \in \Delta^+ \cap w\Delta^-$ such that $\beta = \gamma + \delta$, then $\operatorname{supp} \beta = \operatorname{supp} \gamma \cup \operatorname{supp} \delta$, so ($\alpha_i \in \operatorname{supp} \gamma$ or $\alpha_i \in \operatorname{supp} \delta$). Without loss of generality, $\alpha_i \in \operatorname{supp} \gamma$. Then $\gamma \in A$, We have $\delta \in w\Delta^-$, so $w^{-1}\delta \in \Delta^-$, and $w^{-1}\gamma - w^{-1}\beta = -w^{-1}\delta \in \Delta^+$, so $\beta \prec_w \gamma$. A contradiction with the $\prec_w$-maximality of β.

If there exists $\gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta \prec \gamma$, $(\beta, \gamma) = 1$, and $\gamma - \beta \notin \Delta^+ \cap w\Delta^-$, then $\gamma - \beta \in \Delta$ by Lemma 2.5, $\gamma - \beta \in \Delta^+$ since $\beta \prec \gamma$, but $\gamma - \beta \notin \Delta^+ \cap w\Delta^-$, so $\gamma - \beta \notin w\Delta^-$, and $\gamma - \beta \in w\Delta^+$. Then $\beta \prec_w \gamma$.

$\beta \prec \gamma$, so $\operatorname{supp} \beta \subseteq \operatorname{supp} \gamma$, and $\alpha_i \in \operatorname{supp} \gamma$. Therefore, $\gamma \in A$. A contradiction with the $\prec_w$-maximality of β. □

The following lemma illustrates an advantage of antisimple sorting reflections.

Lemma 3.13. *Let $w \in W$. If $\alpha \in \Delta^+ \cap w\Delta^-$ is such that σ_α is an antisimple sorting reflection, then $\Delta^+ \cap \sigma_\alpha w\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \alpha$.*

Proof. We use Lemma 3.7. We have to check that case 2 never occurs.

Assume that case 2 occurs for some $\beta \in \Delta^+ \cap w\Delta^-$. This means that $\gamma = \beta - \alpha \in w\Delta^+$, $w^{-1}\gamma = w^{-1}\beta - w^{-1}\alpha \in \Delta^+$, and $\alpha \prec_w \beta$. But then α is not a maximal element of $\Delta^+ \cap w\Delta^-$ with respect to $\prec_w$, a contradiction with Lemma 3.10. □

To use Chevalley-Pieri formula, we will use the following terminology.

Definition 3.14. Let $w \in W$, $n = \ell(w)$. We say that *a process of sorting of* w is a sequence of roots $\beta_1, \dots, \beta_n$ such that:

1. $w = \sigma_{\beta_1}\sigma_{\beta_2}\dots\sigma_{\beta_n}$.
2. Denote $w_i = \sigma_{\beta_i}\dots\sigma_{\beta_1}w = \sigma_{\beta_{i+1}}\dots\sigma_{\beta_n}$ $(0 \le i \le n)$. Then for each i, $0 \le i < n$, $\sigma_{\beta_{i+1}}$ has to be an admissible sorting reflection for w_i. In other words, $\ell(w_i)$ has to be $n - i$ for $0 \le i \le n$.

We say that the ith *step* $(1 \le i \le n)$ of the sorting process is the reflection σ_{β_i}, and that the *current* element of W after the ith step of the process (before the $(i+1)$st step of the process) is $w_i = \sigma_{\beta_i}\dots\sigma_{\beta_1}w = \sigma_{\beta_{i+1}}\dots\sigma_{\beta_n}$.

We say that the sorting process is *antireduced*, and the equality $w = \sigma_{\beta_1}\sigma_{\beta_2}\dots\sigma_{\beta_n}$ is an *antireduced expression for* w, if $\sigma\beta_i$ is an antisimple reflection for w_{i-1} for all i, $1 \le i \le n$.

If we only know for some i, $1 \le i \le n$, that $\sigma\beta_i$ is an antisimple reflection for w_{i-1}, we will say that *the ith step of the sorting process is antisimple.*

Definition 3.15. Let $w \in W$, $n = \ell(w)$. Similarly, we say that *a sorting process prefix of* w is a sequence of roots $\beta_1, \dots, \beta_k$ $(k \le n)$ such that:

> Denote $w_i = \sigma_{\beta_i}\dots\sigma_{\beta_1}w$ $(0 \le i \le k)$. Then for each i, $0 \le i < k$, $\sigma_{\beta_{i+1}}$ has to be an admissible sorting reflection for w_i. In other words, $\ell(w_i)$ has to be $n - i$ for $0 \le i \le k$.

We say that the sorting process prefix is *antireduced*, if $\sigma\beta_i$ is an antisimple reflection for w_{i-1} for all i, $1 \le i \le k$.

Lemma 3.16. *If $\beta_1, \dots, \beta_n$ is an antireduced sorting process (resp. antireduced sorting process prefix) for $w \in W$, then $\{\beta_1, \dots, \beta_n\} = \Delta^+ \cap w\Delta^-$ (resp. $\{\beta_1, \dots, \beta_n\} \subseteq \Delta^+ \cap w\Delta^-$).*

Moreover, if $\beta_1, \dots, \beta_k$ is an antireduced sorting process prefix for $w \in W$, and $w_k = \sigma_{\beta_k}\dots\sigma_{\beta_1}w$, then $\Delta^+ \cap w_k\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \{\beta_1, \dots, \beta_k\}$.

Proof. This follows directly from Lemma 3.13 and the definition of an antisimple sorting process. □

Corollary 3.17. *If $\beta_1, \dots, \beta_n$ is an antireduced sorting process prefix (including an antireduced sorting process) for $w \in W$, then there are no coinciding roots among $\beta_1, \dots, \beta_n$.* □

Example 3.18. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \dots, s_{r+1})$, and we have a sorting process of w, then the sequence of the current elements of W is a sequence of $(r+1)$-tuples ("arrays") of numbers, where each next $(r+1)$-tuple is obtained from the previous one by interchanging two numbers so that this interchange is an admissible sorting reflection (see Example 3.5). In the end, our $(r+1)$-tuple has to become $(1, 2, \dots, r+1)$.

Such a sorting process is antireduced if at each step we actually interchange a number i with $i+1$, and $i+1$ has to be located to the left of i immediately before this interchange.

(Remark about relation to programming, we will not need it later: An antireduced sorting process is *not* what is called "bubble sorting" in programming. Bubble sorting can be obtained from a certain *reduced* expression for w (but not from any reduced expression, only from a certain one)).

Definition 3.19. Given a set of positive roots $A \subseteq \Delta^+$ we call a function $f\colon A \to \Pi$ a *distribution of simple roots* on A if $f(\alpha) \in \operatorname{supp} \alpha$ for each $\alpha \in A$

For a given simple root α_i, the number of roots $\alpha \in A$ such that $f(\alpha) = \alpha_i$ is called the *D-multiplicity* of α_i in the distribution.

If we have a distribution with $f(\alpha) = \alpha_i$, we say that the distribution *assigns* the simple root α_i to α.

Definition 3.20. Given a list of positive roots $\beta_1, \ldots, \beta_n$, i. e. order matters, multiple occurrences allowed, we call a function $f\colon \{1, \ldots, n\} \to \Pi$ a *distribution of simple roots* on $\beta_1, \ldots, \beta_n$ if $f(k) \in \operatorname{supp} \beta_k$ for each k, $1 \le k \le n$.

Sometimes we will treat this function as a list (an n-tuple) of its values: $f(1), \ldots, f(k)$. This is convenient, for example, if we want to remove some roots from the list $\beta_1, \ldots, \beta_n$, and at the same time remove the corresponding simple roots from the list $f(1), \ldots, f(k)$.

For a given simple root α_i, the number of indices k, $1 \le k \le n$ such that $f(k) = \alpha_i$ is called the *D-multiplicity* of α_i in the distribution.

If we have a distribution with $f(k) = \alpha_i$, we say that the distribution *assigns* the simple root α_i to the kth root in the list, β_k.

If we need to know the D-multiplicities of all simple roots in a distribution, we briefly say "a distribution with D-multiplicities $n_1, \ldots, n_r$" instead of "a distribution with D-multiplicities $n_1, \ldots, n_r$ of simple roots $\alpha_1, \ldots, \alpha_r$, respectively".

Definition 3.21. We call a tuple $w, n_1, \ldots, n_r$, where $w \in W$, $n_i \in \mathbb{Z}_{\ge 0}$, $n_1 + \ldots + n_r = \ell(w)$, a *configuration of D-multiplicities*.

Definition 3.22. Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. We say that a simple root α_i is *involved* into this configuration if $n_i > 0$.

Definition 3.23. Let $w \in W$. We say that a *labeled sorting process* of w is a sorting process $\beta_1, \ldots, \beta_n$ of w with the following additional information:

We have a simple root distribution on the list $\beta_1, \ldots, \beta_n$.

This distribution will be called the *distribution of labels*, or the *list of labels*, of the labeled sorting process. The simple root it assigns to β_k will be called the *label* at β_k.

In other words, when, at a certain (kth) step of the sorting process, we perform an admissible sorting reflection along a root (β_k), we assign to this step a label, which is a simple root from $\operatorname{supp} \beta_k$.

Note that the distribution of labels is actually a function from $\{1, \ldots, n\}$ to Π (i. e. just an n-tuple of simple roots), so it makes sense, for example, to speak about "two different labeled sorting processes with the same distribution of labels".

Instead of "a labeled sorting process of w with distribution of labels that has D-multiplicities $n_1, \ldots, n_r$ of simple roots", we briefly say "*a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels*".

Definition 3.24. Let $w \in W$. Let $\beta_1, \ldots, \beta_n$ be a labeled sorting process of w with distribution of labels f.

Since $f(k) \in \operatorname{supp} \beta_i$, $f(k)$ is present in the decomposition of β_i into a linear combination of simple roots. Let a_i be the coefficient in front of $f(k)$ in this linear combination.

The *X-multiplicity* of the sorting process (not to be confused with the D-multiplicity of a simple root in a list of simple roots) is the product $a_1 \ldots a_n$.

Definition 3.25. Let $w \in W$. We say that a *labeled sorting process prefix* of w is a sorting process prefix $\beta_1, \ldots, \beta_k$ of w with the following additional information:

We have a simple root distribution on the list $\beta_1, \ldots, \beta_k$.

Instead of "a labeled sorting process prefix of w with distribution of labels that has D-multiplicities $m_1, \ldots, m_r$ of simple roots", we briefly say "*a labeled sorting process prefix of w with D-multiplicities $m_1, \ldots, m_r$ of labels*".

Lemma 3.26. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

$C_{w,n_1,\ldots,n_r}$, the coefficient in front of Z_w in the decomposition of $D_1^{n_1} \ldots D_r^{n_r}$ into a linear combination of Schubert classes, can be computed as follows.

Choose any function $f\colon \{1, \ldots, \ell(w)\} \to \Pi$ that takes each value α_j exactly n_j times for all j, $1 \le j \le r$.

Then $C_{w,n_1,\ldots,n_r}$ is the number of [labeled sorting processes of w with the distribution of labels f], counting their X-multiplicities.

Proof. Induction on $\ell(w)$. For $\ell(w) = 0$, this is clear.

If $w \ne \mathrm{id}$, denote by $\gamma_1, \ldots, \gamma_m$ all of the roots from $\Delta^+ \cap w\Delta^-$ such that σ_{γ_j} is an admissible reflection for w. Also denote by g_j the coefficient in front of $f(1)$ in the decomposition of γ_j into a linear combination of simple roots. (Note that g_j may be 0.)

Then the set of all labeled sorting processes of w with distribution of labels f is split into the disjoint union of m subsets: the sorting processes starting with γ_1, …, the sorting processes starting with γ_m.

If we remove the first root (let it be γ_j) and its label $f(1)$ from a labeled sorting process of w, we will get a sorting process of $\sigma_{\gamma_j} w$ with list of labels $f(2), \ldots, f(\ell(w))$. And the X-multiplicity of this sorting process of w equals g_j times the X-multiplicity of this sorting process of $\sigma_{\gamma_j} w$.

So, using the induction hypothesis, it suffices to prove that

$$C_{w,n_1,\ldots,n_r} = \sum_{j=1}^{m} g_j C_{\sigma_{\gamma_j} w, n_1, \ldots, n_{i_1}-1, n_r}.$$

By the definition of $C_{v,n_1,\ldots,n_{i_1}-1,n_r}$, we have

$$D_{\varpi_1}^{n_1} \ldots D_{\varpi_{i_1}}^{n_{i_1}-1} \ldots D_{\varpi_r}^{n_r} = \sum_{v \in W : \ell(v) = \ell(w)-1} C_{v,n_1,\ldots,n_{i_1}-1,\ldots,n_r} Z_v.$$

Proposition 2.1 applied to each Z_v occurring on the right gives:

$$D_{\varpi_i} Z_v = \sum_{\substack{\alpha \in \Delta^+ \\ \ell(\sigma_\alpha v) = \ell(v)+1}} \varpi_i(\alpha) Z_{\sigma_\alpha v}.$$

Z_w appears on the right-hand side if and only if $\sigma_\alpha v = w$ for some $\alpha \in \Delta^+$, i. e. $v = \sigma_\alpha w$ for some $\alpha \in \Delta^+$. Since $\ell(v) = \ell(w) - 1$, the equality $v = \sigma_\alpha w$ implies that σ_α is an admissible reflection for w, and $\alpha = \gamma_j$ for some j. The coefficient in front of this Z_w in the Pieri formula is $\varpi_i(\gamma_j) = g_j$.

Now let us take the linear combination of all Pieri formulas we wrote for all Z_vs with coefficients $C_{v,n_1,\ldots,n_{i_1}-1,\ldots,n_r}$.

On the left, we will get $D_{\varpi_1}^{n_1} \ldots D_{\varpi_{i_1}}^{n_{i_1}} \ldots D_{\varpi_r}^{n_r}$.

On the right, we will get a linear combination of Schubert classes with some coefficients, and the coefficient in front of Z_w will be $\sum_j g_j C_{\sigma_{\gamma_j} w, n_1, \ldots, n_{i_1}-1, n_r}$. But this coefficient also equals $C_{w,n_1,\ldots,n_r}$. □

Corollary 3.27. *Given $w \in W$, the number of labeled sorting processes with a distribution of labels f counting the X-multiplicities of processes, depends only on the D-multiplicities of simple roots in the distribution f, but not on the distribution f itself itself.* □

Lemma 3.28. *For each $w \in W$, there exists at least one antireduced sorting process.*

Proof. Induction on $\ell(w)$. Trivial for $w = \mathrm{id}$.

By Corollary 3.11, there exists a root $\beta_1 \in \Delta^+ \cap w\Delta^-$ such that σ_{β_1} is an antisimple reflection for w.

Let us try to begin the sorting process with β_1. Set $w_1 = \sigma_{\beta_1} w$. $\ell(w_1) = \ell(w) - 1$. There exists an antireduced sorting process for w_1, denote it by $\beta_2, \ldots, \beta_n$. Then $\beta_1, \beta_2, \ldots, \beta_n$ is an antireduced sorting process for w, because the products $\sigma_{\beta_{k+1}} \ldots \sigma_{\beta_n}$ occurring in the definitions of antireduced sorting processes for w and for w_1 are exactly the same (with the addition of w itself to the sorting process of w, but we have checked explicitly that σ_{β_1} is an antisimple reflection for w). □

4 Criterion of sortability

For each $A \subseteq \Delta^+$, for each $I \subseteq \{1,\dots,r\}$, denote by $R_I(A)$ the set of all roots $\alpha \in A$ such that $\operatorname{supp}\alpha$ contains at least one simple root α_i with $i \in I$. For each $w \in W$, for each $I \subseteq \{1,\dots,r\}$, we briefly write $R_I(w) = R_I(\Delta^+ \cap w\Delta^-)$.

Lemma 4.1. *Let $w \in W$.*

Let $I \subseteq \{1,\dots,r\}$.

Set $m = |R_I(w)|$.

Then there exists an antireduced sorting process prefix $\beta_1,\dots,\beta_m$ of w such that $R_I(w) = \{\beta_1,\dots,\beta_m\}$ (all roots β_i are different by Corollary 3.17).

Proof. Induction on m.

If $m = 0$, everything is clear (we take the empty list of roots).

If $m > 0$, then there exists $\alpha \in \Delta^+ \cap w\Delta^-$ and $i \in I$ such that $\alpha_i \in \operatorname{supp}\alpha$. By Corollary 3.12, there exists $\beta_1 \in \Delta^+ \cap w\Delta^-$ such that $\alpha_i \in \operatorname{supp}\beta_1$ and σ_{β_1} is an antisimple sorting reflection for w. $\alpha_i \in \operatorname{supp}\beta_1$, so $\beta_1 \in R_I(w)$.

Let us try to begin the sorting process prefix with β_1. Set $w_1 = \sigma_{\beta_1} w$. Then $\Delta^+ \cap w_1\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \beta_1$ by Lemma 3.13, so $R_I(w_1) = R_I(w) \setminus \beta_1$.

By induction hypothesis, there exists an antireduced sorting process prefix of w_1 (denote it by $\beta_2,\dots,\beta_m$) such that $R_I(w_1) = \{\beta_2,\dots,\beta_m\}$.

Then $\beta_1,\beta_2,\dots,\beta_n$ is an antireduced sorting process prefix for w, because the products $\sigma_{\beta_k}\dots\sigma_{\beta_2}\sigma_{\beta_1}w = \sigma_{\beta_k}\dots\sigma_{\beta_2}w_1$ occurring in the definitions of antireduced sorting processes for w and for w_1 are exactly the same (with the addition of w itself to the sorting process prefix of w, but we have checked explicitly that σ_{β_1} is an antisimple reflection for w).

We also know that $\beta_1 \in R_I(w)$, $R_I(w_1) = R_I(w) \setminus \beta_1$, and $R_I(w_1) = \{\beta_2,\dots,\beta_m\}$. Therefore, $R_I(w) = \{\beta_1,\beta_2,\dots,\beta_m\}$. ☐

Lemma 4.2. *Let $A \subseteq \Delta^+$, and let $n_1,\dots,n_r \in \mathbb{Z}_{\ge 0}$ be such that $n_1 + \dots + n_r = |A|$.*

Denote by J the set of indices i ($1 \le i \le r$) such that $n_i > 0$.

The following conditions are equivalent:

1. *For each $I \subseteq J$, $|R_I(A)| \ge \sum_{i\in I} n_i$.*
2. *There exists a simple root distribution on A with D-multiplicities $n_1,\dots,n_r$.*
3. *For each $I \subseteq \{1,\dots,r\}$, $|R_I(A)| \ge \sum_{i\in I} n_i$.*

Proof. Note that for each $I \subseteq \{1,\dots,r\}$, by definition of $R_I(A)$,

$$R_I(A) = \bigcup_{i\in I} R_{\{i\}}(A).$$

$1 \Rightarrow 2$

Condition 1 is equivalent to the hypothesis of generalized Hall representative lemma (Lemma 2.3) applied to the $|J|$ sets: $R_{\{j\}}(A)$ for each $j \in J$.

And Lemma 2.3 says that for each $j \in J$, we can choose n_j elements of $R_{\{j\}}(A)$, i. e. n_j roots $\alpha \in A$ such that $\alpha_j \in \operatorname{supp}\alpha$, and all chosen roots (for different values of j) are different. In total, we chose $\sum_{j\in J} n_j$ roots, and, by the definition of J, $\sum_{j\in J} n_j = n_1 + \dots + n_r = |A|$. So, each root from A was chosen exactly once, and we can set $f(\alpha) = \alpha_j$ if α was chosen as an element of $R_{\{j\}}(A)$. This is a simple root distribution on A, and it clearly has D-multiplicities $n_1,\dots,n_r$ of simple roots.

$2 \Rightarrow 3$

Let f be a simple root distribution. Then for each i, $1 \le i \le r$, $f^{-1}(\alpha_i) \subseteq R_{\{i\}}(A)$ and $n_i = |f^{-1}(\alpha_i)|$. So, for each $I \subseteq \{1,\dots,r\}$,

$$\bigcup_{i\in I} f^{-1}(\alpha_i) \subseteq R_I(A).$$

and

$$\sum_{i\in I} n_i = \left|\bigcup_{i\in I} f^{-1}(\alpha_i)\right|$$

Therefore, $\sum_{i\in I} n_i \leq |R_I(A)|$.

$3 \Rightarrow 1$

Follows directly. $\square$

Corollary 4.3. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

Denote by J the set of indices of involved roots, i. e. of indices i ($1 \leq i \leq r$) such that $n_i > 0$.

The following conditions are equivalent:

1. *For each $I \subseteq J$, $|R_I(w)| \geq \sum_{i\in I} n_i$.*
2. *There exists a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$.*
3. *For each $I \subseteq \{1, \ldots, r\}$, $|R_I(w)| \geq \sum_{i\in I} n_i$.* $\square$

Proposition 4.4. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

Then the following conditions are equivalent:

1. *There exists a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.*
2. *There exists a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$.*

If these conditions are satisfied, then there actually exists an antireduced *labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.*

Moreover, if there exists a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels that starts with $\beta \in \Delta^+$ with label $\alpha_i \in \Pi$, then $\beta \in \Delta^+ \cap w\Delta^-$ there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ such that $f(\beta) = \alpha_i$.

Proof. $1 \Rightarrow 2$. Induction on $\ell(w)$. Suppose that there exists a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.

It has to start with some admissible sorting reflection, and all admissible sorting reflections are reflections along some of the roots from $\Delta^+ \cap w\Delta^-$. Suppose that the sorting process starts with $\beta \in \Delta^+ \cap w\Delta^-$ (this is exactly the β from the "moreover" part), and the label assigned to the first step of the sorting process is α_i. Denote $w_1 = \sigma_\beta w$.

The rest of the labeled sorting process of w actually gives us a labeled sorting process of w_1 with D-multiplicities $n_1, \ldots, n_i - 1, \ldots, n_r$ of labels.

Recall that Lemma 3.7 establishes a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \beta$ and $\Delta^+ \cap w_1\Delta^-$. Denote this bijection by $\psi\colon (\Delta^+ \cap w\Delta^-) \setminus \beta \to \Delta^+ \cap w_1\Delta^-$

Lemma 3.7 says that either $\psi(\gamma) = \gamma$, or $\psi(\gamma) = \gamma - \beta$. In both cases, $\psi(\gamma) \preceq \gamma$.

By the induction hypothesis, there exists a simple root distribution on $\Delta^+ \cap w_1\Delta^-$ with D-multiplicities $n_1, \ldots, n_i - 1, \ldots, n_r$ of simple roots. Denote this distribution by $f_1\colon \Delta^+ \cap w_1\Delta^- \to \Pi$.

For each $\gamma \in (\Delta^+ \cap w\Delta^-) \setminus \beta$, since $\psi(\gamma) \preceq \gamma$ and $f_1(\psi(\gamma)) \in \operatorname{supp}\psi(\gamma)$, we have $f_1(\psi(\alpha)) \in \operatorname{supp}\gamma$. Also, $\alpha_i \in \operatorname{supp}\beta$. So, we can define the following simple root distribution f on $\Delta^+ \cap w\Delta^-$: $f(\beta = \alpha_i$, and $f(\gamma) = f_1(\psi(\gamma))$ for $\gamma \neq \beta$.

Note that this f satisfies the statement of the "moreover" part.

$2 \Rightarrow 1$

We are going to construct an antireduced labeled sorting process, then the last claim in the problem statement will be simultaneously proved.

By Lemma 3.28, there exists an antireduced sorting process of w. Denote the roots occurring in this sorting process by $\beta_1, \ldots, \beta_{\ell(w)}$ (in this order). By Lemma 3.16, the set of roots occurring in this antireduced sorting process is exactly $\Delta^+ \cap w\Delta^-$, i. e. $\Delta^+ \cap w\Delta^- = \{\beta_1, \ldots, \beta_{\ell(w)}\}$

We also know that there exists a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$, denote it by $f\colon \Delta^+ \cap w\Delta^- \to \Pi$. Let us assign label $f(\beta_k)$ to the k step of the sorting process, and we will get an antireduced labeled sorting process with D-multiplicities $n_1, \ldots, n_r$ of labels. $\square$

Corollary 4.5. *Let $w \in W$. Suppose we have a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$.*

Then there exists a labeled antireduced sorting process for w such that if at a certain step we make a reflection along $\alpha \in \Delta^+ \cap w\Delta^-$ (we make it only once, see Corollary 3.17), we assign the simple root $f(\alpha)$ to it.

In other words, since all roots occurring in an antireduced sorting process are different, to define a function on the set of occurring roots is equivalent to define a function on $\{1, \ldots, \ell(w)\}$. And the claim is that we can make the latter function, the distribution of labels of the labeled sorting process, the same as the former function, an arbitrary simple root distribution on $\Delta^+ \cap w\Delta^-$.

Proof. The proof exactly repeats the argument $2 \Rightarrow 1$ in the proof of Proposition 4.4. $\square$

Corollary 4.6. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

Then the following conditions are equivalent:

1. *There exists a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.*

2. *For each $I \subseteq \{1, \ldots, r\}$, $|R_I(w)| \geq \sum_{i\in I} n_i$.*

If these conditions are satisfied, then there actually exists an antireduced *labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.*

Proof. The claim follows from Corollary 4.3 and Proposition 4.4. $\square$

Definition 4.7. Let $w \in W$, and let $\alpha \in \Delta^+ \cap w\Delta^-$. A simple root distribution f on $\Delta^+ \cap w\Delta^-$ is called *α-compatible* if σ_α is an admissible sorting reflection for w, and the distribution has the following additional property:

If $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec \beta$, $(\alpha, \beta) = 1$, and $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, then $f(\beta) \notin \operatorname{supp} \alpha$.

Lemma 4.8. *Let $w \in W$, $\alpha \in \Delta^+ \cap w\Delta^-$. Let f be simple root distribution on $\Delta^+ \cap w\Delta^-$.*

The following conditions are equivalent:

1. *f is α-compatible*

2. *For each $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$ and $(\alpha, \beta) = 1$, we have $f(\beta) \notin \operatorname{supp} \alpha$.*

Proof. $1 \Rightarrow 2$

Assume that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$, $(\alpha, \beta) = 1$, and $f(\beta) \in \operatorname{supp} \alpha$. Set $\gamma = \alpha - \beta$ ($\gamma \in \Delta$ by Lemma 2.5). $\alpha \prec_w \beta$, so $\gamma \in w\Delta^-$.

If $\gamma \in \Delta^+$, then σ_α cannot be an admissible reflection for w by Lemma 3.4. If $\gamma \in \Delta^-$, then $\beta \prec \alpha$, and $-\gamma = \beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so we have a contradiction with the definition of α-compatibility.

$2 \Rightarrow 1$

Admissibility of σ_α: assume the contrary. By Lemma 3.4, there exist $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta + \gamma = \alpha$. By Lemma 2.5, $(\beta, \gamma) = -1$, so $(\alpha, \beta) = 1$. $-\gamma = \beta - \alpha \in w\Delta^+$, so $\alpha \prec_w \beta$. Also, $\gamma = \alpha - \beta \in \Delta^+$, so $\beta \prec \alpha$, and $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$. $f(\beta) \in \operatorname{supp} \beta$, so $f(\beta) \in \operatorname{supp} \alpha$, a contradiction.

Now suppose that $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec \beta$, $(\alpha, \beta) = 1$, and $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$.

$\alpha \prec \beta$ and $(\alpha, \beta) = 1$, so $\beta - \alpha \in \Delta^+$.

$\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so $\beta - \alpha \notin w\Delta^-$, so $\beta - \alpha \in w\Delta^+$, and $\alpha \prec_w \beta$.

Condition 2 in the Lemma statement says that $f(\beta) \notin \operatorname{supp} \alpha$, so the definition of α-compatibility holds. $\square$

Lemma 4.9. *Let $w \in W$, $\alpha \in \Delta^+ \cap w\Delta^-$. Let f be simple root distribution on $\Delta^+ \cap w\Delta^-$.*

The following conditions are equivalent:

1. *f is α-compatible*

2. *There are no roots $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$, $(\alpha, \beta) = 1$, $f(\beta) \in \operatorname{supp} \alpha$, and $f(\alpha) \in \operatorname{supp} \beta$.*

Proof. $1 \Rightarrow 2$

Assume that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$, $(\alpha, \beta) = 1$, $f(\beta) \in \operatorname{supp} \alpha$, and $f(\alpha) \in \operatorname{supp} \beta$. Set $\gamma = \alpha - \beta$ ($\gamma \in \Delta$ by Lemma 2.5). $\alpha \prec_w \beta$, so $\gamma \in w\Delta^-$.

If $\gamma \in \Delta^+$, then σ_α cannot be an admissible reflection for w by Lemma 3.4. If $\gamma \in \Delta^-$, then $\beta \prec \alpha$, and $-\gamma = \beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so we have a contradiction with the definition of α-compatibility.

$2 \Rightarrow 1$

Admissibility of σ_α: assume the contrary. By Lemma 3.4, there exist $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta + \gamma = \alpha$.

$\alpha, \beta, \gamma \in \Delta^+$, so $\operatorname{supp} \alpha = \operatorname{supp} \beta \cup \operatorname{supp} \gamma$.

$f(\alpha) \in \operatorname{supp} \alpha$, so we may assume without loss of generality (after a possible interchange of β and γ) that $f(\alpha) \in \operatorname{supp} \beta$.

By Lemma 2.5, $(\beta, \gamma) = -1$, so $(\alpha, \beta) = 1$. $-\gamma = \beta - \alpha \in w\Delta^+$, so $\alpha \prec_w \beta$. Also, $\gamma = \alpha - \beta \in \Delta^+$, so $\beta \prec \alpha$, and $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$. $f(\beta) \in \operatorname{supp} \beta$, so $f(\beta) \in \operatorname{supp} \alpha$, a contradiction.

Now suppose that $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec \beta$, $(\alpha, \beta) = 1$, and $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$.

$\alpha \prec \beta$ and $(\alpha, \beta) = 1$, so $\beta - \alpha \in \Delta^+$.

$\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so $\beta - \alpha \notin w\Delta^-$, so $\beta - \alpha \in w\Delta^+$, and $\alpha \prec_w \beta$.

$\alpha \prec \beta$, so $\operatorname{supp} \alpha \subseteq \operatorname{supp} \beta$. $f(\alpha) \in \operatorname{supp} \alpha$, so $f(\alpha) \in \operatorname{supp} \beta$.

Condition 2 in the Lemma statement says that $f(\beta) \notin \operatorname{supp} \alpha$, so the definition of α-compatibility holds. $\square$

Corollary 4.10. *Let $w \in W$, and let $\alpha \in \Delta^+ \cap w\Delta^-$ be such that $w^{-1}\alpha \in -\Pi$.*

Then every simple root distribution on $\Delta^+ \cap w\Delta^-$ is α-compatible.

Proof. Since $w^{-1}\alpha \in -\Pi$, there are no roots $\beta \in w\Delta^-$ such that $\alpha \prec_w \beta$. $\square$

Lemma 4.11. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities, and let $\alpha \in \Delta^+ \cap w\Delta^-$.*

Suppose that there exists an α-compatible distribution f of simple roots on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots. Suppose that $f(\alpha) = \alpha_i$

Then there exists a labeled sorting process for w that starts with α, the label at this α is $f(\alpha)$, and the whole list of labels is $\alpha_i, \alpha_1, \ldots, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_i, \ldots, \alpha_r, \ldots, \alpha_r$, where, after (excluding) the first α_i, [each α_j is written n_j times, except for α_i, which is written $n_i - 1$ times].

In particular, there exists [a labeled sorting process for w with D-multiplicities $n_1, \ldots, n_i, \ldots, n_r$ of labels] that starts with α, and the label at this α is $f(\alpha)$.

Proof. We start our sorting process with α. Set $w_1 = \sigma_\alpha w$.

By Lemma 3.7 establishes a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \beta$ and $\Delta^+ \cap w_1\Delta^-$. Denote this bijection by $\psi\colon (\Delta^+ \cap w\Delta^-) \setminus \beta \to \Delta^+ \cap w_1\Delta^-$

The definition of α-compatibility says, in terms of Lemma 3.7, that if case 2 of the procedure in Lemma 3.7 holds for some $\beta \in \Delta^+ \cap w\Delta^-$, then $f(\beta) \notin \operatorname{supp} \alpha$. Since $f(\beta) \in \operatorname{supp} \alpha$ for such β, then also $f(\beta) \in \operatorname{supp}(\beta - \alpha) = \operatorname{supp}(\psi(\beta))$.

And if case 3 holds in the procedure in Lemma 3.7 for some $\beta \in \Delta^+ \cap w\Delta^-$, then $\psi(\beta) = \beta$, so clearly, $f(\beta) \in \operatorname{supp}(\psi(\beta))$.

So, $f(\beta) \in \operatorname{supp}(\psi(\beta))$ for all $\beta \in (\Delta^+ \cap w\Delta^-) \setminus \alpha$, and we can set $f_1\colon \Delta^+ \cap w_1\Delta^-$, $f_1(\gamma) = f(\psi^{-1}(\gamma))$. Then $f_1(\gamma) \in \operatorname{supp} \gamma$, so f_1 is a simple root distribution on $\Delta^+ \cap w_1\Delta^-$ with with D-multiplicities $n_1, \ldots, n_i - 1, \ldots, n_r$ of simple roots.

By Proposition 4.4, there exists a labeled sorting process of w_1 with D-multiplicities $n_1, \ldots, n_i - 1, \ldots, n_r$ of labels.

By Corollary 3.27, there exists a labeled sorting process of w_1 with the list of labels $\alpha_1, \ldots, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_i, \ldots, \alpha_r, \ldots, \alpha_r$, where each α_j is written n_j times, except for α_i, which is written $n_i - 1$ times.

We write α with label α_i in front of this sorting process, and we get the claim. $\square$

5 Clusters and excessive configurations

Definition 5.1. Let $I \subseteq \Pi$ be a set of simple roots.

A subset $A \subseteq \Delta^+$ is called a *cluster with set of essential roots* I (or, briefly, an I-cluster) if the following conditions hold:

1. If $\alpha \in A$ and $\alpha_i \in I$, then the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is at most 1.
2. If $\alpha, \beta \in A$, $\alpha \neq \beta$, then (α, β) can be equal to 1 or 0, but not -1.
3. If $\alpha, \beta \in A$ and $(\alpha, \beta) = 0$, then $\operatorname{supp} \alpha \cap \operatorname{supp} \beta \cap I = \varnothing$. In other words, $\operatorname{supp} \alpha$ and $\operatorname{supp} \beta$ don't have essential roots in common.

Lemma 5.2. *A subset of an I-cluster is an I-cluster again. Moreover, if $I' \subseteq I$, then every I-cluster is also an I'-cluster.*

Proof. Obviously follows from the definition. □

Definition 5.3. TODO: invent an appropriate word

A *A-configuration* is a sequence $A, n_1, \ldots, n_r$, where $A \subseteq \Delta^+$, $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}$, and $n_1 + \ldots + n_r = |B|$.

Definition 5.4. Let $A, n_1, \ldots, n_r$ be an A-configuration.

Denote by I the set of simple roots α_i such that $n_i > 0$.

$A, n_1, \ldots, n_r$ is *excessive* if:

$|R_I(A)| = \sum n_i$

and

For each $J \subset I$, $J \neq I$, $J \neq \varnothing$, one has: $|R_J(A)| > \sum_{i \in J} n_i$.

Definition 5.5. Let $A, n_1, \ldots, n_r$ be an A-configuration.

Denote by I the set of simple roots α_i such that $n_i > 0$.

$A, n_1, \ldots, n_r$ is called an *excessive cluster* if:

A is an I-cluster

and

$A, n_1, \ldots, n_r$ is excessive.

We introduce the following definition by induction on n.

Definition 5.6. BASE

An A-configuration $\varnothing, 0, \ldots, 0$ with $|\varnothing| = n = 0$ is always called *excessively clusterizable.*

STEP

An A-configuration $A, n_1, \ldots, n_r$ with $|A| = n > 0$ is called *excessively clusterizable* if:

there exists a subset $I \subseteq \{1, \ldots, r\}$ such that:

denote $k_i = n_i$ if $i \in I$, $k_i = 0$ it $i \notin I$

then, in terms of this notation:

$k_i > 0$ if $i \in I$ and

$\sum k_i > 0$ and

$|R_I(A)| = \sum k_i$ (note that this implies that $(A \setminus R_I(A)), n_1 - k_1, \ldots, n_r - k_r$ is an A-configuration) and

$R_I(A), k_1, \ldots, k_r$ is an excessive cluster and

$(A \setminus R_I(A)), n_1 - k_1, \ldots, n_r - k_r$ is excessively clusterizable.

Lemma 5.7. *Let $A, n_1, \ldots, n_r$ be an excessively clusterizable A-configuration, and let $A', n'_1, \ldots, n'_r$ be another excessively clusterizable A-configuration.*

Denote by J the set of simple roots α_i such that $n_i > 0$.

Suppose that:

$A \cap A' = \varnothing$ and if $\alpha \in A'$, then $\operatorname{supp} \alpha \cap J = \varnothing$ and for each i $(1 \leq i \leq r)$, $(n_i = 0$ or $n'_i = 0)$.

Then $A \cup A', n_1 + n'_1, \ldots, n_r + n'_r$ is an excessively clusterizable A-configuration.

Proof. Induction on $|A|$. If $A = \varnothing$, everything is clear.

Otherwise, there exists a subset $I \subseteq \{1, \dots, r\}$ such that:
denote $k_i = n_i$ if $i \in I$, $k_i = 0$ it $i \notin I$
then, in terms of this notation:
$k_i > 0$ if $i \in I$ and
$\sum k_i > 0$ and
$|R_I(A)| = \sum k_i$ and
$R_I(A), k_1, \dots, k_r$ is an excessive cluster and
$(A \setminus R_I(A)), n_1 - k_1, \dots, n_r - k_r$ is excessively clusterizable.

We are going to use the induction hypothesis for $(A \setminus R_I(A)), n_1 - k_1, \dots, n_r - k_r$ and $A', n'_1, \dots, n'_r$. Let us check that we can use it.

$A \cap A' = \varnothing$, so $(A \setminus R_I(A)) \cap A' = \varnothing$.

Denote $J_1 = J \setminus I$. Clearly, $\alpha_i \in J_1$ if and only if $n_i - k_i > 0$.
If $\alpha \in A'$, then $\operatorname{supp} \alpha \cap J = \varnothing$.
$J_1 \subseteq J$, so, if $\alpha \in A'$, then $\operatorname{supp} \alpha \cap J_1 = \varnothing$.

Clearly, if $n_i = 0$, then $i \notin I$, $k_i = 0$, and $n_i - k_i = 0$.
We know that for all i, $n_i = 0$ or $n'_i = 0$.
So, for all i, $n_i - k_i = 0$ or $n'_i = 0$.

By the induction hypothesis, $(A \setminus R_I(A)) \cup A', n_1 - k_1 + n'_1, \dots, n_r - k_r + n'_r$ is an excessively clusterizable A-configuration.

Note that $A \cap A' = \varnothing$, $R_I(A) \subseteq A$, so $(A \setminus R_I(A)) \cup A' = (A \cup A') \setminus R_I(A)$.

Let us check that $R_I(A) = R_I(A \cup A')$.
Indeed, $I \subseteq J$ since if $\alpha_i \in I$, then $k_i > 0$ and hence $n_i > 0$.
So, if $\alpha \in A'$, then $\operatorname{supp} \alpha \cap I = \varnothing$.
So, $R_I(A') = \varnothing$, and $R_I(A) = R_I(A \cup A')$.

The previous conclusion can be rewritten as follows: $(A \cup A') \setminus R_I(A \cup A'), n_1 - k_1 + n'_1, \dots, n_r - k_r + n'_r$ is an excessively clusterizable A-configuration.

For all $i \in \{1, \dots, r\}$, ($n_i = 0$ or $n'_i = 0$).
If $i \in I$, then $k_i = n_i > 0$, so $n'_i = 0$, and $k_i = n_i + n'_i$.
Recall that if $i \notin I$, then $k_i = 0$.

Summarizing, we know the following: $k_i > 0$ if $i \in I$ and
$\sum k_i > 0$ and
$|R_I(A \cup A')| = |R_I(A)| = \sum k_i$ and
$R_I(A \cup A') = R_I(A), k_1, \dots, k_r$ is an excessive cluster and
$(A \cup A') \setminus R_I(A \cup A'), n_1 - k_1 + n'_1, \dots, n_r - k_r + n'_r$ is excessively clusterizable.

By definition, this means that $A \cup A', n_1 + n'_1, \dots, n_r + n'_r$ is an excessively clusterizable A-configuration. $\square$

Lemma 5.8. *Let $A, n_1, \dots, n_r$ be an A-configuration. Denote by I the set of simple roots α_i such that $n_i > 0$.*

Suppose that A is an I-cluster and
for each $J \subseteq \{1, \dots, r\}$: $|R_J(A)| \geq \sum_{i \in J} n_i$.

Then $A, n_1, \dots, n_r$ is an excessively clusterizable A-configuration.

Proof. Induction on $|A|$. For $A = \varnothing$, everything is clear.

Let J be a minimal by inclusion nonempty subset of I such that $|R_J(A)| = \sum_{i \in J} n_i$.
Then for each $J' \subset J$, $J' \neq J$, $J' \neq \varnothing$ we have $|R_{J'}(A)| > \sum_{i \in J'} n_i$.

Let us try to use this J for the definition of an excessively clusterizable A-configuration. Denote $k_i = n_i$ if $i \in J$, $k_i = 0$ otherwise.

$J \subseteq I$, so if $i \in J$, then $k_i = n_i > 0$.

J is nonempty, so $\sum k_i > 0$.

$|R_J(A)| = \sum_{i \in J} n_i = \sum k_i$ by the choice of J.

By Lemma 5.2, $R_J(A)$ is an I-cluster and a J-cluster. It follows from the choice of J that $R_J(A), k_1, \dots, k_r$ is an excessive cluster.

Finally, we have to check that $(A \setminus R_J(A)), n_1 - k_1, \dots, n_r - k_r$ is excessively clusterizable.
To use the induction hypothesis, denote $I_0 = I \setminus J$.
Then $n_i - k_i > 0$ if and only if $i \in I_0$.

We have to check that $A \setminus R_J(A)$ is an I_0-cluster and for each $J' \subseteq \{1, \dots, r\}$: $|R_{J'}(A \setminus R_J(A))| \geq \sum_{i \in J'}(n_i - k_i)$.

By Lemma 5.2, $A \setminus R_J(A)$ is an I_0-cluster.

Let us prove that for each $J' \subseteq \{1, \dots, r\}$: $|R_{J'}(A \setminus R_J(A))| \geq \sum_{i \in J'}(n_i - k_i)$.

First, consider an arbitrary subset $I_0' \subseteq I_0$ and denote $I_1' = I_0' \cup J$. By the definition of $R_J(A)$, if $\alpha \in A \setminus R_J(A)$, then $\operatorname{supp} \alpha \cap J = \varnothing$.
Therefore, $R_J(A \setminus R_J(A)) = \varnothing$ and
$R_{I_1'}(A \setminus R_J(A)) = R_{I_0'}(A \setminus R_J(A)) \cup R_J(A \setminus R_J(A)) = R_{I_0'}(A \setminus R_J(A))$.
$J \subseteq I_1'$, so $R_J(A) = R_J(R_J(A)) \subseteq R_{I_1'}(R_J(A)) \subseteq R_J(A)$.
So, $R_J(A) = R_{I_1'}(R_J(A))$.
Clearly, $R_{I_1'}(A)$ is the disjoint union of $R_{I_1'}(A \setminus R_J(A))$ and $R_{I_1'}(R_J(A))$.
So, $R_{I_1'}(A)$ is the disjoint union of $R_{I_0'}(A \setminus R_J(A))$ and $R_J(A)$.
Therefore, $|R_{I_1'}(A)| = |R_{I_0'}(A \setminus R_J(A))| + |R_J(A)|$.

The hypothesis of the lemma says that $|R_{I_1'}(A)| \geq \sum_{i \in I_1'} n_i$.
We can write $\sum_{i \in I_1'} n_i \geq (\sum_{i \in J} n_i) + (\sum_{i \in I_0'} n_i)$ and
$|R_{I_1'}(A)| \geq (\sum_{i \in J} n_i) + (\sum_{i \in I_0'} n_i)$ and
$|R_{I_0'}(A \setminus R_J(A))| + |R_J(A)| \geq (\sum_{i \in J} n_i) + (\sum_{i \in I_0'} n_i)$.
By the choice of J, $|R_J(A)| = \sum_{i \in J} n_i$.
So, $|R_{I_0'}(A \setminus R_J(A))| \geq \sum_{i \in I_0'} n_i$.
Finally, $k_i > 0$ if and only if $i \in J$, otherwise $k_i = 0$, so we can write $|R_{I_0'}(A \setminus R_J(A))| \geq \sum_{i \in I_0'}(n_i - k_i)$.

Now, take an arbitrary $I' \subseteq \{1, \dots, r\}$. Set $I_0' = I_0 \cap I'$.
Then $R_{I_0'}(A \setminus R_J(A)) \subseteq R_{I'}(A \setminus R_J(A))$.
So, $|R_{I'}(A \setminus R_J(A))| \geq |R_{I_0'}(A \setminus R_J(A))|$.
Also we can write $\sum_{i \in I'}(n_i - k_i) = (\sum_{i \in I_0'}(n_i - k_i)) + (\sum_{i \in I' \setminus I_0}(n_i - k_i))$.
We have already seen that $n_i - k_i > 0$ if and only if $i \in I_0$.
So, $\sum_{i \in I' \setminus I_0}(n_i - k_i) = 0$ and
$\sum_{i \in I'}(n_i - k_i) = \sum_{i \in I_0'}(n_i - k_i)$.
We already know that for I_0': $|R_{I_0'}(A \setminus R_J(A))| \geq \sum_{i \in I_0'}(n_i - k_i)$.
So, $|R_{I'}(A \setminus R_J(A))| \geq |R_{I_0'}(A \setminus R_J(A))| \geq \sum_{i \in I_0'}(n_i - k_i) = \sum_{i \in I'}(n_i - k_i)$.

Therefore, for every $I' \subseteq \{1, \dots, r\}$: $|R_{I'}(A \setminus R_J(A))| \geq \sum_{i \in I'}(n_i - k_i)$.
By induction hypothesis, $A \setminus R_J(A), n_1 - k_1, \dots, n_r - k_r$ is excessively clusterizable.

We have verified all of the conditions in the definition of an excessively clusterizable A-configuration, so $A, n_1, \dots, n_r$ is an excessively clusterizable A-configuration. $\square$

Lemma 5.9. *Let $A, n_1, \dots, n_r$ be an excessive cluster, $A \neq \varnothing$.*

Let $\alpha \in A$ and α_j be such that $\alpha_j \in \operatorname{supp} \alpha$ and $n_j > 0$.

Then $A \setminus \{\alpha\}, n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_r$ is an excessively clusterizable A-configuration.

Proof. Denote $m_j = n_j - 1$, $m_i = n_i$ for $i \neq j$.
Denote by I (resp. I') the set of simple roots α_i such that $n_i > 0$ (resp. $m_i > 0$). Clearly, $I' \subseteq I$.

We are going to use Lemma 5.8.

By the definition of an excessive cluster, A is an I-cluster. By Lemma 5.2, $A \setminus \{\alpha\}$ is an I'-cluster.

Let $I_0 \subseteq I$. Clearly, $\sum_{i \in I_0} n_i \geq \sum_{i \in I_0} m_i$ and $|R_{I_0}(A \setminus \{\alpha\})| \geq |R_{I_0}(A)| - 1$.

If $I_0 = \varnothing$, then $|R_{I_0}(A \setminus \{\alpha\})| = 0$ and $\sum_{i \in I_0} m_i = 0$.

If $I_0 \neq I$ and $I_0 \neq \varnothing$:
By the definition of an excessive cluster, $|R_{I_0}(A)| > \sum_{i \in I_0} n_i$.
Then $|R_{I_0}(A \setminus \{\alpha\})| \geq |R_{I_0}(A)| - 1 > (\sum_{i \in I_0} n_i) - 1 \geq (\sum_{i \in I_0} m_i) - 1$.
Since all number here are integers, $|R_{I_0}(A)| \geq \sum_{i \in I_0} m_i$.

If $I_0 = I$:
By the definition of an excessive cluster, $|R_{I_0}(A)| = \sum_{i \in I_0} n_i$.

$\sum_{i\in I_0} m_i = (\sum_{i\in I} n_i) - 1$ and $\alpha \in R_{I_0}(A)$, so
$|R_{I_0}(A \setminus \{\alpha\})| = |R_{I_0}(A)| - 1 = (\sum_{i\in I_0} n_i) - 1 = \sum_{i\in I_0} m_j$.

So, for all $I_0 \subseteq I$ we have $|R_{I_0}(A)| \geq \sum_{i\in I_0} m_i$.

Now let $I' \subseteq \{1,\dots,r\}$ be arbitrary. Set $I_0 = I' \cap I$.
$I_0 \subseteq I'$, so $|R_{I'}(A \setminus \{\alpha\})| \geq |R_{I_0}(A \setminus \{\alpha\})|$.
We already know that $|R_{I_0}(A)| \geq \sum_{i\in I_0} m_i$.
And if $i \notin I$, then $n_i = 0$ and $i \neq j$, so $m_i = 0$. So, $\sum_{i\in I_0} m_i = \sum_{i\in I'} m_i$.

Therefore, for all $I' \subseteq \{1,\dots,r\}$ we have $|R_{I'}(A \setminus \{\alpha\})| \geq \sum_{i\in I'} m_i$.

By Lemma 5.8, $A \setminus \{\alpha\}, n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_r$ is an excessively clusterizable A-configuration. □

Proposition 5.10. *Let $A, n_1, \dots, n_r$ be an excessively clusterizable A-configuration with $A \neq \varnothing$*

Let $I \subseteq \{1,\dots,r\}$ be a subset such that:

denote $k_i = n_i$ if $i \in I$, $k_i = 0$ it $i \notin I$

then, in terms of this notation:
$k_i > 0$ if $i \in I$ and
$\sum k_i > 0$ and
$|R_I(A)| = \sum k_i$
$R_I(A), k_1, \dots, k_r$ is an excessive cluster and
$(A \setminus R_I(A)), n_1 - k_1, \dots, n_r - k_r$ is excessively clusterizable.

(such I exists by the definition of an excessively clusterizable A-configuration)

Claim of the proposition: if $\alpha \in R_I(A)$, $j \in I$, and $\alpha_j \in \operatorname{supp}\alpha$, then

$A \setminus \{\alpha\}, n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_r$ is an excessively clusterizable A-configuration.

Proof. We know that $R_I(A), k_1, \dots, k_r$ is an excessive cluster, $\alpha \in R_I(A)$, $j \in I$ (so, $k_j > 0$), and $\alpha_j \in \operatorname{supp}\alpha$.
By Lemma 5.9, $R_I(A \setminus \{\alpha\}), k_1, \dots, k_{j-1}, k_j - 1, k_{j+1}, \dots, k_r$ is an excessively clusterizable A-configuration.

We are going to use Lemma 5.7.

Denote $m_j = k_j - 1$, $m_i = k_i$ for $i \neq j$. Then we can say that $R_I(A \setminus \{\alpha\}), m_1, \dots, m_r$ is an excessively clusterizable A-configuration.

Set $n'_i = n_i - k_i$.
Then $n'_i = n_i$ if $i \notin I$ and $n'_i = 0$ if $i \in I$.
On the other hand, if $i \notin I$, then $k_i = 0$ and $m_i = 0$ (recall that $j \in I$).
So, for all $i \in \{1,\dots,r\}$ we have ($n'_i = 0$ or $m_i = 0$).
Also, note that $m_j + n'_j = n_j - 1$ and $m_i + n'_i = n_i$ if $i \neq j$.
So, we want to prove that $A \setminus \{\alpha\}, m_1 + n'_1, \dots, m_r + n'_r$ is an excessively clusterizable A-configuration.

The hypothesis of the proposition also says that $(A \setminus R_I(A)), n_1 - k_1, \dots, n_r - k_r$ is excessively clusterizable. In other words, $(A \setminus R_I(A)), n'_1, \dots, n'_r$ is excessively clusterizable.

The set of simple roots α_i such that $m_i > 0$ (denote it by J) is either I, or $I \setminus \{\alpha_j\}$. In both cases, $J \subseteq I$.
By the definition of $R_I(A)$, if $\beta \in A$ and $\operatorname{supp}\beta \cap I \neq \varnothing$, then $\beta \in R_I(A)$, and $\beta \notin A \setminus R_I(A)$.
So, if $\beta \in A \setminus R_I(A)$, then $\operatorname{supp}\beta \cap I = \varnothing$ and $\operatorname{supp}\beta \cap J = \varnothing$ since $J \subseteq I$.

Finally, $R_I(A) \cap (A \setminus R_I(A)) = \varnothing$ and $R_I(A \setminus \{\alpha\}) \subseteq R_I(A)$, so $R_I(A \setminus \{\alpha\}) \cap (A \setminus R_I(A)) = \varnothing$.
Therefore, we can apply Lemma 5.7 to $R_I(A \setminus \{\alpha\}), m_1, \dots, m_r$ and $(A \setminus R_I(A)), n'_1, \dots, n'_r$.
It states that $R_I(A \setminus \{\alpha\}) \cup (A \setminus R_I(A)), m_1 + n'_1, \dots, m_r + n'_r$ is an excessively clusterizable A-configuration.

Finally, $\alpha \in R_I(A)$, $\alpha \in A$, so $R_I(A \setminus \{\alpha\}) = R_I(A) \setminus \{\alpha\}$.
Again, $\alpha \in R_I(A)$, $\alpha \in A$, so $A \setminus R_I(A) = (A \setminus \{\alpha\}) \setminus (R_I(A) \setminus \alpha) = (A \setminus \{\alpha\}) \setminus R_I(A \setminus \{\alpha\})$.
So, $R_I(A \setminus \{\alpha\}) \cup (A \setminus R_I(A)) = R_I(A \setminus \{\alpha\}) \cup [(A \setminus \{\alpha\}) \setminus R_I(A \setminus \{\alpha\})]$.
And $R_I(A \setminus \{\alpha\}) \subseteq (A \setminus \{\alpha\})$, so $R_I(A \setminus \{\alpha\}) \cup [(A \setminus \{\alpha\}) \setminus R_I(A \setminus \{\alpha\})] = (A \setminus \{\alpha\})$.

Therefore, $A \setminus \{\alpha\}, m_1 + n'_1, \dots, m_r + n'_r$ is an excessively clusterizable A-configuration. □

Lemma 5.11. *Let $A, n_1, \dots, n_r$ be an excessive A-configuration. Then for each $\alpha \in \Delta^+ \cap w\Delta^-$, there exists a simple root $\alpha_i \in \operatorname{supp}\alpha$ such that $n_i > 0$.*

Proof. By Lemma 4.2, there exists a simple roots distribution f on A with D-multiplicities $n_1, \dots, n_r$ of simple roots.

Set $\alpha_i = f(\alpha)$. Then $\alpha_i \in \operatorname{supp} \alpha$ and f takes value α_i at least once, so $n_i > 0$. □

Lemma 5.12. *Let $A, n_1, \dots, n_r$ be an excessive A-configuration.*

If $n_i > 0$, then there exists a $\prec$-maximal root α such that $\alpha_i \in \operatorname{supp} \alpha$.

Proof. By the definition of an excessive configuration, if $n_i > 0$, then $|R_{\{i\}}(A)| \ge n_i > 0$, so there exists an element $\beta \in R_{\{i\}}(A)$, in other words, there exists a root $\beta \in A$ such that $\alpha_i \in \operatorname{supp} \beta$.

Since A is a finite partially ordered set with order $\prec$, there exists a $\prec$-maximal element α of A such that $\beta \preceq \alpha$. Then $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$ and $\alpha_i \in \operatorname{supp} \alpha$. □

Lemma 5.13. *Let $A, n_1, \dots, n_r$ be an excessive A-configuration. Denote by I the set of simple roots α_i such that $n_i > 0$.*

Suppose that the following is true: If $\beta_1, \beta_2 \in A$ are two different $\prec$-maximal elements of A, then $\operatorname{supp} \beta_1 \cap \operatorname{supp} \beta_2 \cap I = \varnothing$.

Then, in fact, A has a unique $\prec$-maximal element.

Proof. Denote all $\prec$-maximal elements of A by $\beta_1, \dots, \beta_m$.

Assume that $m > 1$.

Denote by I_j $(1 \le j \le m)$ the set of all indices i $(1 \le i \le r)$ such that $n_i > 0$ and $\alpha_i \in \operatorname{supp} \beta_j$.

By the Lemma hypothesis, all sets I_j are disjoint. By Lemma 5.11, all of them are non-empty.

Clearly, $I_j \subseteq I$ for all j. Moreover, since $m > 1$, actually, $I_j \ne I$. Also, it now follows from Lemma 5.12 that $J = \bigcup I_j$.

By the definition of an excessive configuration, $|R_{I_j}(A)| > \sum_{i \in I_j} n_i$.

For each $\alpha \in A$ there exists a $\prec$-maximal root $\beta_j \in A$ such that $\alpha \preceq \beta_j$. This is always true for finite partially ordered sets. And then $\operatorname{supp} \alpha \subseteq \operatorname{supp} \beta_j$, and it follows from Lemma 5.11 applied to α that $\alpha \in R_{I_j}(w)$.

Moreover, if for some $\alpha \in \Delta^+ \cap w\Delta^-$ we have $\alpha \in R_{I_j}(w)$, then there exists $\alpha_i \in \operatorname{supp} \alpha$ such that $n_i > 0$ and $\alpha_i \in \operatorname{supp} \beta_j$. Then we cannot have $\alpha \preceq \beta_k$ for $k \ne j$, otherwise α_i would be in $\operatorname{supp} \beta_k$, and this would be a contradiction with the Lemma hypothesis.

So, for each $\alpha \in A$ there is a unique index j $(1 \le j \le m)$ such that $\alpha \in R_{I_j}(A)$.

In other words, A is a disjoint union of the sets $R_{I_j}(A)$ for all values of j $(1 \le j \le m)$.

So,

$$|A| = \sum_{j=1}^{m} |R_{I_j}(A)|.$$

On the other hand,

$$\sum_{j=1}^{m} |R_{I_j}(A)| > \sum_{j=1}^{m} \sum_{i \in I_j} n_i,$$

and the right-hand side contains each index i such that $n_i > 0$ exactly once since $J = \bigcup I_j$. So,

$$|A| > \sum_{i \in I} n_i = \sum_{i=1}^{r} n_i = |A|,$$

a contradiction. □

Definition 5.14. Let $A, n_1, \dots, n_r$ be an A-configuration.

Denote by I the set of simple roots α_i such that $n_i > 0$.

$A, n_1, \dots, n_r$ is called a *simple excessive cluster* if:

$|I| = 1$ and

$A, n_1, \dots, n_r$ is an excessive cluster.

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Lemma 5.15. *Let $A, n_1, \ldots, n_r$ be an A-configuration. It is a simple excessive cluster if and only if:*

there exists a number i, $1 \le i \le r$, such that:

$n_j = 0$ for $j \ne i$, $n_i > 0$, and

$|A| = n_i$, and

$\alpha_i \in \operatorname{supp} \beta$ for all $\beta \in A$, and

$(\beta, \gamma) = 1$ for all $\beta, \gamma \in A$, $\beta \ne \gamma$, and

for each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots equals 1.

Proof. $\Rightarrow$. Denote by A the set of simple roots α_j such that $n_j > 0$. $|A| = 1$, so there exists a unique index i such that $n_i > 0$, and $n_j = 0$ for $j \ne i$. Then $A = \{\alpha_i\}$

The definition of an excessive cluster also says that $A, n_1, \ldots, n_r$ is an excessive A-configuration, in particular this implies that $|R_{\{i\}}(A)| = \sum n_j = n_i$. The definition of an A-configuration also says that $|A| = \sum n_j = n_i$, so $A = R_{\{i\}}(A)$, and $\alpha_i \in \operatorname{supp} \beta$ for all $\beta \in A$.

The definition of an excessive cluster also says that A is an $\{i\}$-cluster.

Then for each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots is at most 1. But we also know that $\alpha_i \in \operatorname{supp} \beta$, so this coefficient equals precisely 1.

Finally, if $\beta, \gamma \in A$, $\beta \ne \gamma$, then $\operatorname{supp} \alpha \cap \operatorname{supp} \beta \cap \{\alpha_i\} = \{\alpha_i\} \ne \varnothing$, so the only possible value for (β, γ) is 1.

$\Leftarrow$. The set of simple roots α_j such that $n_j > 0$ is $\{\alpha_i\}$.

$(\beta, \gamma) = 1$ for all $\beta, \gamma \in A$, $\beta \ne \gamma$, and

for each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots equals 1,

so A is a $\{i\}$-cluster.

For the definition of an excessive A-configuration, there are no sets $I \subset \{i\}$ such that $I \ne \{i\}$ and $I \ne \varnothing$, so the only condition we have to check in this definition is that $|R_{\{i\}}(A)| = \sum n_j = n_i$. And this is true since for each $\beta \in A$ we have $\alpha_i \in \operatorname{supp} \beta$, so $R_{\{i\}}(A) = A$, and $|A| = \sum n_j = n_i$ by the definition of an A-configuration. □

Lemma 5.16. *Let $A, n_1, \ldots, n_r$ be an A-configuration. It is a simple excessive cluster if and only if:*

there exists a number i, $1 \le i \le r$, such that:

$n_j = 0$ for $j \ne i$, $n_i > 0$, and

$|A| = n_i$, and

$\alpha_i \in \operatorname{supp} \beta$ for all $\beta \in A$, and

A is an $\{i\}$-cluster.

Proof. $\Rightarrow$. Denote by A the set of simple roots α_j such that $n_j > 0$. $|A| = 1$, so there exists a unique index i such that $n_i > 0$, and $n_j = 0$ for $j \ne i$. Then $A = \{\alpha_i\}$

The definition of an excessive cluster also says that $A, n_1, \ldots, n_r$ is an excessive A-configuration, in particular this implies that $|R_{\{i\}}(A)| = \sum n_j = n_i$. The definition of an A-configuration also says that $|A| = \sum n_j = n_i$, so $A = R_{\{i\}}(A)$, and $\alpha_i \in \operatorname{supp} \beta$ for all $\beta \in A$.

The definition of an excessive cluster also says that A is an $\{i\}$-cluster.

$\Leftarrow$. The set of simple roots α_j such that $n_j > 0$ is $\{\alpha_i\}$.

A is a $\{i\}$-cluster.

For the definition of an excessive A-configuration, there are no sets $I \subset \{i\}$ such that $I \ne \{i\}$ and $I \ne \varnothing$, so the only condition we have to check in this definition is that $|R_{\{i\}}(A)| = \sum n_j = n_i$. And this is true since for each $\beta \in A$ we have $\alpha_i \in \operatorname{supp} \beta$, so $R_{\{i\}}(A) = A$, and $|A| = \sum n_j = n_i$ by the definition of an A-configuration. □

Lemma 5.17. *Let $A \subseteq \Delta^+$, $\alpha_i \in \Pi$ Suppose that $\alpha_i \in \operatorname{supp} \beta$ for all $\beta \in A$.*

Then A is an $\{i\}$-cluster if and only if

$(\beta, \gamma) = 1$ for all $\beta, \gamma \in A$, $\beta \ne \gamma$, and

for each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots equals 1.

Proof. $\Rightarrow$. For each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots is at most 1. But we also know that $\alpha_i \in \operatorname{supp}\beta$, so this coefficient equals precisely 1.

If $\beta, \gamma \in A$, $\beta \neq \gamma$, then $\operatorname{supp}\alpha \cap \operatorname{supp}\beta \cap \{\alpha_i\} = \{\alpha_i\} \neq \varnothing$, so the only possible value for (β, γ) is 1.

$\Leftarrow$. $(\beta, \gamma) = 1$ for all $\beta, \gamma \in A$, $\beta \neq \gamma$, and

for each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots equals 1,

so A is a $\{i\}$-cluster. □

We introduce the following definition by induction on n.

Definition 5.18. BASE

An A-configuration $\varnothing, 0, \ldots, 0$ with $|\varnothing| = n = 0$ is always called *simply excessively clusterizable.*

STEP

An A-configuration $A, n_1, \ldots, n_r$ with $|A| = n > 0$ is called *simply excessively clusterizable* if:

there exists an index $i \in \{1, \ldots, r\}$ such that:

denote $k_i = n_i$ and $k_j = 0$ if $j \neq i$

then, in terms of this notation:

$k_i > 0$ and

$|R_{\{i\}}(A)| = k_i$ (note that this implies that $(A \setminus R_{\{i\}}(A)), n_1 - k_1, \ldots, n_r - k_r$ is an A-configuration) and

$R_{\{i\}}(A), k_1, \ldots, k_r$ is a simple excessive cluster and

$(A \setminus R_{\{i\}}(A)), n_1 - k_1, \ldots, n_r - k_r$ is simply excessively clusterizable.

Lemma 5.19. *If an A-configuration $A, n_1, \ldots, n_r$ is simply excessively clusterizable, then it is excessively clusterizable.*

Proof. Follows directly from the definition of an excessively clusterizable A-configuration for $I = \{i\}$, the fact that a simple excessive cluster is an excessive cluster, and induction on $|A|$. □

Lemma 5.20. *Let $A, n_1, \ldots, n_r$ be a simply excessively clusterizable A-configuration, and let $A', n'_1, \ldots, n'_r$ be another simply excessively clusterizable A-configuration.*

Denote by J the set of simple roots α_i such that $n_i > 0$.

Suppose that:

$A \cap A' = \varnothing$ and if $\alpha \in A'$, then $\operatorname{supp}\alpha \cap J = \varnothing$ and for each i $(1 \leq i \leq r)$, $(n_i = 0$ or $n'_i = 0)$.

Then $A \cup A', n_1 + n'_1, \ldots, n_r + n'_r$ is a simply excessively clusterizable A-configuration.

Proof. Induction on $|A|$. If $A = \varnothing$, everything is clear.

Otherwise, there exists an index $i \in \{1, \ldots, r\}$ such that:

denote $k_i = n_i$ and $k_j = 0$ if $i \neq j$

then, in terms of this notation:

$k_i > 0$ and

$|R_{\{i\}}(A)| = k_i$ and

$R_{\{i\}}(A), k_1, \ldots, k_r$ is an excessive cluster and

$(A \setminus R_{\{i\}}(A)), n_1 - k_1, \ldots, n_r - k_r$ is excessively clusterizable.

We are going to use the induction hypothesis for $(A \setminus R_I(A)), n_1 - k_1, \ldots, n_r - k_r$ and $A', n'_1, \ldots, n'_r$. Let us check that we can use it.

$A \cap A' = \varnothing$, so $(A \setminus R_{\{i\}}(A)) \cap A' = \varnothing$.

Denote $J_1 = J \setminus \{i\}$. Clearly, $\alpha_j \in J_1$ if and only if $n_j - k_j > 0$.

If $\alpha \in A'$, then $\operatorname{supp}\alpha \cap J = \varnothing$.

$J_1 \subseteq J$, so, if $\alpha \in A'$, then $\operatorname{supp}\alpha \cap J_1 = \varnothing$.

Clearly, if $n_j = 0$, then $j \neq i$, $k_j = 0$, and $n_j - k_j = 0$.

We know that for all j, $n_j = 0$ or $n'_j = 0$.

So, for all j, $n_j - k_j = 0$ or $n'_j = 0$.

By the induction hypothesis, $(A \setminus R_{\{i\}}(A)) \cup A', n_1 - k_1 + n'_1, \ldots, n_r - k_r + n'_r$ is a simply excessively clusterizable A-configuration.

Note that $A \cap A' = \varnothing$, $R_{\{i\}}(A) \subseteq A$, so $(A \setminus R_{\{i\}}(A)) \cup A' = (A \cup A') \setminus R_{\{i\}}(A)$.

Let us check that $R_{\{i\}}(A) = R_{\{i\}}(A \cup A')$.

Indeed, $i \in J$ since $k_i > 0$ and hence $n_i > 0$.

So, if $\alpha \in A'$, then $\alpha_i \notin \operatorname{supp} \alpha$ since $\operatorname{supp} \alpha \cap J = \varnothing$.

So, $R_{\{i\}}(A') = \varnothing$, and $R_{\{i\}}(A) = R_{\{i\}}(A \cup A')$.

The previous conclusion can be rewritten as follows: $(A \cup A') \setminus R_{\{i\}}(A \cup A'), n_1 - k_1 + n'_1, \ldots, n_r - k_r + n'_r$ is an excessively clusterizable A-configuration.

For all $j \in \{1, \ldots, r\}$, ($n_j = 0$ or $n'_j = 0$).

$k_i = n_i > 0$, so $n'_i = 0$, and $k_i = n_i + n'_i$.

Recall that if $j \neq i$, then $k_j = 0$.

Summarizing, we know the following: $k_i > 0$ and

$|R_{\{i\}}(A \cup A')| = |R_{\{i\}}(A)| = k_i$ and

$R_{\{i\}}(A \cup A') = R_{\{i\}}(A), k_1, \ldots, k_r$ is a simple excessive cluster and

$(A \cup A') \setminus R_{\{i\}}(A \cup A'), n_1 - k_1 + n'_1, \ldots, n_r - k_r + n'_r$ is simply excessively clusterizable.

By definition, this means that $A \cup A', n_1 + n'_1, \ldots, n_r + n'_r$ is a simply excessively clusterizable A-configuration. □

Lemma 5.21. *Let $A, n_1, \ldots, n_r$ be a simply excessively A-clusterizable configuration. Let $I \subseteq \{1, \ldots, r\}$ be the set of indices i such that $n_i > 0$. Then $R_I(A) = A$.*

Proof. Induction on $|A|$. If $A = \varnothing$, then everything is clear. Suppose that $A \neq \varnothing$.

There exists an index $i \in \{1, \ldots, r\}$ such that:

denote $k_i = n_i$ and $k_j = 0$ if $j \neq i$

then, in terms of this notation:

$k_i > 0$ and

$|R_{\{i\}}(A)| = k_i$

$R_{\{i\}}(A), k_1, \ldots, k_r$ is a simple excessive cluster and

$(A \setminus R_{\{i\}}(A)), n_1 - k_1, \ldots, n_r - k_r$ is simply excessively clusterizable.

Denote $J = I \setminus \{i\}$. It follows from the definitions of I and of k_j that $n_j - k_j > 0$ if and only if $j \in J$.

By the induction hypothesis, $A \setminus R_{\{i\}}(A) = R_J(A \setminus R_{\{i\}}(A))$.

By the definition of notation R, this means that for each $\beta \in A \setminus R_{\{i\}}(A)$, there exists $j \in J$ such that $\alpha_j \in \operatorname{supp} \beta$. Also by the definition of notation R, for each $\beta \in R_{\{i\}}(A)$, we have $\alpha_i \in \operatorname{supp} \beta$.

So, for each $\beta \in A$ there exists $j \in J \cup \{i\} = I$ such that $\alpha_j \in \operatorname{supp} \beta$. So, $A = R_I(A)$. □

Lemma 5.22. *Let $I \subseteq \{1, \ldots, r\}$, and let A be an I-cluster. Suppose that $A = R_I(A)$.*

Then there exist numbers $n_1, \ldots, n_r$ such that:

$n_j = 0$ if $j \notin I$, and

$n_1 + \ldots + n_r = |A|$, and

$A, n_1, \ldots, n_r$ is a simply excessively clusterizable A-configuration.

Proof. Induction on $|A|$. If $A = \varnothing$, everything is clear. Suppose $|A| > 0$.

We know that $A = R_I(A)$ and $|A| > 0$, in particular, there exists a simple root α_i and $\beta \in A$ such that $i \in I$ and $\alpha_i \in \operatorname{supp} \beta$. Fix this i until the end of the proof.

By Lemma 5.2, $R_{\{i\}}(A)$ is a $\{i\}$-cluster. Set $n_i = |R_{\{i\}}(A)|$. By Lemma 5.16, $R_{\{i\}}(A), 0, \ldots, 0, n_i, 0, \ldots, 0$, where n_i occurs at the ith position, is a simple excessive cluster.

Set $I' = \setminus\{i\}$, $A' = A \setminus R_I(A)$. By Lemma 5.2, A' is an I'-cluster.

Also, if $\beta \in A'$, then $\beta \in A$, $\beta \in R_I(A)$, and $\operatorname{supp} \beta \cap I \neq \varnothing$. If $\beta \in A'$, then $\beta \notin R_{\{i\}}(A)$, and $\alpha_i \notin \operatorname{supp} \beta$. So, in fact $\operatorname{supp} \beta \cap I' \neq \varnothing$, and $\beta \in R_{I'}(A')$.

Therefore, $A' = R_{I'}(A')$.

By the induction hypothesis, there exist numbers $n'_1, \ldots, n'_r$ such that

$n'_j = 0$ if $j \notin I'$, and

$n'_1 + \ldots + n'_r = |A'|$, and

$A', n'_1, \ldots, n'_r$ is a simply excessively clusterizable A-configuration.

Set $n_j = n'_j$ for $j \neq i$. Then, if $j \notin I$, then $j \neq i$ and $j \notin I'$, so $n_j = 0$.

$n_1 + \ldots + n_r = n'_1 + \ldots + n'_r + n_i = |A'| + |R_{\{i\}}(A)| = |A|$.

We have already verified all conditions in the definition that $A, n_1, \ldots, n_r$ is a simply excessively clusterizable A-configuration. $\square$

Proposition 5.23. *Let $A, n_1, \ldots, n_r$ be an excessively clusterizable A-configuration.*
Then there exist numbers $m_1, \ldots, m_r$ such that
if $n_i = 0$, then $m_i = 0$, and
$m_1 + \ldots + m_r = |A|$, and
$A, m_1, \ldots, m_r$ is a simply excessively clusterizable A-configuration.

Proof. Induction on $|A|$. If $A = \varnothing$, everything is clear. Suppose $|A| > 0$.
There exists a subset $I \subseteq \{1, \ldots, r\}$ such that:
denote $k_i = n_i$ if $i \in I$, $k_i = 0$ it $i \notin I$
then, in terms of this notation:
$k_i > 0$ if $i \in I$ and
$\sum k_i > 0$ and
$|R_I(A)| = \sum k_i$ and
$R_I(A), k_1, \ldots, k_r$ is an excessive cluster and
$(A \setminus R_I(A)), n_1 - k_1, \ldots, n_r - k_r$ is excessively clusterizable.

$R_I(A), k_1, \ldots, k_r$ is an excessive cluster, and the set of indices i such that $k_i > 0$ is exactly I, so $R_I(A)$ is an I-cluster. By the definition of notation R, $R_I(R_I(A)) = R_I(A)$. So, by Lemma 5.22, there exist numbers $m'_1, \ldots, m'_r$ such that
if $i \notin I$, then $m'_i = 0$,
and $m'_1 + \ldots + m'_r = |R_I(A)|$, and
$R_I(A), m'_1, \ldots, m'_r$ is a simply excessively clusterizable A-configuration.
$(A \setminus R_I(A)), n_1 - k_1, \ldots, n_r - k_r$ is excessively clusterizable.
By the induction hypothesis, there exist numbers $m''_1, \ldots, m''_r$ such that
if $n_i - k_i = 0$, then $m''_i = 0$, and
$m''_1 + \ldots + m''_r = |A \setminus R_I(A)|$, and
$A \setminus R_I(A), m''_1, \ldots, m''_r$ is a simply excessively clusterizable A-configuration.

We are going to use Lemma 5.20. We have two simply clusterizable A-configurations: $R_I(A), m'_1, \ldots, m'_r$ and $A \setminus R_I(A), m''_1, \ldots, m''_r$.

Denote by J the set of simple roots α_i such that $m'_i > 0$. We know that if $i \notin I$, then $m'_i = 0$, so $J \subseteq I$. Clearly, if $\beta \in A \setminus R_I(A)$, then $\operatorname{supp} \beta \cap I = \varnothing$, so $\operatorname{supp} \beta \cap J = \varnothing$.

Now, for each i, $1 \le i \le r$, we have: if $i \notin I$, then $m'_i = 0$; if $i \in I$, then $k_i = n_i$, $n_i - k_i = 0$, and $m''_i = 0$. So, ($m'_i = 0$ or $m''_i = 0$).

Set $m_i = m'_i + m''_i$ (in other words, $m_i = m'_i$ for $i : \alpha_i \in I$ and $m_i = m''_i$ for $i : \alpha_i \notin I$). Then $m_1 + \ldots + m_r = m'_1 + \ldots + m'_r + m''_1 + \ldots + m''_r = |R_I(A)| + |A \setminus R_I(A)| = |A|$.

By Lemma 5.20, $A, m_1, \ldots, m_r$ is a simply excessively clusterizable A-configuration.

Finally, if $n_i = 0$, then $i \notin I$, $m'_i = 0$, also $k_i = 0$, so $m''_i = 0$, and $m_i = 0$. $\square$

Lemma 5.24. *Let $A, n_1, \ldots, n_r$ be a simple excessive cluster. Let $w \in W$.*
Denote by i (the only existing by Lemma 5.16) index such that $n_i > 0$.
Suppose that:
$wA \in \Delta^+$
and
for each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots
=
the coefficient in front of α_i in the decomposition of $w\beta$ into a linear combination of simple roots.
Then $wA, n_1, \ldots, n_r$ is a simple excessive cluster.

Proof. We use Lemma 5.16. We know that:
$n_j = 0$ for $j \ne i$, $n_i > 0$, and
$|A| = n_i$, and
$\alpha_i \in \operatorname{supp} \beta$ for all $\beta \in A$, and

A is an $\{i\}$-cluster.

By Lemma 5.17,

the coefficients in front of α_i in the decompositions of all roots $\beta \in A$ into linear combinations of simple roots are all 1, and

for each $\beta, \gamma \in A$, $\beta \neq \gamma$, we have $(\beta, \gamma) = 1$.

So, it follows from the lemma hypothesis that

the coefficients in front of α_i in the decompositions of all roots $\beta \in wA$ into linear combinations of simple roots are all 1,

and, since the action of W preserves scalar products,

for each $\beta, \gamma \in A$, $\beta \neq \gamma$, we have $(\beta, \gamma) = (w^{-1}\beta, w^{-1}\gamma) = 1$.

In particular, for each $\beta \in wA$ we have $\alpha_i \in \operatorname{supp} \beta$.

By Lemma 5.17 again, wA is an $\{i\}$-cluster.

We already know that $\alpha_i \in \operatorname{supp} \beta$ for all $\beta \in wA$.

$|wA| = |A| = n_i$.

The fact that [$n_j = 0$ for $j \neq i$, $n_i > 0$] does not depend on w.

By Lemma 5.16, wA is an excessive cluster. $\square$

Lemma 5.25. *Let $A, n_1, \ldots, n_r$ be a simply excessively clusterizable A-configuration. Let $w \in W$.*

Denote by I the set of simple roots α_i such that $n_i > 0$.

Suppose that:

$wA \subseteq \Delta^+$

and

for each $\beta \in A$, for each $\alpha_i \in I$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots

$=$

the coefficient in front of α_i in the decomposition of $w\beta$ into a linear combination of simple roots.

Then $wA, n_1, \ldots, n_r$ is a simply excessively clusterizable A-configuration.

Proof. Induction on $|A|$. If $A = \varnothing$, everything is clear. Suppose $A \neq \varnothing$.

By definition, there exists an index $i \in \{1, \ldots, r\}$ such that:

denote $k_i = n_i$ and $k_j = 0$ if $j \neq i$

then, in terms of this notation:

$k_i > 0$ and

$|R_{\{i\}}(A)| = k_i$ and

$R_{\{i\}}(A), k_1, \ldots, k_r$ is a simple excessive cluster and

$(A \setminus R_{\{i\}}(A)), n_1 - k_1, \ldots, n_r - k_r$ is simply excessively clusterizable.

By the definition of notation R, for each $\beta \in R_{\{i\}}(A)$ we have $\alpha_i \in \operatorname{supp} \beta$.

$R_{\{i\}}(A) \subseteq A$, and $\alpha_i \in I$ since $k_i > 0$, so

for each $\beta \in R_{\{i\}}(A)$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots

$=$

the coefficient in front of α_i in the decomposition of $w\beta$ into a linear combination of simple roots.

By Lemma 5.24,

$wR_{\{i\}}(A), k_1, \ldots, k_r$ is a simple excessive cluster.

Again, for each $\beta \in A$, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots

$=$

the coefficient in front of α_i in the decomposition of $w\beta$ into a linear combination of simple roots.

So, by the definition of notation R, for each $\beta \in A$, we have [$w\beta \in R_{\{i\}}(wA)$ iff $\beta \in R_{\{i\}}(A)$.] In other words, $wR_{\{i\}}(A) = R_{\{i\}}(wA)$.

Therefore, $w(A \setminus R_{\{i\}}(A)) = wA \setminus R_{\{i\}}(wA)$.

So, $R_{\{i\}}(wA), k_1, \ldots, k_r$ is a simple excessive cluster.

Recall that $(A \setminus R_{\{i\}}(A)), n_1 - k_1, \ldots, n_r - k_r$ is simply excessively clusterizable.

By the induction hypothesis, $(w(A \setminus R_{\{i\}}(A)) = wA \setminus R_{\{i\}}(wA)), n_1 - k_1, \ldots, n_r - k_r$ is simply excessively clusterizable.

By the definition of a simply excessively clusterizable A-configuration, $wA, n_1, \dots, n_r$ is a simply excessively clusterizable A-configuration. $\square$

6 Necessary condition of unique sortability

6.1 Basic sufficient conditions for non-unique sortability

Lemma 6.1. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that there exists a root $\alpha \in \Delta^+ \cap w\Delta^-$ such that the [coefficient in front of $f(\alpha)$ in the decomposition of α into a linear combination of simple roots] is at least 2,

then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. By Corollary 4.5, there exists an antireduced labeled sorting process such that when we perform a reflection along a root $\beta \in \Delta^+ \cap w\Delta^-$, the label at this root is $f(\beta)$. Denote the corresponding distribution of simple roots $\{1, \dots, \ell(w)\} \to \Pi$ by f_1. The D-multiplicities of labels of this sorting process are $n_1, \dots, n_r$.

In particular, when we perform the reflection σ_α, the label is $f(\alpha)$. By the definition of X-multiplicity, that means that the X-multiplicity of this sorting process is at least 2 (more precisely, it is a positive integer divisible by 2).

By Lemma 3.26, $C_{w,n_1,\dots,n_r}$ is the number of [labeled sorting processes of w with the distribution of labels f_1], counting their X-multiplicities, so, it is at least 2 since we have a labeled sorting process with distribution of labels f_1 and X-multiplicity at least 2. $\square$

Lemma 6.2. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = -1$ and $f(\alpha) = f(\beta)$,

then $C_{w,n_1,\dots,n_r} \geq 2$

Proof. $\alpha, \beta \in \Delta$, $(\alpha, \beta) = -1$, so $\alpha + \beta \in \Delta$.

$\alpha, \beta \in \Delta^+$, so $\alpha + \beta \in \Delta^+$.

$\alpha, \beta \in w\Delta^-$, so $w^{-1}\alpha, w^{-1}\beta \in \Delta^-$, so $w^{-1}(\alpha + \beta) = w^{-1}\alpha + w^{-1}\beta \in \Delta^-$, so $\alpha + \beta \in w\Delta^-$.

Therefore, $\alpha + \beta \in \Delta^+ \cap w\Delta^-$.

Denote $f(\alpha) = f(\beta) = \alpha_i$, $f(\alpha + \beta) = \alpha_j$.

Clearly, $\operatorname{supp}(\alpha+\beta) = \operatorname{supp}\alpha \cup \operatorname{supp}\beta$. $\alpha_j \in \operatorname{supp}(\alpha+\beta)$, so α_j is in at least one of ($\operatorname{supp}\alpha$, $\operatorname{supp}\beta$).

Without loss of generality, suppose that $\alpha_j \in \operatorname{supp}\alpha$.

Consider the following new simple root distribution g on $\Delta^+ \cap w\Delta^-$:

$g(\alpha + \beta) = \alpha_i$, $g(\alpha) = \alpha_j$, and $g(\gamma) = f(\gamma)$ for all other $\gamma \in \Delta^+ \cap w\Delta^-$.

$\alpha_i \in \operatorname{supp}\alpha$ and $\alpha_i \in \operatorname{supp}\beta$, so the coefficient in front of $g(\alpha + \beta) = \alpha_i$ in the decomposition of $\alpha + \beta$ into a linear combination of simple roots is at least 2.

The claim follows from Lemma 6.1. $\square$

Lemma 6.3. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.*

If there exist two simple root distributions $f, g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $\alpha \neq \beta$ such that f is α-compatible, g is β-compatible, and $f(\alpha) = g(\beta)$,

then $C_{w,n_1,\dots,n_r} \geq 2$

Proof. Denote $\alpha_i = g(\alpha) = f(\beta)$. Denote by L the following list of simple roots (i. e. a function $\{1, \dots, \ell(w)\} \to \Pi$): $\alpha_i, \alpha_1, \dots, \alpha_1, \dots, \alpha_i, \dots, \alpha_i, \dots, \alpha_r, \dots, \alpha_r$, where, after (excluding) the first α_i, [each α_j is written n_j times, except for α_i, which is written $n_i - 1$ times].

By Lemma 4.11, there exists a labeled sorting process for w that starts with α, the label at this α is $f(\alpha)$, and the whole list of labels is L.

And there is another labeled sorting process for w that starts with β, the label at this β is $g(\beta)$, and the whole list of labels is L.

By Lemma 3.26, $C_{w,n_1,\dots,n_r} \geq 2$. $\square$

Corollary 6.4. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $\alpha \neq \beta$ such that f is both α-compatible and β-compatible and $f(\alpha) = f(\beta)$,

then $C_{w,n_1,\dots,n_r} \geq 2$

Proof. This is the previous lemma with $f = g$. □

Lemma 6.5. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities, let $0 \leq k \leq \ell(w)$.*

Let $\beta_1, \dots, \beta_k$ be a labeled sorting process prefix of w with D-multiplicities $m_1, \dots, m_r$ of labels. Suppose that $m_i \leq n_i$. Denote $w_k = \sigma_{\beta_k} \dots \sigma_{\beta_1} w$.

Then $C_{w,n_1,\dots,n_r} \geq C_{w_k,n_1-m_1,\dots,n_r-m_r}$.

In particular, if $C_{w_k,n_1-m_1,\dots,n_r-m_r} \geq 2$, then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. Denote the list of labels of the labeled sorting process prefix $\beta_1, \dots, \beta_k$ by L.

Fix a function $\{k+1, \dots, \ell(w)\} \to \Pi$ with D-multiplicities $n_1 - m_1, \dots, n_r - m_r$ of simple roots. For example, fix the following list of simple roots: $\alpha_1, \dots, \alpha_1, \dots, \alpha_r, \dots, \alpha_r$, where α_i is repeated $n_i - m_i$ times. Denote this list by L'.

For each labeled sorting process of w_k with distribution of labels L, do the following. Denote this sorting process by $\beta_{k+1}, \dots, \beta_{\ell(w)}$. Write $\beta_1, \dots, \beta_k$ in front of $\beta_{k+1}, \dots, \beta_{\ell(w)}$, and assign the original labels to these $\beta_1, \dots, \beta_k$. We get a labeled sorting process of w with list of labels L, L'. The D-multiplicities of labels in L, L' are $n_1, \dots, n_r$. And the X-multiplicity of this sorting process of w is divisible by the X-multiplicity of the sorting process of w_k.

Note that we will get different labeled sorting process of w for different labeled sorting processes of w_k.

By Lemma 3.26, $C_{w_k,n_1-m_1,\dots,n_r-m_r}$ is the number of labeled sorting processes of w_k with list of labels L', counting their X-multiplicities, and $C_{w,n_1,\dots,n_r}$ is the number of labeled sorting processes of w with list of labels L, L', counting their X-multiplicities. So, $C_{w,n_1,\dots,n_r} \geq C_{w_k,n_1-m_1,\dots,n_r-m_r}$. □

Corollary 6.6. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities, let $0 \leq k \leq \ell(w)$.*

Let f be a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots.

Let $\beta_1, \dots, \beta_k$ be an antireduced labeled sorting process prefix with the label $f(\beta_i)$ at each β_i (this is well-defined by Lemma 3.16 and Corollary 3.17). Denote $w_k = \sigma_{\beta_k} \dots \sigma_{\beta_1} w$.

Denote by g the restriction of f onto $\Delta^+ \cap w_k\Delta^-$ (this is well-defined by the "moreover" part of Lemma 3.16), and denote by $p_1, \dots, p_r$ the D-multiplicities of simple roots in g.

Then $C_{w,n_1,\dots,n_r} \geq C_{w_k,p_1,\dots,p_r} > 0$.

In particular, if $C_{w_k,p_1,\dots,p_r} \geq 2$, then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. Clearly, if we denote the D-multiplicities of simple roots of the distribution of labels on $\beta_1, \dots, b_k$ by $m_1, \dots, m_r$, then $p_i = n_i - m_i$.

The fact that $C_{w_k,p_1,\dots,p_r} \geq 0$ follows from the presence of g, Proposition 4.4 and Lemma 3.26.

The rest of the claim now follows from Lemma 6.5. □

Lemma 6.7. *Let $w \in W$. Suppose that $\Delta^+ \cap w\Delta^-$ contains exactly one root α such that $w^{-1}\alpha \in -\Pi$.*

Then for every $\beta \in \Delta^+ \cap w\Delta^-$, $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$.

Proof. Fix $\beta \in \Delta^+ \cap w\Delta^-$. Denote $w^{-1}\alpha = -\alpha_i$ and $w^{-1}\beta = -\sum_j a_j \alpha_j$.

Clearly, $\operatorname{supp} w(a_i\alpha_i) = \operatorname{supp} \alpha$. Since α is the only root in $\Delta^\cap w\Delta^-$ such that $w^{-1}\alpha \in -Pi$, for all other roots α_j with $j \neq i$ we have $w(-\alpha_j) \notin \Delta^+ \cap w\Delta^-$. Clearly, $w(-\alpha_j) \in w\Delta^-$, so $w(-\alpha_j) \notin \Delta^+$ if $i \neq j$, and $w(-\alpha_j) \in \Delta^-$ if $i \neq j$.

Therefore, all coefficients in the decomposition of $w(-\sum_{j\neq i} a_j\alpha_j)$ into a linear combination of simple roots are nonpositive.

We also know that $\beta \in \Delta^+$, so all coefficients in its decomposition into a linear combination of simple roots are nonnegative.

Since all coefficients in the decomposition of $w(-\sum_{j\neq i} a_j\alpha_j)$ into a linear combination of simple roots are nonpositive, the (nonnegative) coefficients in the decomposition of β into a linear combination of simple roots are smaller than or equal to the corresponding (also nonnegative) coefficients in the decomposition of $w(a_i\alpha_i)$ into a linear combination of simple roots.

So, $\operatorname{supp}\beta \subseteq \operatorname{supp} w(a_i\alpha_i) = \operatorname{supp}\alpha$. $\square$

Lemma 6.8. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. Suppose that $\Delta^+ \cap w\Delta^-$ contains exactly one root α such that $w^{-1}\alpha \in -\Pi$.*

Suppose that there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$ and $f(\alpha) = f(\beta)$.

Then at least one of the following statements is true:

1. *$C_{w,n_1,\ldots,n_r} \geq 2$.*
2. *There exists a (possibly different) simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with (the same) D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exist $\beta', \beta'' \in \Delta^+ \cap w\Delta^-$ such that $\alpha \neq \beta'$, $\alpha \neq \beta''$, $(\beta', \beta'') = 0$ and $g(\beta') = g(\beta'') = f(\alpha)$.*

Proof. First, until the end of the proof, call a root $\gamma \in \Delta^+ \cap w\Delta^-$ *red* if $\gamma \neq \alpha$ and there exists a simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $g(\alpha) = g(\gamma) = f(\alpha)$.

Clearly, β is a red root.

Without loss of generality (after a possible change of f) we may assume that β is a [maximal in the sense of $\prec_w$] element of the set of {red roots γ such that $(\gamma, \alpha) = 0$}.

Suppose first that there exists a red root γ such that $(\gamma, \alpha) = -1$.

This means that there exists a simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $g(\alpha) = g(\gamma) = f(\alpha)$. By Lemma 6.2 (applied to the distribution g), $C_{w,n_1,\ldots,n_r} \geq 2$.

Now we suppose until the end of the proof that if γ is a red root, then $(\gamma, \alpha) = 0$ or $(\gamma, \alpha) = 1$.

Similarly, note that if there exists a red root γ such that the coefficient in front of $f(\alpha)$ in the decomposition of γ into a linear combination of simple roots is at least 2, then $C_{w,n_1,\ldots,n_r} \geq 2$ by lemma 6.1.

So, we also suppose until the end of the proof that if γ is a red root, then the coefficient in front of $f(\alpha)$ in the decomposition of γ into a linear combination of simple roots is 1.

Also, if the coefficient in front of $f(\alpha)$ in the decomposition of α into a linear combination of simple roots is at least 2, then $C_{w,n_1,\ldots,n_r} \geq 2$ by lemma 6.1.

So, we also suppose until the end of the proof that the coefficient in front of $f(\alpha)$ in the decomposition of α into a linear combination of simple roots is 1.

1. Consider the case when f is a β-compatible distribution.

   By Lemma 4.10, f is also an α-compatible distribution. By Corollary 6.4, $C_{w,n_1,\ldots,n_r} \geq 2$.

   END Consider the case when f is a β-compatible distribution.

2. Now consider the case that f is not a β-compatible distribution.

   By Lemma 4.9, this means that there exists a root $\delta \in \Delta^+ \cap w\Delta^-$ such that $\beta \prec_w \delta$, $(\beta, \delta) = 1$, $f(\delta) \in \operatorname{supp}\beta$, and $f(\beta) \in \operatorname{supp}\delta$.

   $(\beta, \delta) = 1$, so $\delta \neq \alpha$ since $(\beta, \alpha) = 0$.

   Since $f(\beta) \in \operatorname{supp}\delta$, $f(\delta) \in \operatorname{supp}\beta$, we can consider a new simple root distribution h on $\Delta^+ \cap w\Delta^-$: $h(\beta) = f(\delta)$, $h(\delta) = f(\beta)$, and $h(\epsilon) = \epsilon$ for all $\epsilon \in \Delta^+ \cap w\Delta^-$, $\epsilon \neq \beta$, $\epsilon \neq \delta$. Clearly, h has D-multiplicities $n_1, \ldots, n_r$ of simple roots as well as f. Note also that $h(\delta) = f(\beta) = f(\alpha) = h(\alpha)$. Therefore, δ is a red root, and there are only two possibilities for (δ, α): $(\delta, \alpha) = 0$ and $(\delta, \alpha) = 1$.

   In fact, $(\delta, \alpha) = 0$ is also impossible, because $\beta \prec_w \delta$, and β is a maximal with respect to $\prec_w$ element of the set of red roots orthogonal to α.

   So, $(\delta, \alpha) = 1$.

   By Lemma 2.7, $\alpha - \delta + \beta \in \Delta$. Denote $\beta' = \alpha - \delta + \beta$. Lemma 2.7 also says that $(\beta', \delta) = 0$. It also says that $(\beta', \alpha) = 1$, so $\alpha \neq \beta'$.

We are now supposing that the coefficients in front of $f(\alpha)$ in the decompositions of α and of all red roots into linear combinations of simple roots are all 1. By Lemma 2.8, $\beta' \in \Delta^+$, and the coefficient in front of $f(\alpha)$ in the decomposition of β' into a linear combination of simple roots is 1. In particular, $f(\alpha) \in \operatorname{supp} \beta'$.

$\beta \prec_w \delta$, so, by Lemma 2.15, $\beta' \in w\Delta^-$.

By Lemma 6.7, $\operatorname{supp} \beta' \subseteq \operatorname{supp} \alpha$, so $f(\beta') \in \operatorname{supp} \alpha$.

Set $\beta'' = \delta$ and define a new simple root distribution g on $\Delta^+ \cap w\Delta^-$ as follows:

$g(\alpha) = f(\beta')$.

$g(\beta') = g(\beta'') = f(\alpha) = f(\beta)$.

$g(\beta) = f(\beta'')$.

Clearly, g has D-multiplicities $n_1, \ldots, n_r$ of simple roots as well as f.

END consider the case that f is not a β-compatible distribution.

□

Lemma 6.9. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f \colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exist roots $\delta', \delta'' \in \Delta^+ \cap w\Delta^-$ such that $(\delta', \delta'') = 0$ and $f(\delta') = f(\delta'')$,

then $C_{w,n_1,\ldots,n_r} \geq 2$

Proof. We are going to construct two different labeled sorting processes with the same list of labels.

Both sorting processes will begin in the same way and proceed in the same way, while possible.

Set $w_0 = w$.

We perform the following *antisimple* reflections while we don't say we want to stop. We will denote the current element of W after i reflections by w_i.

While we perform these reflections, we will sometimes need to modify the distribution f. In rigorous terms, we will have several simple root distributions $f_0 = f, f_1, \ldots, f_k$ $(0 \leq k < \ell(w))$ such that when we perform the ith reflection (and it will be the ith reflection in both of the sorting processes we will construct), and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$ (recall that we are doing antisimple reflections, see Lemma 3.13), we assign (in both processes) the label $f_i(\gamma)$ to it. And when we modify our distribution later, i. e. when we define f_j with $j > i$, we don't change its value that was already assigned to a step of the sorting process, i. e. $f_j(\gamma)$ will be the same as $f_i(\gamma)$.

Also, all distributions f_i will have the same D-multiplicities of simple roots as f.

In the end, when we stop after k steps, it will be true that when we performed the ith reflection and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$, the label assigned to this reflection was $f_k(\gamma)$.

Also, while we perform this reflections, we will sometimes need to modify the values of δ' and δ''. Again, in rigorous terms, we will have two sequences of roots, $\delta'_0 = \delta', \delta'_1 \ldots, \delta'_k$ and $\delta''_0 = \delta'', \delta'_1 \ldots, \delta''_k$ such that $(\delta'_i, \delta''_i) = 0$, $f_i(\delta'_i) = f_i(\delta''_i) = f(\delta')$, and $\delta'_i, \delta''_i \in \Delta^+ \cap w_i\Delta^-$. In particular, this means that $|\Delta^+ \cap w_i\Delta^-| = \ell(w_i) \geq 2$, and this means that at a certain point we will have to stop explicitly, we cannot exhaust the whole $|\Delta^+ \cap w\Delta^-|$.

For each $i \in \mathbb{N}$, starting from $i = 1$.

1. If there exists $\gamma \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\gamma \in -\Pi$, $\gamma \neq \delta'_{i-1}$, $\gamma \neq \delta''_{i-1}$,

   then:

   Set $f_i = f_{i-1}$, $\delta'_i = \delta'_{i-1}$, $\delta''_i = \delta''_{i-1}$

   We are only performing antisimple reflections now, so by Lemma 3.13, $\Delta^+ \cap w_{i-1}\Delta^- \subseteq \Delta^+ \cap w\Delta^-$, and $\gamma \in \Delta^+ \cap w\Delta^-$, and f is defined on γ.

   we say that the ith step of both sorting processes will be $\beta_i = \gamma$ with label $f_i(\gamma)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

   $\beta_i \neq \delta'_i$, $\beta_i \neq \delta''_i$, so $\delta'_i, \delta''_i \in \Delta^+ \cap w_i\Delta^-$.

   And we CONTINUE with the next step of the sorting process (with the next value of i).

2. Otherwise, if ($w_{i-1}^{-1}\delta'_{i-1} \in -\Pi$ and $w_{i-1}^{-1}\delta''_{i-1} \in -\Pi$), then we say that we WANT TO STOP.

3. Otherwise, there is only one $\gamma \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\gamma \in -\Pi$, and this γ is either δ'_{i-1} or δ''_{i-1}.

   Without loss of generality, suppose that $\gamma = \delta'_{i-1}$.

   Restrict f_{i-1} onto $\Delta^+ \cap w_{i-1}\Delta^-$, and denote the result by g_{i-1}. Temporarily (until the end of this step of the sorting process) denote the D-multiplicities of simple roots in g_{i-1} by $m_1, \ldots, m_r$.

   Let us apply Lemma 6.8 to w_{i-1}, to the distribution g_{i-1}, and to δ'_{i-1} and δ''_{i-1}.

   Lemma 6.8 may tell us $C_{w_{i-1},m_1,\ldots,m_r} \geq 2$. Then by Corollary 6.6, $C_{w,n_1,\ldots,n_r} \geq 2$. Stop everything, we are done.

   Otherwise, Lemma 6.8 gives us a new simple root distribution, which we denote by g_i, on $\Delta^+ \cap w_{i-1}\Delta^-$ and a new pair of roots, which we denote by δ'_i and δ''_i, such that:

   the D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \ldots, m_r$.

   $\delta'_i, \delta''_i \in \Delta^+ \cap w_{i-1}\Delta^-$,

   $(\delta'_i, \delta''_i) = 0$,

   $g_i(\delta'_i) = g_i(\delta''_i) = g_{i-1}(\delta'_{i-1}) = f(\delta')$

   $\delta'_i \neq \gamma$, $\delta''_i \neq \gamma$.

   Expand this new distribution g_i to the whole $\Delta^+ \cap w\Delta^-$ using f_{i-1}. In rigorous terms, define the following new distribution f_i on $\Delta^+ \cap w\Delta^-$: $f_i(\alpha) = g_i(\alpha)$ if $\alpha \in \Delta^+ \cap w_{i-1}\Delta^-$, and $f_i(\alpha) = f_{i-1}(\alpha)$ otherwise.

   The D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \ldots, m_r$, so the D-multiplicities of simple roots in f_i are the same as the D-multiplicities of simple roots in f_{i-1}, they are $n_1, \ldots, n_r$.

   Now we again say that the ith step of both sorting processes will be $\beta_i = \gamma$ with label $f_i(\gamma)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

   Again, $\beta_i \neq \delta'_i$, $\beta_i \neq \delta''_i$, so $\delta'_i, \delta''_i \in \Delta^+ \cap w_i\Delta^-$.

   And we CONTINUE with the next step of the sorting process (with the next value of i).

END For each $i \in \mathbb{N}$, starting from $i = 1$.

After a certain number (denote it by k) of steps, we will stop. At this point we will have a simple root distribution f_k on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots, a sequence $\beta_1, \ldots, \beta_k$ of elements of $\Delta^+ \cap w\Delta^-$, a sequence $w_0 = w, w_1, \ldots, w_k$ of elements of W such that

[σ_{β_i} is an antisimple sorting reflection for w_{i-1}, and $w_i = \sigma_{\beta_i} w_i$],

and two roots $\delta'_k, \delta''_k \in \Delta^+ \cap w_k\Delta^-$ such that $(\delta'_k, \delta''_k) = 0$, $f_k(\delta'_k) = f_k(\delta''_k) = f(\delta')$, and $w^{-1}\delta'_k, w^{-1}\delta''_k \in -\Pi$.

Again restrict f_k onto $\Delta^+ \cap w_k\Delta^-$, and denote the result by g_k. Denote the D-multiplicities of simple roots in g_k by $m_1, \ldots, m_r$.

By Corollary 4.10, g_k is both δ'_k-compatible and δ''_k-compatible. By Corollary 6.4, $C_{w_k,m_1,\ldots,m_r} \geq 2$. By Corollary 6.6, $C_{w,n_1,\ldots,n_r} \geq 2$. □

6.2 Uniqueness and non-uniqueness of sortability in case of excessive configuration

Definition 6.10. Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.

We say that it is *excessive* if $\Delta^+ \cap w\Delta^-, n_1, \ldots, n_r$ is an excessive A-configuration.

Definition 6.11. Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.

We say that it is a *free-first-choice configuration* if for each $\alpha \in \Delta^+ \cap w\Delta^-$ and for each $\alpha_i \in \operatorname{supp}\alpha$ such that $n_i > 0$ there exists a simple root distribution f on $\Delta^+ \cap w\Delta-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $f(\alpha) = \alpha_i$

Lemma 6.12. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. If it is excessive, then it is a free-first-choice configuration.*

Proof. Fix $\alpha \in \Delta^+ \cap w\Delta^-$ and an involved root $\alpha_i \in \operatorname{supp} \alpha$. Set $A = (\Delta^+ \cap w\Delta^-) \setminus \alpha$.

Denote by J the set of indices j $(1 \le j \le r)$ such that $n_j > 0$. Note that $i \in J$.

Set $m_j = n_j$ for $j \ne i$ and $m_i = n_i - 1$. Since $n_i > 0$, $m_j \ge 0$ for all j $(1 \le j \le r)$.

Let $I \subseteq J$. Clearly, $\sum_{j \in I} n_j \ge \sum_{j \in I} m_j$ and $|R_I(A)| \ge |R_I(w)| - 1$.

If $I \ne J$, then $|R_I(A)| \ge |R_I(w)| - 1 > (\sum_{j \in I} n_j) - 1 \ge (\sum_{j \in I} m_j) - 1$. Since all number here are integers, $|R_I(A)| \ge \sum_{j \in I} m_j$.

If $I = J$, then $\sum_{j \in I} m_j = (\sum_{j \in J} n_j) - 1$, and $|R_I(A)| \ge |R_I(w)| - 1 \ge (\sum_{j \in I} n_j) - 1 = \sum_{j \in I} m_j$.

So, for all $I \subseteq J$ we have $|R_I(A)| \ge \sum_{j \in I} m_j$.

Denote by J' the set of indices $j \in \{1, \ldots, r\}$ such that $m_j > 0$. Clearly, $J' \subseteq J$. So, for all $I \subseteq J'$ we also have $|R_I(A)| \ge \sum_{j \in I} m_j$. By Lemma 4.2, there exists a simple root distribution g on A with D-multiplicities $m_1, \ldots, m_r$.

Set $f(\alpha) = \alpha_i$ and $f(\beta) = g(\beta)$ for $\beta \in A$. This is a distribution of simple roots on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$. ☐

Definition 6.13. Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.

We say that this configuration *has large essential coordinates* if there exists $\alpha \in \Delta^+ \cap w\Delta^-$ and $\alpha_i \in \Pi$ such that $n_i > 0$ and the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is at least 2.

We say that this configuration *has small essential coordinates* if it does not have large essential coordinates.

Lemma 6.14. *Let $w, n_1, \ldots, n_r$ be a free-first-choice configuration of D-multiplicities. If it has large essential coordinates,*

then $C_{w,n_1,\ldots,n_r} \ge 2$.

Proof. Since the configuration has large essential coordinates, there exists $\alpha \in \Delta^+ \cap w\Delta^-$ and $\alpha_i \in \Pi$ such that $n_i > 0$ and the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is at least 2.

By the definition of a free-first-choice configuration, there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $f(\alpha) = \alpha_i$. The claim follows from Lemma 6.1. ☐

Lemma 6.15. *Let $w, n_1, \ldots, n_r$ be a free-first-choice configuration of D-multiplicities.*

If there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = -1$ and an involved *simple root $\alpha_i \in \operatorname{supp} \alpha \cap \operatorname{supp} \beta$,*

then $C_{w,n_1,\ldots,n_r} \ge 2$.

Proof. $\alpha, \beta \in \Delta$, $(\alpha, \beta) = -1$, so $\alpha + \beta \in \Delta$.

$\alpha, \beta \in \Delta^+$, so $\alpha + \beta \in \Delta^+$.

$\alpha, \beta \in w\Delta^-$, so $w^{-1}\alpha, w^{-1}\beta \in \Delta^-$, so $w^{-1}(\alpha + \beta) = w^{-1}\alpha + w^{-1}\beta \in \Delta^-$, so $\alpha + \beta \in w\Delta^-$.

Therefore, $\gamma = \alpha + \beta \in \Delta^+ \cap w\Delta^-$.

Since $\alpha_i \in \operatorname{supp} \alpha$ and $\alpha_i \in \operatorname{supp} \beta$, the coefficient in front of α_i in the decomposition of $\gamma = \alpha + \beta$ into a linear combination of simple roots is at least 2.

α_i is an involved root, so the configuration $w, n_1, \ldots, n_r$ has large essential coordinates. The claim follows from Lemma 6.15. ☐

Definition 6.16. Let $w \in W$. We call a simple root distribution f on $\Delta^+ \cap w\Delta^-$ *flexible* if there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$, $f(\beta) \in \operatorname{supp} \alpha$, and $f(\alpha) \in \operatorname{supp} \beta$.

Lemma 6.17. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. If there exist two simple root distributions f and g on $\Delta^+ \cap w\Delta^-$, both with D-multiplicities $n_1, \ldots, n_r$ of simple roots, and roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $w^{-1}\alpha, w^{-1}\beta \in -\Pi$, $\alpha \ne \beta$, $f(\alpha) = g(\beta)$,*

then $C_{w,n_1,\ldots,n_r} \ge 2$.

Proof. By Lemma 4.10, f is α-compatible, and g is β-compatible.

By Lemma 6.3, $C_{w,n_1,\ldots,n_r} \geq 2$. □

Lemma 6.18. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. If there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots and roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $\alpha \neq \beta$ such that $w^{-1}\alpha, w^{-1}\beta \in -\Pi$, $f(\alpha) \in \operatorname{supp} \beta$, and $f(\beta) \in \operatorname{supp} \alpha$,*

then $C_{w,n_1,\ldots,n_r} \geq 2$.

Proof. Consider (possibly) another simple roots distribution g on $\Delta^+ \cap w\Delta^-$: $g(\alpha) = f(\beta)$, $g(\beta) = f(\alpha)$, and $g(\gamma) = f(\gamma)$ for all other $\gamma \in \Delta^+ \cap w\Delta^-$. Since $f(\alpha) \in \operatorname{supp} \beta$ and $f(\beta) \in \operatorname{supp} \alpha$, this is really a simple root distribution. Clearly, it also has D-multiplicities $n_1, \ldots, n_r$ of simple roots. The claim follows from Lemma 6.17. □

Lemma 6.19. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities that has small essential coordinates. Suppose that $\Delta^+ \cap w\Delta^-$ contains exactly one root α such that $w^{-1}\alpha \in -\Pi$.*

Suppose that there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$ and $f(\alpha) \in \operatorname{supp} \beta$.

Then at least one of the following statements is true:

1. *$C_{w,n_1,\ldots,n_r} \geq 2$.*
2. *There exists a simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ whose restriction to $\Delta^+ \cap (\sigma_\alpha w)\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \alpha$ is flexible.*

Proof. The proof is very similar to the proof of Lemma 6.8.

First, until the end of the proof, call a root $\gamma \in \Delta^+ \cap w\Delta^-$ *red* if $\gamma \neq \alpha$ and $f(\alpha) \in \operatorname{supp} \gamma$.

Clearly, β is a red root.

Without loss of generality we may assume that β is a [maximal in the sense of $\prec_w$] element of the set of {red roots γ such that $(\gamma, \alpha) = 0$}.

Denote $\alpha_i = f(\alpha)$. Since f is a simple root distribution with D-multiplicities $n_1, \ldots, n_r$ of simple roots and $f(\alpha) = \alpha_i$, $n_i > 0$.

Assume that there exists a red root γ such that $(\gamma, \alpha) = -1$.

This means that $f(\gamma) = f(\alpha)$, in particular, $f(\alpha) \in \operatorname{supp} \alpha$, $f(\alpha) \in \operatorname{supp} \gamma$.

By Lemma 2.5, $\alpha + \gamma \in \Delta$.

$\alpha, \gamma \in \Delta^+$, so $\alpha + \gamma \in \Delta^+$.

$\alpha, \gamma \in w\Delta^-$, so $\alpha + \gamma \in w\Delta^-$.

Therefore, $\alpha + \gamma \in \Delta^+ \cap w\Delta^-$.

$f(\alpha) \in \operatorname{supp} \alpha$, $f(\alpha) \in \operatorname{supp} \gamma$, so, the coefficient in front of $f(\alpha)$ in the decomposition of $\alpha + \gamma$ into the linear combination of simple roots is at least 2. We know that $n_i > 0$, so $w, n_1, \ldots, n_r$ is actually a configuration that has large essential coordinates. A contradiction.

Therefore, if γ is a red root, then $(\gamma, \alpha) = 0$ or $(\gamma, \alpha) = 1$.

By Lemma 6.7, $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$, so $f(\beta) \in \operatorname{supp} \alpha$. We also know that $f(\alpha) \in \operatorname{supp} \beta$.

Consider another simple roots distribution h on $\Delta^+ \cap w\Delta^-$: $h(\alpha) = f(\beta)$, $h(\beta) = f(\alpha)$, and $h(\gamma) = f(\gamma)$ for all other $\gamma \in \Delta^+ \cap w\Delta^-$. Since $f(\alpha) \in \operatorname{supp} \beta$ and $f(\beta) \in \operatorname{supp} \alpha$, this is really a simple root distribution. Clearly, it also has D-multiplicities $n_1, \ldots, n_r$ of simple roots.

1. Consider the case when h is a β-compatible distribution.

   By Lemma 4.10, f is an α-compatible distribution. By Lemma 6.3, $C_{w,n_1,\ldots,n_r} \geq 2$.

   END Consider the case when h is a β-compatible distribution.

2. Now consider the case that h is not a β-compatible distribution.

   By Lemma 4.9, this means that there exists a root $\gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta \prec_w \gamma$, $(\beta, \gamma) = 1$, $h(\gamma) \in \operatorname{supp} \beta$, and $h(\beta) \in \operatorname{supp} \gamma$.

   $(\beta, \gamma) = 1$, so $\gamma \neq \alpha$ since $(\beta, \alpha) = 0$.

   $\gamma \neq \alpha$, $\gamma \neq \beta$, so $f(\gamma) = h(\gamma) \in \operatorname{supp} \beta$, and $h(\beta) = f(\alpha) \in \operatorname{supp} \gamma$.

$f(\alpha) \in \operatorname{supp} \gamma$, so γ is a red root, and $(\gamma, \alpha) ne - 1$.

$(\gamma, \alpha) = 0$ is also impossible since $\beta \prec_w \gamma$, and we would have a contradiction with the minimality of β with respect to $\prec_w$ in the set of red roots orthogonal to α.

So, $(\gamma, \alpha) = 1$. Recall that $(\alpha, \beta) = 0$.

Set $\delta = \alpha - \gamma + \beta$. By Lemma 2.7, $\delta \in \Delta$ and $(\delta, \gamma) = 0$.

By Lemma 2.7, $\alpha - \delta + \beta \in \Delta$. Lemma 2.7 also says that $(\gamma, \delta) = 0$. It also says that $(\beta, \delta) = 1$, $(\delta, \alpha) = 1$, so $\alpha \neq \delta$.

Since $w, n_1, \ldots, n_r$ has small essential coordinates, and $n_i > 0$ that the coefficients in front of $f(\alpha) = \alpha_i$ in the decompositions of α and of all red roots into linear combinations of simple roots are all 1. By Lemma 2.8, $\delta \in \Delta^+$, and the coefficient in front of α_i in the decomposition of δ into a linear combination of simple roots is 1. In particular, $f(\alpha) \in \operatorname{supp} \delta$.

$\beta \prec_w \gamma$, so, by Lemma 2.15, $\delta \in w\Delta^-$. Therefore, $\delta \in \Delta^+ \cap w\Delta^-$.

Now let us check that $f(\gamma) \in \operatorname{supp} \delta$ or $f(\delta) \in \operatorname{supp} \gamma$.

Assume the contrary: $f(\gamma) \notin \operatorname{supp} \delta$ and $f(\delta) \notin \operatorname{supp} \gamma$. Recall that $f(\gamma) \in \operatorname{supp} \beta$. Recall also that $(\delta, \beta) = 1$. So, $\beta - \delta \in \Delta$, and either $\beta - \delta \in \Delta^-$, or $\beta - \delta \in \Delta^+$.

But if $\beta - \delta \in \Delta^-$, then $\beta \prec \delta$, so $\operatorname{supp} \beta \subseteq \operatorname{supp} \delta$, and it is impossible to have $f(\gamma) \in \operatorname{supp} \beta$ and $f(\gamma) \notin \operatorname{supp} \delta$, a contradiction.

So, $\beta - \delta \in \Delta^+$. Then $\alpha \prec \gamma = \beta - \delta + \alpha$, so $\operatorname{supp} \alpha \subseteq \operatorname{supp} \gamma$. By Lemma 6.7, $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$, so $\operatorname{supp} \beta \subseteq \operatorname{supp} \gamma$.

Also, $\beta - \delta \in \Delta^+$, so $\delta \prec \beta$, and $\operatorname{supp} \delta \subseteq \operatorname{supp} \beta$. We know that $f(\delta) \in \operatorname{supp} \delta$, so $f(\delta) \in \operatorname{supp} \beta$. We know that $\operatorname{supp} \beta \subseteq \operatorname{supp} \gamma$, so $f(\delta) \in \operatorname{supp} \gamma$, a contradiction.

Therefore, $f(\gamma) \in \operatorname{supp} \delta$ or $f(\delta) \in \operatorname{supp} \gamma$.

Let us consider 3 cases:

(a) $f(\gamma) \in \operatorname{supp} \delta$ and $f(\delta) \in \operatorname{supp} \gamma$. Set $g = f$. Then $g(\delta) \in \operatorname{supp} \gamma$, $g(\gamma) \in \operatorname{supp} \delta$.

(b) $f(\gamma) \in \operatorname{supp} \delta$, but $f(\delta) \notin \operatorname{supp} \gamma$. Recall that $f(\alpha) \in \operatorname{supp} \delta$. By Lemma 6.7, $\operatorname{supp} \delta \subseteq \operatorname{supp} \alpha$, so $f(\delta) \in \operatorname{supp} \alpha$.

   Set $g(\alpha) = f(\delta)$, $g(\delta) = f(\alpha)$, and $g(\epsilon) = f(\epsilon)$ for all other $\epsilon \in \Delta^+ \cap w\Delta^-$. This is a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots.

   Recall also that $f(\alpha) \in \operatorname{supp} \gamma$.

   Summarizing, $g(\delta) = f(\alpha) \in \operatorname{supp} \gamma$, $g(\gamma) = f(\gamma) \in \operatorname{supp} \delta$.

(c) $f(\delta) \in \operatorname{supp} \gamma$, but $f(\gamma) \notin \operatorname{supp} \delta$. Similarly to the previous case:

   Recall that $f(\alpha) \in \operatorname{supp} \gamma$. By Lemma 6.7, $\operatorname{supp} \gamma \subseteq \operatorname{supp} \alpha$, so $f(\gamma) \in \operatorname{supp} \alpha$.

   Set $g(\alpha) = f(\gamma)$, $g(\gamma) = f(\alpha)$, and $g(\epsilon) = f(\epsilon)$ for all other $\epsilon \in \Delta^+ \cap w\Delta^-$. This is a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots.

   Recall also that $f(\alpha) \in \operatorname{supp} \delta$.

   Summarizing, $g(\gamma) = f(\alpha) \in \operatorname{supp} \delta$, $g(\delta) = f(\delta) \in \operatorname{supp} \gamma$.

END consider 3 cases.

So, we have constructed a simple root distribution g on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $g(\delta) \in \operatorname{supp} \gamma$, $g(\gamma) \in \operatorname{supp} \delta$.

Recall that $\alpha \neq \delta$, $\alpha \neq \gamma$, and $(\gamma, \delta) = 0$ so the restriction of g to $(\Delta^+ \cap w\Delta^-) \setminus \alpha$ is flexible.

END consider the case that h is not a β-compatible distribution.

$\square$

Lemma 6.20. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities that has small essential coordinates.*

If there exists a flexible simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots,

then $C_{w,n_1,\ldots,n_r} \geq 2$

Proof. The proof is very similar to the proof of Lemma 6.9

Set $w_0 = w$.

We perform the following *antisimple* reflections while we don't say we want to stop. This way we construct a labeled antisimple sorting process prefix. Again, we will denote the current element of W after i reflections by w_i.

Again, we will have several simple root distributions $f_0 = f, f_1, \ldots, f_k$ $(0 \leq k < \ell(w))$ such that when we perform the ith reflection (and it will be the ith reflection in both of the sorting processes we will construct), and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$ (recall that we are doing antisimple reflections, see Lemma 3.13), we assign the label $f_i(\gamma)$ to it. And when we modify our distribution later, i. e. when we define f_j with $j > i$, we don't change its value that was already assigned to a step of the sorting process, i. e. $f_j(\gamma)$ will be the same as $f_i(\gamma)$.

Also, all distributions f_i will have the same D-multiplicities of simple roots as f.

In the end, when we stop after k steps, it will be true that when we performed the ith reflection and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$, the label assigned to this reflection was $f_k(\gamma)$.

We will also maintain the following fact: the restriction of f_i onto $\Delta^+ \cap w_i\Delta^-$ $(i \geq 0)$ is flexible.

For each $i \in \mathbb{N}$, starting from $i = 1$.

1. If there exist two roots $\gamma, \gamma' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\gamma, w_{i-1}^{-1}\gamma' \in -\Pi$, $f_{i-1}(\gamma) \in \operatorname{supp}\gamma'$, $f_{i-1}(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$ then we say that we WANT TO STOP.

2. Otherwise, if there exist three *different* roots $\alpha, \gamma, \gamma' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\alpha \in -Pi$, $f_{i-1}(\gamma) \in \operatorname{supp}\gamma'$, $f_{i-1}(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$, then:

   Set $f_i = f_{i-1}$

   we say that the ith step of the sorting process prefix will be $\beta_i = \alpha$ with label $f_i(\alpha)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

   $\Delta^+ \cap w_i\Delta^-$ still contains γ and γ', so the restriction of f_i to $\Delta^+ \cap w_i\Delta^-$ is flexible.

   And we CONTINUE with the next step of the sorting process (with the next value of i).

3. Otherwise:

   We know that (we are maintaining the fact that) the restriction of f_{i-1} to $\Delta^+ \cap w_{i-1}\Delta^-$ is flexible. So, there exist $\gamma, \gamma' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $f_{i-1}(\gamma) \in \operatorname{supp}\gamma'$, $f_{i-1}(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$.

   By Lemma 3.11, there exists $\alpha \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\alpha \in -Pi$.

   All three roots α, γ, γ' cannot be different, this would be case 2. But $\gamma \neq \gamma'$ since $(\gamma, \gamma') = 0$. So, $\alpha = \gamma$ or $\alpha = \gamma'$, without loss of generality let us suppose that $\alpha = \gamma$.

   Note that $w_{i-1}^{-1}\gamma' \notin -\Pi$, otherwise this would be case 1.

   Also, we cannot have another root $\alpha' \in \Delta^+ \cap w_{i-1}\Delta^-$, different from $\alpha = \gamma$, such that $w_{i-1}^{-1}\alpha' \in -\Pi$, this would also be case 2. In other words, there exists exactly one root $\alpha' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\alpha' \in -\Pi$, and this root is α.

   Restrict f_{i-1} onto $\Delta^+ \cap w_{i-1}\Delta^-$, and denote the result by g_{i-1}. Temporarily (until the end of this step of the sorting process) denote the D-multiplicities of simple roots in g_{i-1} by $m_1, \ldots, m_r$.

   We are going to apply Lemma 6.19 to w_{i-1}. The only condition we have to check is that the configuration $w_{i-1}, m_1, \ldots, m_r$ has small essential coordinates. But we are doing only antisimple reflections, so $\Delta^+ \cap w_{i-1}\Delta^- \subseteq \Delta^+ \cap w\Delta^-$. Also, $n_j \geq m_j$ by the definition of m_j. So, if for some $\delta \in \Delta^+ \cap w_{i-1}\Delta^-$, the coefficient in front of some α_j in the decomposition of δ into a linear combination of simple roots is at least 2, and $m_j > 0$, then $n_j > 0$, and $\delta \in \Delta^+ \cap w\Delta^-$. But this is impossible since $w, n_1, \ldots, n_r$ is a configuration with small essential coordinates.

So, the configuration $w_{i-1}, m_1, \dots, m_r$ has small essential coordinates, and we can use Lemma 6.19.

Lemma 6.19 may tell us $C_{w_{i-1},m_1,\dots,m_r} \geq 2$. Then by Corollary 6.6, $C_{w,n_1,\dots,n_r} \geq 2$. Stop everything, we are done.

Otherwise, Lemma 6.8 gives us a new simple root distribution, which we denote by g_i, on $\Delta^+ \cap w_{i-1}\Delta^-$ such that:

the D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \dots, m_r$,

and the restriction of g_i to $(\Delta^+ \cap w_{i-1}\Delta^-) \setminus \alpha$ is flexible.

Expand this new distribution g_i to the whole $\Delta^+ \cap w\Delta^-$ using f_{i-1}. In rigorous terms, define the following new distribution f_i on $\Delta^+ \cap w\Delta^-$: $f_i(\delta) = g_i(\delta)$ if $\delta \in \Delta^+ \cap w_{i-1}\Delta^-$, and $f_i(\delta) = f_{i-1}(\delta)$ otherwise.

The D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \dots, m_r$, so the D-multiplicities of simple roots in f_i are the same as the D-multiplicities of simple roots in f_{i-1}, they are $n_1, \dots, n_r$.

Now we again say that the ith step of both sorting processes will be $\beta_i = \alpha$ with label $f_i(\alpha)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

The restriction of f_i to $\Delta^+ \cap w_i\Delta^-$ is the same as the restriction of g_i to $(\Delta^+ \cap w_{i-1}\Delta^-) \setminus \alpha$, it is flexible.

And we CONTINUE with the next step of the sorting process (with the next value of i).

END For each $i \in \mathbb{N}$, starting from $i = 1$.

After a certain number (denote it by k) of steps, we will stop. At this point we will have a simple root distribution f_k on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots, a sequence $\beta_1, \dots, \beta_k$ of elements of $\Delta^+ \cap w\Delta^-$, a sequence $w_0 = w, w_1, \dots, w_k$ of elements of W such that σ_{β_i} is an antisimple sorting reflection for w_{i-1}, and $w_i = \sigma_{\beta_i} w_i$, and two roots $\gamma, \gamma' \in \Delta^+ \cap w_k\Delta^-$ such that $w_k^{-1}\gamma, w_k^{-1}\gamma' \in -\Pi$, $f_k(\gamma) \in \operatorname{supp}\gamma'$, $f_k(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$.

Again restrict f_k onto $\Delta^+ \cap w_k\Delta^-$, and denote the result by g_k. We know (we were maintaining the fact that) g_k is flexible. Denote the D-multiplicities of simple roots in g_k by $m_1, \dots, m_r$.

By Lemma 6.18, $C_{w_k,m_1,\dots,m_r} \geq 2$. By Corollary 6.6, $C_{w,n_1,\dots,n_r} \geq 2$. □

Lemma 6.21. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities.*
If there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$ and $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$,
then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. If the configuration has large essential coordinates, $C_{w,n_1,\dots,n_r} \geq 2$ by lemma 6.14.

Suppose that the configuration has small essential coordinates. By Lemma 5.11, there exists a simple root $\alpha_i \in \operatorname{supp}\beta$ involved in $w, n_1, \dots, n_r$.

By Lemma 6.12, $w, n_1, \dots, n_r$ is a free-first-choice configuration, so there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that $f(\alpha) = \alpha_i$.

So, $f(\alpha) \in \operatorname{supp}\beta$. Also, $f(\beta) \in \operatorname{supp}\alpha$ since $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$. So, f is a flexible distribution.

By Lemma 6.20, $C_{w,n_1,\dots,n_r} \geq 2$. □

Lemma 6.22. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities.*
If there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $\alpha \neq \beta$ such that $w^{-1}\alpha, w^{-1}\beta \in -\Pi$, and an involved *simple root $\alpha_i \in \operatorname{supp}\alpha \cap \operatorname{supp}\beta$,*
then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. By Lemma 6.12, $w, n_1, \dots, n_r$ is a free-first-choice configuration, so there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that $f(\alpha) = \alpha_i$ and (possibly) another simple root distribution g on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that $g(\beta) = \alpha_i$.

The claim follows from Lemma 6.17. □

Lemma 6.23. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\dots,n_r} = 1$, Let $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $\alpha \neq \beta$, $\operatorname{supp}\alpha \subseteq \operatorname{supp}\beta$.*

Then $(\alpha, \beta) = 1$.

Proof. If $(\alpha, \beta) = -1$, then:

By Lemma 5.11, there exists a simple root $\alpha_i \in \operatorname{supp}\alpha$ involved in $w, n_1, \dots, n_r$. Then $\alpha_i \in \operatorname{supp}\beta$, and we have a contradiction with Lemma 6.15.

If $(\alpha, \beta) = 0$, then we have a contradiction with Lemma 6.21. □

Lemma 6.24. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\dots,n_r} = 1$, Let $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $(\alpha, \beta) = -1$.*

Then $\gamma = \alpha + \beta$ is a maximal (in the sense of $\prec$) element of $\Delta^+ \cap w\Delta^-$.

Proof. $\alpha, \beta \in \Delta$, $(\alpha, \beta) = -1$, so $\alpha + \beta \in \Delta$.

$\alpha, \beta \in \Delta^+$, so $\alpha + \beta \in \Delta^+$.

$\alpha, \beta \in w\Delta^-$, so $w^{-1}\alpha, w^{-1}\beta \in \Delta^-$, so $w^{-1}(\alpha + \beta) = w^{-1}\alpha + w^{-1}\beta \in \Delta^-$, so $\alpha + \beta \in w\Delta^-$.

Therefore, $\gamma = \alpha + \beta \in \Delta^+ \cap w\Delta^-$. Clearly, $\alpha \prec \gamma$, $\beta \prec \gamma$.

Assume that there exists $\delta \in \Delta^+ \cap w\Delta^-$, $\gamma \prec \delta$. Then $\alpha \prec \delta$ and $\beta \prec \delta$. So, $\operatorname{supp}\alpha \subseteq \operatorname{supp}\delta$ and $\operatorname{supp}\beta \subseteq \operatorname{supp}\delta$. By Lemma 6.23, $(\alpha, \delta) = 1$ and $(\beta, \delta) = 1$. So $(\gamma, \delta) = 2$, a contradiction. □

Lemma 6.25. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\dots,n_r} = 1$,*

Let α be a maximal (in the sense of $\prec$) element of $\Delta^+ \cap w\Delta^-$.

If $w^{-1}\alpha \notin -\Pi$, then there exist roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

Proof. By Lemma 3.10, there are two possibilities:

Either there exists $\delta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec \delta$, $(\delta, \alpha) = 1$, and $\delta - \alpha \notin \Delta^+ \cap w\Delta^-$,

or there exist roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

But the existence of such a δ is impossible since α is a maximal (in the sense of $\prec$) element of $\Delta^+ \cap w\Delta^-$. □

Lemma 6.26. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\dots,n_r} = 1$, and let α be a $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$.*

Suppose that there exist roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

If $\delta \in \Delta^+ \cap w\Delta^-$, $\delta \prec \alpha$, $\delta \neq \beta$, $\delta \neq \gamma$, then there are exactly two possibilities:

1. $(\delta, \beta) = 1$, $(\delta, \gamma) = 0$, $(\delta, \alpha) = 1$.

2. $(\delta, \gamma) = 1$, $(\delta, \beta) = 0$, $(\delta, \alpha) = 1$.

Proof. $\delta \prec \alpha$, so by Lemma 6.23, $(\delta, \alpha) = 1$, and $(\delta, \beta) + (\delta, \gamma) = 1$

Since $\delta \neq \beta$ and $\delta \neq \gamma$, each of the numbers (δ, β) and (δ, γ) can be either 1, or 0, or -1.

The sum of two numbers from the set $\{1, 0, -1\}$ can equal 1 only if one of these numbers is 1, and the other one is 0. □

Lemma 6.27. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\dots,n_r} = 1$, and let α be a $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$.*

Suppose that there exist roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

Denote by L the set consisting of β and all roots $\delta \in \Delta^+ \cap w\Delta^-$, such that $\delta \prec \alpha$, $\delta \neq \beta$, $\delta \neq \gamma$, and $(\delta, \beta) = 1$, $(\delta, \gamma) = 0$, $(\delta, \alpha) = 1$.

Let β' be a $\prec$-maximal element of L.

Then $w^{-1}\beta' \in -\Pi$ and $\beta \preceq \beta'$.

Proof. First, note that $\beta - \alpha = -\gamma \in w\Delta^+$, so $\alpha \prec_w \beta$.

Our next goal is to check that $\beta \preceq_w \beta'$ and $\beta \preceq \beta'$. If $\beta = \beta'$, this is clear, suppose that $\beta \neq \beta'$ (until we say this assumption is over).

Then by construction, $(\beta, \beta') = 1$, $\beta' - \beta \in \Delta$, and β and β' are $\prec$-comparable. $\beta' \prec \beta$ is impossible since we chose a maximal element of L, so $\beta \prec \beta'$.

Now, $\beta' - \beta \in \Delta^+ \cap w\Delta^-$ is impossible by Lemma 6.24 since $\beta' \prec \alpha$ and β' cannot be a $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$. But we already know that $\beta \prec \beta'$, so $\beta' - \beta \in \Delta^+$, so $\beta' - \beta \notin w\Delta^-$, $\beta' - \beta \in w\Delta^+$, and $\beta \prec_w \beta'$.

END suppose that $\beta \neq \beta'$.

So, we see that in both cases, $\beta \preceq_w \beta'$ and $\beta \preceq \beta'$.

$\alpha \prec_w \beta$, $\beta \preceq_w \beta$, so $\alpha \prec_w \beta'$.

$\beta' \prec \alpha$ by construction, so $(\beta', \alpha) = 1$ by Lemma 6.23, and $\alpha - \beta' \in \Delta$ by Lemma 2.5. Denote $\gamma' = \alpha - \beta'$. $\beta' \prec \alpha$, so $\gamma' \in \Delta^+$. $\alpha \prec_w \beta'$, so $\gamma' \in w\Delta^-$, and $\gamma' \in \Delta^+ \cap w\Delta^-$.

Assume that $w^{-1}\beta' \notin -\Pi$.

Then by Lemma 3.10, there are two possibilities:

Either there exist roots $\delta', \delta'' \in \Delta^+ \cap w\Delta^-$ such that $\beta' = \delta' + \delta''$, but this is impossible by Lemma 6.24 since $\beta' \prec \alpha$ and β' cannot be a $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$.

Or there exists $\beta'' \in \Delta^+ \cap w\Delta^-$ such that $\beta' \prec \beta''$, $(\beta', \beta'') = 1$, and $\beta'' - \beta' \notin \Delta^+ \cap w\Delta^-$. We have to consider this possibility in more details.

First, $(\beta', \beta'') = 1$, so $\beta'' - \beta' \in \Delta$.

$\beta' \prec \beta''$, so $\beta'' - \beta' \in \Delta^+$.

$\beta'' - \beta' \notin \Delta^+ \cap w\Delta^-$, so $\beta'' - \beta' \notin w\Delta^-$, $\beta'' - \beta' \in w\Delta^+$, and $\beta' \prec_w \beta''$.

Recall that $\alpha \prec_w \beta'$, so $\alpha \prec_w \beta''$, and $\alpha \neq \beta''$.

Let us find (β'', γ').

If $(\beta'', \gamma') = 1$, then $(\beta'', \alpha) = (\beta'', \beta') + (\beta'', \gamma') = 2$, but $\alpha \neq \beta''$, so this is impossible.

If $(\beta'', \gamma') = -1$, then $\beta'' + \gamma' \in \Delta$ by Lemma 2.5, $\beta'' + \gamma' \in \Delta^+$ since $\beta'', \gamma' \in \Delta^+$, $\beta'' + \gamma' \in w\Delta^-$ since $\beta'', \gamma' \in w\Delta^-$, and $\alpha = \beta' + \gamma' \prec \beta'' + \gamma'$ since $\beta' \prec \beta''$. A contradiction with the $\prec$-maximality of α.

Therefore, $(\beta'', \gamma') = 0$.

Then $(\beta'', \alpha) = (\beta'', \beta') + (\beta'', \gamma') = 1$, $\alpha - \beta'' \in \Delta$, and α is $\prec$-comparable with β''. Since α is a $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$, $\beta'' \prec \alpha$.

$\beta' \prec \beta''$, $\beta \prec \beta'$, so $\beta \prec \beta''$. By Lemma 6.23, $(\beta, \beta'') = 1$. In particular, $\beta \neq \beta''$ Note that $\beta + \gamma \in \Delta$, so $(\beta, \gamma) = -1$ by Lemma 2.5. So, $\beta'' \neq \gamma$.

By Lemma 6.26, $(\beta, \beta'') = 1$, $(\beta'', \gamma) = 0$, and $(\beta'', \alpha) = 1$.

Summarizing, $\beta'' \prec \alpha$, $\beta \neq \beta''$, $\beta'' \neq \gamma$, $(\beta, \beta'') = 1$, $(\beta'', \gamma) = 0$, and $(\beta'', \alpha) = 1$, and $\beta' \prec_w \beta''$ This is a contradiction with the $\prec$-maximality of β'.

Therefore, $w^{-1}\beta' \in -\Pi$. $\square$

Lemma 6.28. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$,*

Let α be a maximal (in the sense of $\prec$) element of $\Delta^+ \cap w\Delta^-$.

Let $\alpha_i \in \operatorname{supp} \alpha$.

Then there exists a root $\beta' \in \Delta^+ \cap w\Delta^-$ such that: $w^{-1}\beta' \in -\Pi$, $\alpha_i \in \operatorname{supp} \beta$, $\beta' \preceq \alpha$.

Proof. If $w^{-1}\alpha \in -\Pi$, we are done.

Otherwise, by Lemma 6.25, there exist $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

Then ($\alpha_i \in \operatorname{supp} \beta$ or $\alpha_i \in \operatorname{supp} \gamma$), because it is not possible to have $\alpha_i \notin \operatorname{supp} \beta$, $\alpha_i \notin \operatorname{supp} \gamma$, and $\alpha_i \in \operatorname{supp}(\beta + \gamma)$.

Without loss of generality, $\alpha_i \in \operatorname{supp} \beta$.

By Lemma 6.25, there exists $\beta' \in \Delta^+ \cap w\Delta^-$ such that $\beta' \prec \alpha$, $w^{-1}\beta' \in -\Pi$, and $\beta \prec \beta'$.

$\beta \prec \beta'$, so $\operatorname{supp} \beta \subseteq \operatorname{supp} \beta'$.

$\alpha_i \in \operatorname{supp} \beta$, so $\alpha_i \in \operatorname{supp} \beta'$. $\square$

Lemma 6.29. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$,*

Let α, α' be two different maximal (in the sense of $\prec$) elements of $\Delta^+ \cap w\Delta^-$.

Then $\operatorname{supp} \alpha \cap \operatorname{supp} \alpha'$ does not contain involved roots.

Proof. Assume the contrary. Assume there exists α_i such that $n_i > 0$ and $\alpha_i \in \operatorname{supp}\alpha$, $\alpha_i \in \operatorname{supp}\alpha'$.

By Lemma 6.28, there exist $\beta, \beta' \in \Delta^+ \cap w\Delta^-$ such that $w^{-1}\beta, w^{-1}\beta' \in -\Pi$, $\beta \prec \alpha$, $\beta' \prec \alpha'$, $\alpha_i \in \operatorname{supp}\beta$, $\alpha_i \in \operatorname{supp}\beta'$.

Then by Lemma 6.22, this is possible only if $\beta = \beta'$.

By Lemma 6.23, $(\alpha, \beta) = (\alpha, \beta') = 1$.

$\alpha_i \in \operatorname{supp}\alpha \cap \operatorname{supp}\alpha'$, so by Lemma 6.15, (α, α') cannot be -1.

(α, α') cannot be 1, otherwise $\alpha - \alpha' \in \Delta$, and α and α' would be $\prec$-comparable, they would not be able to be both $\prec$-maximal.

So, $(\alpha, \alpha') = 0$.

By Lemma 2.7, $\gamma = \alpha - \beta + \alpha' \in \Delta$.

$(\alpha, \beta) = 1$, so $\alpha - \beta \in \Delta$, $w^{-1}\alpha - w^{-1}\beta \in \Delta$, and α and β are $\prec_w$-comparable. By Lemma 3.10, β is a $\prec_w$-maximal element of $\Delta^+ \cap w\Delta^-$, so we cannot have $\beta \prec_w \alpha$. Hence, $\alpha \prec_w \beta$.

By Lemma 2.15, $\gamma \in w\Delta^-$.

$\alpha_i \in \operatorname{supp}\alpha$, $\alpha_i \in \operatorname{supp}\alpha'$, $\alpha_i \in \operatorname{supp}\beta$. By Lemma 6.14, the coefficients in front of α_i in the decompositions of α, α', and β into linear combinations of simple roots are all 1. So, by Lemma 2.8, $\gamma \in \Delta^+$.

Therefore, $\gamma \in \Delta^+ \cap w\Delta^-$.

$\beta \prec \alpha$, so $0 \prec \alpha - \beta$, and $\alpha' \prec \gamma$. A contradiction with the maximality of α'. $\square$

Lemma 6.30. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$,*

Then there is a unique $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$.

Proof. Denote by J the set of indices i $(1 \le i \le r)$ such that $n_i > 0$.

By Lemma 6.29, if β_1 and β_2 are two different $\prec$-maximal elements of $\Delta^+ \cap w\Delta^-$, then $\operatorname{supp}\beta_1 \cap \operatorname{supp}\beta_2 \cap J = \varnothing$.

So, we can apply Lemma 5.13. By Lemma 5.13, there is a unique $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$. $\square$

Lemma 6.31. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$, and let α be (the unique by Lemma 6.30) $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$.*

Suppose that there exist roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

Denote by L the set consisting of β and all roots $\delta \in \Delta^+ \cap w\Delta^-$, such that $\delta \prec \alpha$, $\delta \ne \beta$, $\delta \ne \gamma$, and $(\delta, \beta) = 1$, $(\delta, \gamma) = 0$, $(\delta, \alpha) = 1$.

Then there exists a unique *$\prec$-maximal element of L.*

Proof. We are going to use Lemma 6.27.

Assume that there exist two different $\prec$-maximal elements of L. Denote them by β' and β''.

By Lemma 6.27, $\beta \prec \beta'$, $\beta \prec \beta''$, $w^{-1}\beta' \in -\Pi$, and $w^{-1}\beta'' \in -\Pi$.

By Lemma 5.11, there exists an involved root $\alpha_i \in \operatorname{supp}\beta$. Then $\alpha_i \in \operatorname{supp}\beta'$, $\alpha_i \in \operatorname{supp}\beta''$. We have a contradiction with Lemma 6.22. $\square$

Lemma 6.32. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$, and let α be (the unique by Lemma 6.30) $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$.*

Then it is impossible to find roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

Proof. Assume the contrary, assume that there exist roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$.

Denote by L the set consisting of β and all roots $\delta \in \Delta^+ \cap w\Delta^-$, such that $\delta \prec \alpha$, $\delta \ne \beta$, $\delta \ne \gamma$, and $(\delta, \beta) = 1$, $(\delta, \gamma) = 0$, $(\delta, \alpha) = 1$.

By Lemma 6.31, there exists a unique $\prec$-maximal element of L, denote it by β'. By Lemma 6.27, $w^{-1}\beta' \in -\Pi$.

Similarly, denote by L' the set consisting of γ and all roots $\delta \in \Delta^+ \cap w\Delta^-$, such that $\delta \prec \alpha$, $\delta \ne \beta$, $\delta \ne \gamma$, and $(\delta, \beta) = 0$, $(\delta, \gamma) = 1$, $(\delta, \alpha) = 1$.

We can apply Lemmas 6.31 and 6.27 to γ instead of β and to L' instead of L. We will find a $\prec$-maximal element of L', denote it by γ', and see that $w^{-1}\gamma' \in -\Pi$.

By Lemma 6.26, $\Delta^+ \cap w\Delta^-$ is a disjoint union of L, L', and $\{\alpha\}$.

The rest of the proof is similar to the proof of Lemma 6.30.

Denote by J the set of indices i $(1 \le i \le r)$ such that $n_i > 0$.

Since α is the unique $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$, it follows from Lemma 5.12 that $J \subseteq \operatorname{supp} \alpha$.

Clearly, $\operatorname{supp} \alpha = \operatorname{supp} \beta \cup \operatorname{supp} \gamma$. Since $\beta \prec \beta'$ and $\gamma \prec \gamma'$, we have $\operatorname{supp} \alpha = \operatorname{supp} \beta' \cup \operatorname{supp} \gamma'$. So, $J = J \cap \operatorname{supp} \alpha = (J \cap \operatorname{supp} \beta') \cup (J \cap \operatorname{supp} \gamma')$.

But $(J \cap \operatorname{supp} \beta) \cap (J \cap \operatorname{supp} \gamma) = \varnothing$ by Lemma 6.22. So, J is the disjoint union of $J \cap \operatorname{supp} \beta'$ and $J \cap \operatorname{supp} \gamma'$.

By the definition of notation R, $R_{J\cap\operatorname{supp}\beta'}(w) \cup R_{J\cap\operatorname{supp}\gamma'}(w) = R_J(w)$. By Lemma 5.11 $R_J(w) = \Delta^+ \cap w\Delta^-$.

Suppose that $\delta \in L$. β' is the unique $\prec$-maximal element of L, so $\delta \prec \beta'$, $\operatorname{supp} \delta \subseteq \operatorname{supp} \beta'$, and $\operatorname{supp} \delta \cap (J \cap \operatorname{supp} \gamma') = \varnothing$. So, $\delta \notin R_{J\cap\operatorname{supp}\gamma'}(w)$. Since $R_{J\cap\operatorname{supp}\beta'}(w) \cup R_{J\cap\operatorname{supp}\gamma'}(w) = \Delta^+ \cap w\Delta^-$, $\delta \in R_{J\cap\operatorname{supp}\beta'}(w)$.

Similarly, if $\delta \in L'$, then , since γ' is the unique $\prec$-maximal element of L', so $\delta \prec \gamma'$. $\operatorname{supp} \delta \subseteq \operatorname{supp} \gamma'$, and $\operatorname{supp} \delta \cap (J \cap \operatorname{supp} \beta') = \varnothing$. So, $\delta \notin R_{J\cap\operatorname{supp}\beta'}(w)$. Since $R_{J\cap\operatorname{supp}\beta'}(w) \cup R_{J\cap\operatorname{supp}\gamma'}(w) = \Delta^+ \cap w\Delta^-$, $\delta \in R_{J\cap\operatorname{supp}\gamma'}(w)$.

Since $\beta' \prec \alpha$ and $\gamma' \prec \alpha$, $\operatorname{supp} \beta' \subseteq \operatorname{supp} \alpha$ and $\operatorname{supp} \gamma' \subseteq \operatorname{supp} \alpha$. So, $\alpha \in R_{J\cap\operatorname{supp}\beta'}(w)$, $\alpha \in R_{J\cap\operatorname{supp}\gamma'}(w)$.

Summarizing, for each $\delta \in \Delta^+ \cap w\Delta^-$:

If $\delta \in L$, then $\delta \in R_{J\cap\operatorname{supp}\beta'}(w)$ and $\delta \notin R_{J\cap\operatorname{supp}\gamma'}(w)$.

If $\delta \in L'$, then $\delta \notin R_{J\cap\operatorname{supp}\beta'}(w)$ and $\delta \in R_{J\cap\operatorname{supp}\gamma'}(w)$.

If $\delta = \alpha$, then $\delta \in R_{J\cap\operatorname{supp}\beta'}(w)$ and $\delta \in R_{J\cap\operatorname{supp}\gamma'}(w)$.

In other words, $R_{J\cap\operatorname{supp}\beta'}(w) = L \cup \{\alpha\}$ and $R_{J\cap\operatorname{supp}\gamma'}(w) = L' \cup \{\alpha\}$.

Therefore, $|R_{J\cap\operatorname{supp}\beta'}(w)| + |R_{J\cap\operatorname{supp}\gamma'}(w)| = |L| + |L'| + 2 = |\Delta^+ \cap w\Delta^-| + 1 = \ell(w) + 1$.

On the other hand, by the definition of an excessive configuration, $|R_{J\cap\operatorname{supp}\beta'}(w)| > \sum_{j\in J\cap\operatorname{supp}\beta'} n_j$ and $|R_{J\cap\operatorname{supp}\gamma'}(w)| > \sum_{j\in J\cap\operatorname{supp}\gamma'} n_j$.

Since all numbers here are integers, $|R_{J\cap\operatorname{supp}\beta'}(w)| \ge 1 + \sum_{j\in J\cap\operatorname{supp}\beta'} n_j$ and $|R_{J\cap\operatorname{supp}\gamma'}(w)| \ge 1 + \sum_{j\in J\cap\operatorname{supp}\gamma'} n_j$.

We know that J is the disjoint union of $J \cap \operatorname{supp} \beta'$ and $J \cap \operatorname{supp} \gamma'$, so $\sum_{j\in J\cap\operatorname{supp}\beta'} n_j + \sum_{j\in J\cap\operatorname{supp}\gamma'} n_j = \sum_{j\in J} n_j$. By the definition of J, $\sum_{j\in J} n_j = \sum_{j=1}^{r} n_j$.

The sum of the two last inequalities is: $\ell(w) + 1 \ge 2 + n_1 + \ldots + n_r$. But $n_1 + \ldots + n_r = \ell(w)$, and we get a contradiction. □

Lemma 6.33. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$.*

It is impossible to find two roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $(\beta, \gamma) = 0$ and there exists an involved root $\alpha_i \in \operatorname{supp} \beta \cap \operatorname{supp} \gamma$.

Proof. Assume the contrary.

Let α be the (unique by Lemma 6.30) $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$.

By Lemma 6.23, $(\alpha, \beta) = (\alpha, \gamma) = 1$.

By Lemma 2.7, $\delta = \beta - \alpha + \gamma \in \Delta$ and $(\alpha, \delta) = 0$.

By Lemma 6.14, the coefficients in front of α_i in the decompositions of α, β, and γ into linear combinations of simple roots are all 1. So, be Lemma 2.8, $\delta \in \Delta^+$.

$(\alpha, \beta) = 1$, so $\alpha - \beta \in \Delta$.

α is the unique $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$, so $\beta \prec \alpha$, and $\beta - \alpha \in \Delta^+$.

By Lemma 6.32, it is impossible to have $\alpha - \beta \in \Delta^+ \cap w\Delta^-$, so $\alpha - \beta \in w\Delta^+$, and $\alpha \prec_w \beta$.

By Lemma 2.15, $\delta \in w\Delta^-$.

So, $\delta \in \Delta^+ \cap w\Delta^-$, $(\delta, \alpha) = 0$, and $\delta \prec \alpha$ since α is the unique $\prec$-maximal element of $\Delta^+ \cap w\Delta^-$. We have a contradiction with Lemma 6.23. □

Proposition 6.34. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$.*

Then $\Delta^+ \cap w\Delta^-, n_1, \ldots, n_r$ is an excessive cluster.

Proof. Denote by J the set of involved simple roots.

Let us check that $\Delta^+ \cap w\Delta^-$ is a J-cluster.

Let $\alpha \in \Delta^+ \cap w\Delta^-$, and let $\alpha_i \in J$. Then the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is at most 1 by lemma 6.14.

Let $\alpha, \beta \in \Delta^+ \cap w\Delta^-$.

(α, β) cannot be equal -1 by Lemmas 6.24 and 6.32.

If $(\alpha, \beta) = 0$, then $\operatorname{supp} \alpha \cap \operatorname{supp} \beta \cap J = \varnothing$ by Lemma 6.33.

So, $\Delta^+ \cap w\Delta^-$ is a J-cluster.

By the definition of a configuration of D-multiplicities, $\ell(w) = |\Delta^+ \cap w\Delta^-| = \sum n_i$, and, by the definition of an involved simple root, $\sum n_i = \sum_{i \in J} n_i$. So, $|\Delta^+ \cap w\Delta^-| = \sum_{i \in J} n_i$.

By the definition of an excessive configuration of D-multiplicities, if $I \subset J$, $I \neq J$, $I \neq \varnothing$, then $|R_I(w)| = |R_I(\Delta^+ \cap w\Delta^-)| > \sum_{i \in I} n_i$, so $\Delta^+ \cap w\Delta^-, n_1, \ldots, n_r$ is an excessive cluster. $\square$

6.3 Reduction of the general case to the case of excessive configuration

Lemma 6.35. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$.*

Then there exists a minimal by inclusion nonempty subset $I \subseteq \{1, \ldots, r\}$ such that $|R_I(w)| = \sum_{i \in I} n_i$ and $n_i > 0$ if $i \in I$.

Proof. $n_1 + \ldots + n_r = \ell(w) > 0$. Denote by $J \subseteq \{1, \ldots, r\}$ the set of i such that α_i is an involved simple root, i. e. $n_i > 0$. Then $\sum_{i \in J} n_i = n_1 + \ldots + n_r = \ell(w)$. Clearly, J is nonempty.

By Lemma 3.26, there exists a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels. By Corollary 4.6, $|R_J(w)| \geq \sum_{i \in J} n_i = \ell(w)$. On the other hand, by the definition of notation R, $R_J(w) \subseteq \Delta^+ \cap w\Delta^-$, so $|R_J(w)| \leq |\Delta^+ \cap w\Delta^-| = \ell(w)$. So, $|R_J(w)| = \ell(w) = \sum_{i \in J} n_i$. In particular, this means that there exist nonempty subsets $J' \subseteq \{1, \ldots, r\}$ such that [$|R_{J'}(w)| = \sum_{i \in J'} n_i$ and $n_i > 0$ for all $i \in J'$]. (J is one of such subsets J'.)

Then there exists a minimal by inclusion nonempty subset $I \subseteq \{1, \ldots, r\}$ such that $|R_I(w)| = \sum_{i \in I} n_i$ and $n_i > 0$ if $i \in I$. $\square$

Lemma 6.36. *Let $w \in W$, let σ_α be an admissible sorting reflection for w.*

Let $I \subseteq \{1, \ldots, r\}$ be a subset such that
$\operatorname{supp} \alpha \cap I = \varnothing$.
Then
$R_I(\sigma_\alpha w) = \sigma_\alpha R_I(w)$,
and for every $j \in I$, for every $\beta \in R_I(w)$:
the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots
$=$
the coefficient in front of α_j in the decomposition of $\sigma_\alpha\beta$ into a linear combination of simple roots.

Moreover, by Lemma 3.7, there exists a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \alpha$ and $(\Delta^+ \cap \sigma_\alpha w\Delta^-)$. Denote it by ψ. Then $\psi(R_I(w)) = R_I(\sigma_\alpha w)$.

Proof. Let $\beta \in R_I(w)$. Let $j \in I$ be an index such that the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots is positive.

Then $\beta - \sigma_\alpha\beta$ is a multiple of α.

Also, $\beta \neq \alpha$ since $\operatorname{supp} \alpha \cap I = \varnothing$. Lemma 3.7 says that $\psi(\beta)$ is either β or $\beta - \alpha$. Again, in both cases, $\beta - \psi(\beta)$ is a multiple of α.

Therefore, all three of the coefficients in front of α_j in the decompositions of β, of $\psi(\beta)$ and of $\sigma_\alpha\beta$ into linear combinations of simple roots coincide, and the coefficients in front of α_j in the decompositions of $\sigma_\alpha\beta$ and of $\psi(\beta)$ into linear combinations of simple roots are positive. In particular, $\sigma_\alpha\beta \in \Delta^+$. Also, $\sigma_\alpha\beta \in \sigma_\alpha w\Delta^-$ since $\beta \in R_I(w)$ and $\beta \in w\Delta^-$. Therefore, $\sigma_\alpha\beta\Delta^+ \cap (\sigma_\alpha w)\Delta^-$, and $\sigma_\alpha\beta \in R_I(\sigma_\alpha w)$. Clearly, $\psi(\beta) \in \Delta^+ \cap (\sigma_\alpha w)\Delta^-$, so $\psi(\beta) \in R_I(\sigma_\alpha w)$.

Similarly, if $\gamma \in R_I(\sigma_\alpha w)$, then there exists $j \in I$ such that the coefficient in front of α_j in the decomposition of γ into a linear combination of simple roots is positive. $\gamma - \sigma_\alpha^{-1}\gamma = \gamma - \sigma_\alpha\gamma$ is a multiple of α. Also, Lemma 3.7 says that $\psi^{-1}(\gamma)$ is either γ or $\gamma + \alpha$.

Again, the coefficients in front of α_j in the decompositions of γ, of $\psi^{-1}(\gamma)$, and of $\sigma_\alpha^{-1}\gamma = \sigma_\alpha\gamma$ into linear combinations of simple roots coincide. And the coefficients in front of α_j in the decompositions of $\sigma_\alpha\gamma$ and of $\psi^{-1}(\gamma)$ into linear combinations of simple roots are positive. So, $\sigma_\alpha\gamma \in \Delta^+$. Also, $\sigma_\alpha\gamma \in \sigma_\alpha\sigma_\alpha w\Delta^- = w\Delta^-$ since $\gamma \in R_I(\sigma_\alpha w)$ and $\gamma \in \sigma_\alpha w\Delta^-$. Therefore, $\sigma_\alpha\gamma\Delta^+ \cap w\Delta^-$, and $\sigma_\alpha\gamma \in R_I(\alpha w)$. Clearly, $\psi^{-1}(\gamma) \in \Delta^+ \cap (\sigma_\alpha w)\Delta^-$, so $\psi(\gamma) \in R_I(\sigma_\alpha w)$.

Hence, σ_α establishes a bijection between $R_I(w)$ and $R_I(\sigma_\alpha w)$.

Finally, let $\beta \in R_I(w)$. Let $j \in I$ be arbitrary. Again we can say that $\beta - \sigma_\alpha\beta$ is a multiple of α. Therefore, the coefficients in front of α_j in the decompositions of β and $\sigma_\alpha\beta$ into linear combinations of simple roots coincide. $\square$

Lemma 6.37. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} \geq 1$.*

Suppose that there exists $I \subseteq \{1, \ldots, r\}$ such that $|R_I(w)| = \sum_{i\in I} n_i < \ell(w)$.

Then there exist $\alpha \in \Delta^+ \cap w\Delta^-$ and α_i such that:

σ_α is an admissible sorting reflection for w, and

$i \notin I$, and

$C_{\sigma_\alpha w,n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_r} \geq 1$, and

$R_I(\sigma_\alpha w) = \sigma_\alpha R_I(w)$,

and for every $j \in I$, for every $\beta \in R_I(w)$:

the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_\alpha\beta$ into a linear combination of simple roots.

Proof. We know that $\sum_{i\in I} n_i < \ell(w)$ and $n_1 + \ldots + n_r = \ell(w)$, so there exists $i \notin I$ such that $n_i > 0$. Fix this i until the end of the proof.

Consider the following list of labels: $L = \alpha_i, \alpha_1, \ldots, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_i, \ldots, \alpha_r, \ldots, \alpha_r$, where, after (excluding) the first α_i, [each α_j is written n_j times, except for α_i, which is written $n_i - 1$ times]. Clearly, it has D-multiplicities $n_1, \ldots, n_r$ of labels. By Lemma 3.26, there exists a labeled sorting process of w with list of labels L. Denote the root it starts with by α. Then, by Proposition 4.4 ("moreover" part), there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ such that $f(\alpha) = \alpha_i$.

σ_α is an admissible sorting reflection for w by the definition of a sorting process.

If we remove α with its label α_i from the beginning of the sorting process, we will get a labeled sorting process for $\sigma_\alpha w$ with D-multiplicities $n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r$ of labels. So, $C_{\sigma_\alpha w,n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_r} \geq 1$.

Let us check that $\alpha \notin R_I(w)$. Assume the contrary. By the definition of notation R, if $f(\beta) \in I$, then $\beta \in R_I(w)$. The number of roots $\beta \in \Delta^+ \cap w\Delta^-$ such that $f(\beta) \in I$, is exactly $\sum_{j\in I} n_j$. So, we have $\sum_{j\in I} n_j$ roots β in $R_I(w)$ such that $f(\beta) \in I$, and one more root $\alpha \in R_I(w)$, which is different because $f(\alpha) = \alpha_i \notin I$. So, $|R_I(w)| \geq 1 + \sum_{j\in I} n_j$, a contradiction with Lemma hypothesis.

So, $\alpha \notin R_I(w)$, and $\operatorname{supp}\alpha \cap I = \varnothing$.

Now the rest of the claim follows from Lemma 6.36. $\square$

Lemma 6.38. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} \geq 1$ and $w \neq \mathrm{id}$.*

Let $I \subseteq \{1, \ldots, r\}$ be a subset such that $|R_I(w)| = \sum_{i\in I} n_i$ (not necessarily minimal by inclusion, not necessarily consisting of involved roots only).

Denote $k_i = n_i$ if $i \notin I$, $k_i = 0$ if $i \in I$. Denote $k = k_1 + \ldots + k_r$.

Then there exists a labeled sorting process prefix $\beta_1, \ldots, \beta_k$ of w with D-multiplicities $k_1, \ldots, k_r$ of labels such that

Denote $w_k = \sigma_{\beta_k} \ldots \sigma_{\beta_1} w$

$C_{w_k,n_1-k_1,\ldots,n_r-k_r} \geq 1$, and

$R_I(w_k) = \sigma_{\beta_k} \ldots \sigma_{\beta_1} R_I(w)$,

and for every $j \in I$, for every $\beta \in R_I(w)$:

the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_k} \ldots \sigma_{\beta_1}\beta$ into a linear combination of simple roots.

Proof. Induction on k. If $k = 0$, everything is clear.

If $k > 0$, then $\sum_{i \in I} n_i < \ell(w)$, and we can use Lemma 6.37.

It says that there exists an index $i \in \{1, \ldots, r\}$, $i \notin I$, and a root $\alpha \in \Delta^+ \cap w\Delta^-$. Denote $\beta_1 = \alpha$ and fix this i until the end of the proof.

Lemma 6.37 also says that:

σ_{β_1} is an admissible sorting reflection for w, and

$C_{\sigma_{\beta_1} w, n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r} \geq 1$, and

$R_I(\sigma_{\beta_1} w) = \sigma_{\beta_1} R_I(w)$.

In particular, $|R_I(\sigma_{\beta_1} w)| = \sum_{j \in I} n_j$.

We can apply the induction hypothesis to the configuration $\sigma_{\beta_1} w, n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r$ of D-multiplicities (recall that $i \notin I$). This induction hypothesis will use numbers $k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r$ instead of $k_1, \ldots, k_r$ and $k - 1$ instead of k.

As an output, the induction hypothesis will give a labeled sorting process prefix of $\sigma_{\beta_1} w$ of length $k - 1$ with D-multiplicities $k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r$ of labels. Denote this sorting process prefix by $\beta_2, \ldots, \beta_k$.

Then, if we denote $w_k = \sigma_{\beta_k} \ldots \sigma_{\beta_2} \sigma_{\beta_1} w$, this sorting process prefix will have the following properties:

$C_{w_k, n_1 - k_1, \ldots, n_{i-1} - k_{k-i}, (n_i - 1) - (k_i - 1), n_{i+1} - k_{i+1}, \ldots, n_r - k_r} = C_{w_k, n_1 - k_1, \ldots, n_r - k_r} \geq 1$, and

$R_I(w_k) = \sigma_{\beta_k} \ldots \sigma_{\beta_2} R_I(\sigma_{\beta_1} w)$,

and for every $j \in I$, for every $\beta \in R_I(\sigma_{\beta_1} w)$:

the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_k} \ldots \sigma_{\beta_2} \beta$ into a linear combination of simple roots.

We already know that $R_I(\sigma_{\beta_1} w) = \sigma_{\beta_1} R_I(w)$, so $R_I(w_k) = \sigma_{\beta_k} \ldots \sigma_{\beta_1} R_I(w)$.

Finally take any $j \in I$ and any $\beta \in R_I(w)$. Recall that Lemma 6.37 also says that

the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_1} \beta$ into a linear combination of simple roots

and that $\sigma_{\beta_1} \beta \in \sigma_{\beta_1} R_I(w) = R_I(\sigma_{\beta_1} w)$, so the conclusion we made from the induction hypothesis says that

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_1} \beta$ into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_k} \ldots \sigma_{\beta_2} \sigma_{\beta_1} \beta$ into a linear combination of simple roots.

Therefore,

the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_k} \ldots \sigma_{\beta_1} \beta$ into a linear combination of simple roots. ☐

Lemma 6.39. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities such that $C_{w, n_1, \ldots, n_r} = 1$ and $w \neq \mathrm{id}$.*

Let $I \subseteq \{1, \ldots, r\}$ be a minimal by inclusion nonempty subset such that $|R_I(w)| = \sum_{i \in I} n_i$ and $n_i > 0$ if $i \in I$.

Denote $m_i = n_i$ if $i \in I$, $m_i = 0$ if $i \notin I$.

Then $R_I(w), m_1, \ldots, m_r$ is an excessive cluster.

Proof. Denote $k_i = n_i$ if $i \notin I$, $k_i = 0$ if $i \in I$. Denote $k = k_1 + \ldots + k_r$. Clearly, $m_i + k_i = n_i$ for all i $(1 \leq i \leq r)$.

By Lemma 6.38, there exists a labeled sorting process prefix $\beta_1, \ldots, \beta_k$ of w with D-multiplicities $k_1, \ldots, k_r$ of labels such that

Denote $w_k = \sigma_{\beta_k} \ldots \sigma_{\beta_1} w$

$C_{w_k, n_1 - k_1, \ldots, n_r - k_r} \geq 1$, and

$R_I(w_k) = \sigma_{\beta_k} \ldots \sigma_{\beta_1} R_I(w)$, and for every $j \in I$, for every $\beta \in R_I(w)$:

the coefficient in front of α_j in the decomposition of β into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_k} \dots \sigma_{\beta_1} \beta$ into a linear combination of simple roots.

First, note that by Lemma 6.5, $C_{w_k, n_1-k_1, \dots, n_r-k_r}$ must be 1, otherwise $C_{w, n_1, \dots, n_r}$ would be bigger than 1.

By the lemma hypothesis, $|R_I(w)| = \sum_{i \in I} n_i$. By the definition of m_i, $\sum_{i \in I} n_i = \sum m_i$, so $|R_I(w)| = \sum m_i$.

$m_i = n_i - k_i$ for all i, and $\sum k_i = k$, and by the definition of a configuration of D-multiplicities, $\sum n_i = \ell(w)$. So, $\sum m_i = \ell(w) - k$, and $|R_I(w)| = \ell(w) - k$.

We also know that $R_I(w_k) = \sigma_{\beta_k} \dots \sigma_{\beta_1} R_I(w)$, so $|R_I(w_k)| = |R_I(w)| = \ell(w) - k$.

On the other hand, since $\beta_1, \dots, \beta_k$ is a sorting process prefix for w, for each j, $1 \le j \le k$, σ_{β_j} is an admissible sorting reflection for $\sigma_{\beta_{j-1}} \dots \sigma_{\beta_1} w$, and $\ell(\sigma_{\beta_j} \dots \sigma_{\beta_1} w) = \ell(\sigma_{\beta_{j-1}} \dots \sigma_{\beta_1} w) - 1$. So, $\ell(w_k) = \ell(\sigma_{\beta_k} \dots \sigma_{\beta_1} w) = \ell(w) - k$.

Therefore, $|R_I(w_k)| = \ell(w) - k = \ell(w_k) = |\Delta^+ \cap w_k \Delta^-|$, and $R_I(w_k) = \Delta^+ \cap w_k \Delta^-$.

Now, choose an arbitrary subset $I_0 \subset I$, $I_0 \ne I$, $I_0 \ne \varnothing$. Clearly, $R_{I_0}(w) \subseteq R_I(w)$.

If $\beta \in R_{I_0}(w)$, then there exists $i \in I_0$ such that the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots is positive. We know that this coefficient equals the coefficient in front of α_i in the decomposition of $\sigma_{\beta_k} \dots \sigma_{\beta_1} \beta$ into a linear combination of simple roots. So, the coefficient in front of α_i in the decomposition of $\sigma_{\beta_k} \dots \sigma_{\beta_1} \beta$ into a linear combination of simple roots is positive. We know that $\sigma_{\beta_k} \dots \sigma_{\beta_1} R_I(w) = R_I(w_k) = \Delta^+ \cap w_k \Delta^-$, so $\sigma_{\beta_k} \dots \sigma_{\beta_1} \beta \in \Delta^+ \cap w_k \Delta^-$, and $\sigma_{\beta_k} \dots \sigma_{\beta_1} \beta \in R_{I_0}(w_k)$.

Similarly, take an arbitrary $\gamma \in R_{I_0}(w_k)$. Then $(\sigma_{\beta_k} \dots \sigma_{\beta_1})^{-1} \gamma \in R_I(w)$. Moreover, there exists $i \in I_0$ such that the coefficient in front of α_i in the decomposition of γ into a linear combination of simple roots is positive. And this coefficient equals the coefficient in front of α_i in the decomposition of $(\sigma_{\beta_k} \dots \sigma_{\beta_1})^{-1} \gamma$ into a linear combination of simple roots. So, the coefficient in front of α_i in the decomposition of $(\sigma_{\beta_k} \dots \sigma_{\beta_1})^{-1} \gamma$ into a linear combination of simple roots is positive, and $(\sigma_{\beta_k} \dots \sigma_{\beta_1})^{-1} \gamma \in R_{I_0}(w)$.

Summarizing, $R_{I_0}(w_k) = \sigma_{\beta_k} \dots \sigma_{\beta_1} R_{I_0}(w)$. Therefore, $|R_{I_0}(w_k)| = |R_{I_0}(w)|$.

We chose I_0 so that $I_0 \ne I$, $I_0 \ne \varnothing$, and I was a minimal by inclusion nonempty subset of $\{1, \dots, r\}$ such that $\sum_{i \in I} n_i = |R_I(w)|$. So, $|R_{I_0}(w)| \ne \sum_{i \in I_0} n_i$.

Also, $C_{w, n_1, \dots, n_r} = 1$, so, by Lemma 3.26, there exists a labeled sorting process of w with D-multiplicities $n_1, \dots, n_r$ of labels. By Corollary 4.6, $|R_{I_0}(w)| \ge \sum_{i \in I_0} n_i$. Therefore, $|R_{I_0}(w_k)| = |R_{I_0}(w)| > \sum_{i \in I_0} n_i$.

Now recall that $m_i = n_i$ for all $i \in I$, and $I_0 \subset I$. So, $\sum_{i \in I_0} m_i > |R_{I_0}(w_k)|$.

We know that if $i \in I$, then $n_i > 0$ by lemma hypothesis and $m_i = n_i$ so $m_i > 0$, and if $i \notin I$, then $m_i = 0$ by the definition of m_i. So, I is the set of involved roots of the configuration $w_k, m_1, \dots, m_r$ of D-multiplicities. And we have checked that $|R_I(w_k)| = \ell(w_k) = \sum_{i \in I} m_i$, and that if $I_0 \subset I$, $I_0 \ne I$, $I_0 \ne \varnothing$, then $|R_{I_0}(w_k)| > \sum_{i \in I_0} m_i$. Together this is exactly the definition of an excessive configuration of D-multiplicities.

Therefore, $w_k, m_1, \dots, m_r$ is an excessive configuration of D-multiplicities.

By Proposition 6.34, $\Delta^+ \cap w_k \Delta^-$ is an excessive cluster.

Let us check that $R_I(w)$ is an I-cluster.

Let $\alpha \in R_I(w)$, and let $i \in I$. Then $\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha \in R_I(w_k) = \Delta^+ \cap w_k \Delta^-$. $\Delta^+ \cap w_k \Delta^-$ is an excessive cluster, so it is an I-cluster, and the coefficient in front of α_i in the decomposition of $\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha$ into a linear combination of simple roots is at most 1. And Lemma 6.38 also says that this coefficient equals the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots. So, the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is at most 1.

Let $\alpha, \beta \in R_I(w)$, $\alpha \ne \beta$. Note that $(\alpha, \beta) = (\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha, \sigma_{\beta_k} \dots \sigma_{\beta_1} \beta)$.

Again, $\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha, \sigma_{\beta_k} \dots \sigma_{\beta_1} \beta \in R_I(w_k) = \Delta^+ \cap w_k \Delta^-$, and $\Delta^+ \cap w_k \Delta^-$ is an I-cluster, so $(\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha, \sigma_{\beta_k} \dots \sigma_{\beta_1} \beta)$ cannot be equal -1, and (α, β) cannot be equal -1.

If $(\alpha, \beta) = 0$, then $(\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha, \sigma_{\beta_k} \dots \sigma_{\beta_1} \beta)$, and $\operatorname{supp}(\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha) \cap \operatorname{supp}(\sigma_{\beta_k} \dots \sigma_{\beta_1} \alpha) \cap I = \varnothing$. If $\operatorname{supp} \alpha \cap \operatorname{supp} \beta \cap I \ne \varnothing$, then there exists $i \in I$ such that both of the coefficients in front of

α_i in the decompositions of α and β into linear combinations of simple roots are positive. But these coefficients equal the coefficients in front of α_i in the decompositions of $\sigma_{\beta_k}\dots\sigma_{\beta_1}\alpha$ and $\sigma_{\beta_k}\dots\sigma_{\beta_1}\beta$ into linear combinations of simple roots, respectively, a contradiction with the fact that $\operatorname{supp}(\sigma_{\beta_k}\dots\sigma_{\beta_1}\alpha)\cap\operatorname{supp}(\sigma_{\beta_k}\dots\sigma_{\beta_1}\alpha)\cap I=\varnothing$.

Therefore, $R_I(w)$ is an I-cluster.

By the definition of a configuration of D-multiplicities, $\ell(w)=|\Delta^+\cap w\Delta^-|=\sum n_i$, and, by the definition of an involved simple root, $\sum n_i=\sum_{i\in J} n_i$. So, $|\Delta^+\cap w\Delta^-|=\sum_{i\in J} n_i$.

By the definition of an excessive configuration of D-multiplicities, if $I\subset J$, $I\neq J$, $I\neq\varnothing$, then $|R_I(w)|=|R_I(\Delta^+\cap w\Delta^-)|>\sum_{i\in I} n_i$, so $\Delta^+\cap w\Delta^-, n_1,\dots,n_r$ is an excessive cluster.

We have already seen that $\sum_{i\in I} n_i=|R_I(w)|$ and if $I_0\subset I$, $I_0\neq I$, and $I_0\neq\varnothing$, then $|R_{I_0}(w)|>\sum_{i\in I_0} n_i$. Now recall that $m_i=n_i$ if $i\in I$, and that $\sum m_i=\sum_{i\in I} n_i$, and that I is exactly the set of indices such that $m_i>0$. So, $|R_I(w)|=\sum m_i$ and if $I_0\subset I$, $I_0\neq I$, and $I_0\neq\varnothing$, then $|R_{I_0}(w)|>\sum_{i\in I_0} m_i$.

Therefore, $R_I(w), m_1,\dots,m_r$ is an excessive cluster. □

Proposition 6.40. *Let $w, n_1,\dots,n_r$ be a configuration of D-multiplicities such that $C_{w,n_1,\dots,n_r}=1$. Then $\Delta^+\cap w\Delta^-, n_1,\dots,n_r$ is an excessively clusterizable A-configuration.*

Proof. Induction on $\ell(w)$.

If $w=\mathrm{id}$, then everything is clear.

Suppose that $w\neq\mathrm{id}$.

By Lemma 3.26, there exists a labeled sorting process of w with D-multiplicities $n_1,\dots,n_r$ of labels. By Proposition 4.4, there exists a distribution f of simple roots on $\Delta^+\cap w\Delta^-$ with D-multiplicities $n_1,\dots,n_r$ of simple roots.

By Lemma 6.35, there exists a minimal by inclusion nonempty subset $I\subseteq\{1,\dots,r\}$ such that $|R_I(w)|=\sum_{i\in I} n_i$ and $n_i>0$ if $i\in I$.

Denote $m_i=n_i$ if $i\in I$, $m_i=0$ if $i\notin I$. Denote also $k_i=n_i-m_i$.

Clearly, $m_i>0$ if and only if $i\in I$. Also, $\sum m_i>0$ since I is nonempty.

We know that $|R_I(w)|=\sum_{i\in I} n_i$ and $n_i=m_i$ if $i\in I$, so $|R_I(w)|=\sum_{i\in I} m_i$.

By Lemma 6.39, $R_I(w), m_1,\dots,m_r$ is an excessive cluster.

To prove that $\Delta^+\cap w\Delta^-, n_1,\dots,n_r$ is excessively clusterizable, it suffices to prove that $(\Delta^+\cap w\Delta^-)\setminus R_I(w), k_1,\dots,k_r$ is excessively clusterizable (we have just checked all other conditions in the definition of an excessively clusterizable A-configuration).

Set $m=|R_I(w)|$.

By Lemma 4.1, there exists an antireduced sorting process prefix $\beta_1,\dots,\beta_m$ of w such that $R_I(w)=\{\beta_1,\dots,\beta_m\}$. Recall that we have a distribution f of simple roots on $\Delta^+\cap w\Delta^-$ with D-multiplicities $n_1,\dots,n_r$ of simple roots. Let us make a labeled antireduced sorting process prefix out of $\beta_1,\dots,\beta_m$: we assign label $f(\beta_i)$ to β_i (this is well-defined by Lemma 3.16 and Corollary 3.17). Denote the D-multiplicities of the labels we have just assigned by $m'_1,\dots,m'_r$. Clearly, $m'_1+\dots+m'_r=m$.

Now note that if $\alpha\in\Delta^+\cap w\Delta^-$ and $f(\alpha)\in I$, then, by the definition of a simple root distribution, $f(\alpha)\in\operatorname{supp}\alpha$, and $\alpha\in R_I(w)$, in other words, $\alpha\in\{\beta_1,\dots,\beta_m\}$. Therefore, if $i\in I$, then the D-multiplicity of label α_i in the antireduced sorting process prefix $\beta_1,\dots,\beta_m$ equals the the D-multiplicity of value α_i in f, i. e. it equals $n_i=m_i$. In other words, $m'_i=m_i$ if $i\in I$.

We know that $m=|R_I(w)|=\sum_{i\in I} m_i$. So, $m=|R_I(w)|=\sum_{i\in I} m'_i$ We also know that $\sum m'_i=m$. Since m'_i are nonnegative integers, $m'_i=0$ for all $i\notin I$. We also know that $m_i=0$ if $i\notin I$, so $m'_i=m_i$ for all $i\notin I$. Therefore, $m_i=m'_i$ for all i, $1\le i\le r$ and $k_i=n_i-m'_i$ for all i, $1\le i\le r$.

So, if we restrict f to $(\Delta^+\cap w\Delta^-)\setminus R_I(w)=(\Delta^+\cap w\Delta^-)\setminus\{\beta_1,\dots,\beta_m\}$, we will get a simple root distribution (denote it by g) with D-multiplicities $k_1,\dots,k_r$ of simple roots.

Denote $w_m=\sigma_{\beta_m}\dots\sigma_{\beta_1}w$. By Lemma 3.16, $(\Delta^+\cap w\Delta^-)\setminus\{\beta_1,\dots,\beta_m\}=\Delta^+\cap w_m\Delta^-$. By Corollary 6.6, $C_{w,n_1,\dots,n_r}\ge C_{w_m,k_1,\dots,k_r}>0$. But $C_{w,n_1,\dots,n_r}=1$, so $C_{w_m,k_1,\dots,k_r}=1$, and we can apply the induction hypothesis. It says that $\Delta^+\cap w_m\Delta^-, k_1,\dots,k_r$ is an excessively clusterizable A-configuration. We have already checked that $(\Delta^+\cap w\Delta^-)\setminus R_I(w)=(\Delta^+\cap w\Delta^-)\setminus\{\beta_1,\dots,\beta_m\}=\Delta^+\cap w_m\Delta^-$, so $(\Delta^+\cap w\Delta^-)\setminus R_I(w), k_1,\dots,k_r$ is an excessively clusterizable A-configuration. And this was the last condition we had to check in the definition of an excessively clusterizable A-configuration for $\Delta^+\cap w\Delta^-, n_1,\dots,n_r$. □

7 Sufficient condition of unique sortability

Lemma 7.1. *Let $w \in W$, and let $I \subseteq \{1, \dots, r\}$ be a nonempty subset such that $R_I(w)$ is an I-cluster. Let $\alpha \in R_I(w)$ be such that σ_α is an admissible sorting reflection for w.*

Suppose that $\operatorname{supp} \alpha \cap I \neq I$.

Denote by A_1 the set of roots $\beta \in R_I(w)$ such that $\alpha \prec_w \beta$.

Denote by A_2 the set of roots $\gamma \in R_I(w)$ such that $(\alpha, \gamma) = 0$.

Then $R_{I \setminus \operatorname{supp} \alpha}(w) \subseteq A_1 \cup A_2$.

Proof. Let $\delta \in R_{I \setminus \operatorname{supp} \alpha}(w)$.

$\delta \neq \alpha$ since $\operatorname{supp} \alpha \cap (I \setminus \operatorname{supp} \alpha) = \varnothing$.

Clearly, $R_{I \setminus \operatorname{supp} \alpha}(w) \subseteq R_I(w)$, so $\delta \in R_I(w)$, and we have two possibilities for δ: either $(\delta, \alpha) = 1$, or [$(\delta, \alpha) = 0$ and $\operatorname{supp} \delta \cap \operatorname{supp} \alpha \cap I = \varnothing$].

Suppose that $(\delta, \alpha) = 1$. Then $\delta - \alpha \in \Delta$, so α and δ are $\prec$-comparable. Moreover in fact, $\alpha \prec \delta$, otherwise $\operatorname{supp} \delta \subseteq \operatorname{supp} \alpha$ and $\operatorname{supp} \delta \cap (I \setminus \operatorname{supp} \alpha) = \varnothing$, a contradiction with $\delta \in R_{I \setminus \operatorname{supp} \alpha}(w)$.

So, $\delta - \alpha \in \Delta^+$. Assume that also $\delta - \alpha \in w\Delta^-$. Let $i \in I \setminus \operatorname{supp} \alpha$ be an index such that $\alpha_i \in \operatorname{supp} \delta$. Then $\alpha_i \notin \operatorname{supp} \alpha$, and $\alpha_i \in \operatorname{supp}(\delta - \alpha)$. So, $\delta - \alpha \in R_I(w)$. But then $(\alpha, \delta - \alpha) = -1$, a contradiction with the fact that $R_I(w)$ is an I-cluster.

So, $\delta - \alpha \in w\Delta^+$, and $w^{-1}\delta - w^{-1}\alpha = w^{-1}(\delta - \alpha) \in \Delta^+$, so $\alpha \prec_w \delta$, and $\delta \in A_1$.

END Suppose that $(\delta, \alpha) = 1$.

If $(\delta, \alpha) = 0$, then $\delta \in A_2$. □

Lemma 7.2. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities. Let $I \subseteq \{1, \dots, r\}$ be a nonempty subset such that:*

denote $k_i = n_i$ for $i \in I$, $k_i = 0$ otherwise

in terms of this notation, suppose that $|R_I(w)| = \sum k_i$ and $R_I(w), k_1, \dots, k_r$ is an excessive cluster.

Let $\alpha \in R_I(w)$.

Suppose that α is not the $\prec_w$-greatest element of $R_I(w)$ (in other words, either α is not a $\prec_w$-maximal element of $R_I(w)$, or it is a $\prec_w$-maximal element, but there are more $\prec_w$-maximal elements in $R_I(w)$).

Denote by A_1 the set of roots $\beta \in R_I(w)$ such that $\alpha \prec_w \beta$.

Denote by A_2 the set of roots $\gamma \in R_I(w)$ such that $(\alpha, \gamma) = 0$.

Then $|A_1 \cup A_2| > \sum_{i \in (I \setminus \operatorname{supp} \alpha)} k_i$.

Proof. First, suppose that $\operatorname{supp} \alpha \cap I = I$, in other words, $I \subseteq \operatorname{supp} \alpha$. Then $I \setminus \operatorname{supp} \alpha = \varnothing$ and $\sum_{i \in (I \setminus \operatorname{supp} \alpha)} k_i = 0$.

α is not the $\prec_w$-greatest element of $R_I(w)$, so either there exists $\beta \in R_I(w)$ such that $\alpha \prec_w \beta$, or there exists $\gamma \in R_I(w)$ such that $\alpha \neq \gamma$ and α and γ are not $\prec_w$-comparable.

If there exists $\beta \in R_I(w)$ such that $\alpha \prec_w \beta$, then $\beta \in A_1$, and $A_1 \neq \varnothing$, and $|A_1 \cup A_2| > 0$.

Suppose that there exists $\gamma \in R_I(w)$ such that β and γ are not $\prec_w$-comparable. Then (α, γ) cannot be 1, otherwise $\gamma - \alpha \in \Delta$ by Lemma 2.5, $w^{-1}(\gamma - \alpha) = w^{-1}\gamma - w^{-1}\alpha \in \Delta$, and α and γ are $\prec_w$-comparable.

$R_I(w)$ is an I-cluster, so (α, γ) cannot be -1. Therefore, $(\alpha, \gamma) = 0$, $\gamma \in A_2$, A_2 is nonempty, and $|A_1 \cup A_2| > 0$.

END suppose that $\operatorname{supp} \alpha \cap I = I$.

Now suppose that $\operatorname{supp} \alpha \cap I \neq I$. By Lemma 7.1, $R_{I \setminus \operatorname{supp} \alpha}(w) \subseteq A_1 \cup A_2$, so $|A_1 \cup A_2| \geq |R_{I \setminus \operatorname{supp} \alpha}(w)|$. By the definition of an excessive cluster, $|R_{I \setminus \operatorname{supp} \alpha}(w)| > \sum_{i \in (I \setminus \operatorname{supp} \alpha)} k_i$. □

Lemma 7.3. *Let $w \in W$, and let $I \subseteq \{1, \dots, r\}$ be a nonempty subset such that $R_I(w)$ is an I-cluster. Let $\alpha \in R_I(w)$ be such that σ_α is an admissible sorting reflection for w.*

Denote by A_1 the set of roots $\beta \in R_I(w)$ such that $\alpha \prec_w \beta$.

Denote by A_2 the set of roots $\gamma \in R_I(w)$ such that $(\alpha, \gamma) = 0$.

Lemma 3.7 establishes a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \alpha$ and $\Delta^+ \cap \sigma_\alpha w\Delta^-$. Denote this bijection by $\psi \colon (\Delta^+ \cap w\Delta^-) \setminus \alpha \to \Delta^+ \cap \sigma_\alpha w\Delta^-$.

Then $R_{\operatorname{supp} \alpha \cap I}(\sigma_\alpha w) \cap \psi(A_1 \cup A_2) = \varnothing$.

Proof. Suppose that $\gamma \in A_2$. Then by Lemma 3.7, $\psi(\gamma) = \gamma$.

$\gamma \in R_I(w)$, and $R_I(w)$ is an I-cluster, so $\operatorname{supp}\alpha \cap \operatorname{supp}\gamma \cap I = \varnothing$. Therefore, $\psi(\gamma) = \gamma \notin R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)$.

It suffices to prove that if $\beta \in A_1$, $\beta \notin A_2$, then $\psi(\beta) \notin R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)$.

So, suppose that $\beta \in A_1$, $\beta \notin A_2$. Then (β, α) cannot be -1 since $R_I(w)$ is an I-cluster, (β, α) cannot be 0 since $\beta \notin A_2$, so $(\beta, \alpha) = 1$.

Then $\beta - \alpha \in \Delta$, and $w^{-1}\beta - w^{-1}\alpha \in \Delta^+$ since $\alpha \prec_w \beta$. So, $\beta - \alpha \in w\Delta^+$ and $\alpha - \beta \in w\Delta^-$.

If $\alpha - \beta \in \Delta^+$, then $\alpha - \beta \in \Delta^+ \cap w\Delta^-$, $\beta \in \Delta^+ \cap w\Delta^-$, and σ_α is not an admissible reflection by Lemma 3.4, a contradiction.

So, $\alpha - \beta \in \Delta^-$, and $\alpha \prec \beta$. Recall that $\beta - \alpha \in w\Delta^+$, so $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$. By Lemma 3.7, $\psi(\beta) = \beta - \alpha$.

Assume that $\psi(\beta) = \beta - \alpha \in R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)$. Then there exists a simple root α_i such that $i \in \operatorname{supp}\alpha \cap I$ and the coefficient in front of α_i in the decomposition of $\beta - \alpha$ into a linear combination of simple roots is at least 1. $i \in \operatorname{supp}\alpha \cap I$, so the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is also at least 1. So, the coefficient in front of α_i in the decomposition of β into a linear combination of simple roots is at least 2, and $i \in I$, a contradiction with the definition of an I-cluster. $\square$

Lemma 7.4. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. Let $I \subseteq \{1, \ldots, r\}$ be a nonempty subset such that:*

denote $k_i = n_i$ for $i \in I$, $k_i = 0$ otherwise

in terms of this notation, suppose that $|R_I(w)| = \sum k_i$ and $R_I(w), k_1, \ldots, k_r$ is an excessive cluster.

Let $\alpha \in R_I(w)$.

Suppose that α is not the $\prec_w$-greatest element of $R_I(w)$.

Let $i \in I$.

Then there are no labeled sorting processes of w with D-multiplicities $n_1, \ldots, n_r$ of labels that start with α with label α_i.

Proof. If $\alpha_i \notin \operatorname{supp}\alpha$, everything is clear. Let $\alpha_i \in \operatorname{supp}\alpha$.

$i \in I$, so $k_i = n_i$. If $n_i = 0$, then everything is also clear. Let $n_i > 0$.

Assume the contrary, assume that there exists a labeled sorting process $\beta_1, \ldots, \beta_{\ell(w)}$ of w with D-multiplicities $n_1, \ldots, n_r$ of labels such that $\beta_1 = \alpha$, and the label at this α is α_i.

Then $\beta_2, \ldots, \beta_{\ell(w)}$ with the same labels form a labeled sorting process of $\sigma_\alpha w$ with D-multiplicities $n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r$ of labels.

Set $n'_j = n_j$ for $j \neq i$, $n'_i = n_i - 1$.

By Corollary 4.6, $|R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)| \geq \sum_{j\in\operatorname{supp}\alpha\cap I} n'_j$. Since $i \in \operatorname{supp}\alpha \cap I$, we can write $|R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)| \geq -1 + \sum_{j\in\operatorname{supp}\alpha\cap I} n_j$.

Lemma 3.7 establishes a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \alpha$ and $\Delta^+ \cap \sigma_\alpha w\Delta^-$. Denote this bijection by $\psi\colon (\Delta^+ \cap w\Delta^-) \setminus \alpha \to \Delta^+ \cap \sigma_\alpha w\Delta^-$.

By Lemma 3.7, if $\beta \in \Delta^+ \cap w\Delta^-$, $\beta \neq \alpha$, then $\psi(\beta)$ equals either β, or $\beta - \alpha$. In particular $\psi(\beta) \preceq \beta$, and $\operatorname{supp}\psi(\beta) \subseteq \operatorname{supp}\beta$. So, if $\gamma \in R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)$, then $\operatorname{supp}\gamma \cap \operatorname{supp}\alpha \cap I \neq \varnothing$, so $\operatorname{supp}\psi^{-1}(\gamma) \cap \operatorname{supp}\alpha \cap I \neq \varnothing$, and $\operatorname{supp}\psi^{-1}(\gamma) \cap I \neq \varnothing$, and $\psi^{-1}(\gamma) \in R_I(w)$. So, $\psi^{-1}(R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)) \subseteq R_I(w)$.

Denote by A_1 the set of roots $\beta \in R_I(w)$ such that $\alpha \prec_w \beta$.

Denote by A_2 the set of roots $\gamma \in R_I(w)$ such that $(\alpha, \gamma) = 0$.

By Lemma 7.3, $R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w) \cap \psi(A_1 \cup A_2) = \varnothing$, so $\psi^{-1}(R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)) \cap A_1 \cup A_2 = \varnothing$.

So, we have three disjoint subsets of $R_I(w)$: $\psi^{-1}(R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w))$, $(A_1 \cup A_2)$, and $\{\alpha\}$. Therefore,

$$|R_I(w)| \geq |\psi^{-1}(R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w))| + |A_1 \cup A_2| + 1 = |R_{\operatorname{supp}\alpha\cap I}(\sigma_\alpha w)| + |A_1 \cup A_2| + 1 \geq -1 + \sum_{j\in\operatorname{supp}\alpha\cap I} n_j + |A_1 \cup A_2| + 1.$$

By Lemma 7.2, $|A_1 \cup A_2| > \sum_{j\in(I\setminus\operatorname{supp}\alpha)} k_j$. So,

$$|R_I(w)| \geq \sum_{j\in\operatorname{supp}\alpha\cap I} n_j + |A_1 \cup A_2| > \sum_{j\in\operatorname{supp}\alpha\cap I} n_j + \sum_{j\in(I\setminus\operatorname{supp}\alpha)} k_j$$

Now, I is the disjoint union of $\operatorname{supp}\alpha \cap I$ and $I \setminus \operatorname{supp}\alpha$, and $k_j = n_j$ for $j \in I$. So, $|R_I(w)| > \sum_{j\in I} n_j$. This is a contradiction with the definition of an excessive cluster. □

Lemma 7.5. *Let $w, n_1, \ldots, n_r$ be an excessively clusterizable configuration of D-multiplicities, $w \neq \mathrm{id}$.*
Let $I \subseteq \{1, \ldots, r\}$ be a subset such that:
denote $k_i = n_i$ if $i \in I$, $k_i = 0$ it $i \notin I$
then, in terms of this notation:
$k_i > 0$ if $i \in I$ and
$\sum k_i > 0$ and
$|R_I(w)| = \sum k_i$
$R_I(w), k_1, \ldots, k_r$ is an excessive cluster.
Then $R_I(w)$ contains a unique $\prec_w$-maximal element α. Moreover, $I \subseteq \operatorname{supp}\alpha$.

Proof. First, assume that there are at least two different $\prec$-maximal elements in $R_I(w)$.

We are going to use Lemma 5.13. Let β_1 and β_2 be two different $\prec$-maximal elements of $R_I(w)$.

Then (β_1, β_2) cannot be -1 since $R_I(w)$ is an I-cluster, and (β_1, β_2) cannot be 1, otherwise they would be $\prec$-comparable. So, $(\beta_1, \beta_2) = 0$, and, by the definition of an I-cluster, $\operatorname{supp}\beta_1 \cap \operatorname{supp}\beta_2 \cap I = \varnothing$.

So, by Lemma 5.13, there is actually a unique $\prec$-maximal element of $R_I(w)$, denote it by β.

Now let $i \in I$. By Lemma 5.12, $\alpha_i \in \operatorname{supp}\beta$. So, $I \subseteq \operatorname{supp}\beta$.

Again let $i \in I$. Assume that there exists a $\prec_w$-maximal element of $\gamma \in R_I(w)$ such that $\alpha_i \notin \operatorname{supp}\gamma$. Clearly, $\gamma \neq \beta$.

β is the unique $\prec$-maximal element of $R_I(w)$, so $\gamma \prec \beta$, and $\operatorname{supp}\gamma \subseteq \operatorname{supp}\beta$. Also, $\operatorname{supp}\gamma \cap I \neq \varnothing$ since $\gamma \in R_I(w)$. So, $\operatorname{supp}\gamma \cap \operatorname{supp}\beta \cap I \neq \varnothing$, and, by the definition of an I-cluster, $(\beta, \gamma) = 1$. Then $\beta - \gamma \in \Delta^+$, and β and γ are $\prec_w$-comparable.

γ is a $\prec_w$-maximal element of $R_I(w)$, so $\beta \prec_w \gamma$, and $\gamma - \beta \in w\Delta^+$, and $\beta - \gamma \in w\Delta^-$, and $\beta - \gamma \in \Delta^+ \cap w\Delta^-$.

$i \in I$, so $\alpha_i \in \operatorname{supp}\beta$, but we have assumed that $\alpha_i \notin \operatorname{supp}\gamma$. So, $\alpha_i \in \operatorname{supp}\beta - \gamma$, and $\beta - \gamma \in R_I(w)$. But $(\gamma, \beta - \gamma) = -1$, a contradiction with the definition of an I-cluster.

Therefore, if γ is a $\prec_w$-maximal element of $R_I(w)$, then $I \subseteq \operatorname{supp}\gamma$.

Finally, assume that there are two different $\prec_w$-maximal elements of $R_I(w)$. Denote them by α and γ. We know that $I \subseteq \operatorname{supp}\alpha$ and $I \subseteq \operatorname{supp}\gamma$, so $\operatorname{supp}\alpha \cap \operatorname{supp}\gamma \cap I \neq \varnothing$. By the definition of an I-cluster, $(\alpha, \gamma) = 1$. Then $\alpha - \gamma \in \Delta$, and α and γ are $\prec_w$-comparable, a contradiction.

So, there exists a unique $\prec_w$-maximal element of $R_I(w)$. □

Lemma 7.6. *Let $w, n_1, \ldots, n_r$ be an excessively clusterizable configuration of D-multiplicities, $n = \ell(w)$.*

Then there exists a function $f\colon \{1, \ldots, n\} \to \Pi$ that takes each value α_i exactly n_i times and such that there exists a unique sorting process of w with list of labels f, moreover, this unique sorting process is in fact antireduced, and its X-multiplicity equals 1.

Proof. Induction on n. If $n = 0$, everything is clear.

If $n > 0$, then, by the definition of an excessively clusterizable configuration,
there exists a subset $I \subseteq \{1, \ldots, r\}$ such that:
denote $k_i = n_i$ if $i \in I$, $k_i = 0$ it $i \notin I$
then, in terms of this notation:
$k_i > 0$ if $i \in I$ and
$\sum k_i > 0$ and
$|R_I(w)| = \sum k_i$
$R_I(w), k_1, \ldots, k_r$ is an excessive cluster and
$((\Delta^+ \cap w\Delta^-) \setminus R_I(w)), n_1 - k_1, \ldots, n_r - k_r$ is excessively clusterizable.

By Lemma 7.5, $R_I(w)$ contains a unique $\prec_w$-maximal element, denote it by β_1, and $I \subseteq \operatorname{supp}\beta_1$.

Choose an arbitrary $i \in I$. Set $f(1) = \alpha_i$.

$\alpha_i \in \operatorname{supp}\beta_1$, so, by Corollary 3.12, there exists $\alpha \in \Delta^+ \cap w\Delta^-$ such that $\alpha_i \in \operatorname{supp}\alpha$, and σ_α is an antisimple sorting reflection for w. $\alpha_i \in \operatorname{supp}\alpha$, so $\alpha \in R_I(w)$. By Lemma 3.10, α is a $\prec_w$-maximal element of $\Delta^+ \cap w\Delta^-$. Then it also $\prec_w$-maximal in $R_I(w)$. But by Lemma 7.5, $R_I(w)$ contains only one $\prec_w$-maximal element, so $\alpha = \beta_1$. In other words, σ_{β_1} is an antisimple sorting reflection for w.

Set $w_1 = \sigma_{\beta_1} w$. By Lemma 3.13, $\Delta^+ \cap w_1\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \beta_1$.

By Proposition 5.10, $(\Delta^+ \cap w\Delta^-) \setminus \beta_1, n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r$ is an excessively clusterizable A-configuration. So, $w_1, n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r$ is an excessively clusterizable configuration of D-multiplicities.

By the induction hypothesis, there exists a function $g \colon \{1, \ldots, n-1\} \to \Pi$ that takes each value α_j with $j \neq i$ exactly n_j times and takes value α_i exactly $n_i - 1$ times and such that there exists a unique sorting process of w_1 with list of labels g. Denote this sorting process by $\beta_2, \ldots, \beta_n$. It is antireduced, and its X-multiplicity equals 1.

For each j, $2 \leq j \neq n$, set $f(j) = g(j-1)$. Then f takes each value α_i exactly n_i times.

Then we can assign label α_i to β_1, and $\beta_1, \beta_2, \ldots, \beta_n$ becomes a labeled antireduced sorting process for w with list of labels f.

By the definition of an I-cluster, the coefficient in front of α_i in the decomposition of β_1 into a linear combination of simple roots equals 1. So, the X-multiplicity of the labeled sorting process we have constructed for w is 1.

Suppose that we have another sorting process for w with list of labels f, denote it by $\gamma_1, \ldots, \gamma_n$.

By Lemma 7.4 $\gamma_1 = \beta_1$. It follows directly from the definition of a labeled sorting process that $\gamma_2, \ldots, \gamma_n$ is a labeled sorting process of w_1 with list of labels g. Therefore, by the induction hypothesis, $\beta_j = \gamma_j$ for $2 \leq j \leq n$, and the labeled sorting process of w with list of labels l is unique. □

Proposition 7.7. *Let $w, n_1, \ldots, n_r$ be an excessively clusterizable configuration of D-multiplicities. Then $C_{w,n_1,\ldots,n_r} = 1$.*

Proof. Follows directly from Lemma 7.6 and Lemma 3.26. □

8 Criterion for unique sortability

Theorem 8.1. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. The following conditions are equivalent:*

1. *$w, n_1, \ldots, n_r$ is excessively clusterizable.*
2. *$C_{w,n_1,\ldots,n_r} = 1$.*

Proof. Follows directly from Proposition 6.40 and Proposition 7.7. □

9 Powers of a single divisor

Definition 9.1. We call a sequence $0 = \beta_0, \beta_1, \ldots, \beta_k$, where $\beta_1, \ldots, \beta_k \in \Delta^+$, *path-originating* if:
$\beta_j - \beta_{j-1}$ for all j $(1 \leq j \leq k)$ are simple roots, denote them by $\alpha_{i_j} = \beta_j - \beta_{j-1}$ $(1 \leq j \leq k)$
$(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$, and $(\alpha_{i_j}, \alpha_{i_{j'}}) = 0$ if $|j - j'| > 1$.

Remark 9.2. *In terms of the notations from Definition 9.1, $\beta_i = \alpha_1 + \ldots + \alpha_i$.*

Remark 9.3. *In terms of the notations from Definition 9.1, there are no coinciding roots among $\alpha_{i_1}, \ldots, \alpha_{i_k}$.*

Remark 9.4. *If $0 = \beta_0, \beta_1, \ldots, \beta_k$ is a path-originating sequence and $j \leq k$, then $0 = \beta_0, \beta_1, \ldots, \beta_j$ is a path-originating sequence of roots.*

Remark 9.5. *In terms of Dynkin diagrams (recall that we are working only with simply-laced Dynkin diagrams), two vertices i and j are connected with an edge if and only if $(\alpha_i, \alpha_j) = -1$. Otherwise, $(\alpha_i, \alpha_j) = 0$ for $i \neq j$, $(\alpha_i, \alpha_i) = 2$.*

So, in terms of Dynkin diagrams, the path-originating sequences of roots are exactly the sequences of roots constructed as follows:

Take any simple path in the Dynkin diagrams, i. e. any sequence $i_1, \ldots, i_k$ of vertices such that each two subsequent vertices are connected with an edge, and the vertices don't reappear.

Then set $\beta_1 = \alpha_{i_1}, \beta_2 = \alpha_{i_1} + \alpha_{i_2}, \ldots, \beta_k = \alpha_{i_1} + \ldots + \alpha_{i_k}$.

Remark 9.6. *If $0 = \beta_0, \beta_1, \dots, \beta_k$ is a path-originating sequence and $j \le k$, then $\beta_1 \prec \beta_2 \prec \dots \prec \beta_k$.*

Lemma 9.7. *Let $0 = \beta_0, \beta_1, \dots, \beta_k$ is a path-originating sequence. Denote $\alpha_{i_j} = \beta_j - \beta_{j-1}$.*

If $\beta_m = \gamma + \delta$, where $1 \le m \le k$ and $\gamma, \delta \in \Delta^+$, then, up to an interchange of γ and δ, there exists an index p ($1 \le p < m$) such that $\gamma = \beta_p$ and $\delta = \alpha_{i_{p+1}} + \dots + \alpha_{i_m}$.

Proof. Note that if $0 = \beta_0, \beta_1, \dots, \beta_k$ is a path-originating sequence and $1 \le m \le k$, then $0 = \beta_0, \beta_1, \dots, \beta_m$ is also a path-originating sequence. So it suffices to consider the case when $k = m$.

Induction on k. If $k = 1$, then $\beta_k = \beta_1 \in \Pi$, and everything is clear.

Suppose that $k > 1$. Then $\beta_{k-1} = \beta_k - \alpha_{i_k} \in \Delta^+$, so, by Lemma 2.5, $(\beta_k, \alpha_{i_k}) = 1$. So, $(\gamma, \alpha_{i_k}) + (\delta, \alpha_{i_k}) = 1$.

If $\gamma = \alpha_{i_k}$ (resp. $\delta = \alpha_{i_k}$)), then $\delta = \beta_{k-1}$ (resp. $\gamma = \beta_{k-1}$), and we are done.

Suppose that $\gamma \ne \alpha_{i_k}$ and $\delta \ne \alpha_{i_k}$. Then one of the products (γ, α_{i_k}) and (δ, α_{i_k}) has to be 1, and the other has to be 0.

Without loss of generality, $(\gamma, \alpha_{i_k}) = 0$ and $(\delta, \alpha_{i_k}) = 1$. Then $\delta - \alpha_{i_k} \in \Delta$.

$\delta \in \Delta^+$ and $\alpha_{i_k} \in \Pi$, so $\alpha_{i_k} - \delta$ cannot be in Δ^+. Hence, $\delta - \alpha_{i_k} \in \Delta^+$.

Now we have $\beta_{k-1} = \beta_k - \alpha_{i_k} = \gamma + (\delta - \alpha_{i_k})$. By the induction hypothesis, there exists p ($1 \le p < k-1$) such that either $\gamma = \beta_p$, or $\delta - \alpha_{i_k} = \beta_p$.

Assume that $\delta - \alpha_{i_k} = \beta_p$. Then by Lemma 2.5, $(\beta_p, \alpha_{i_k}) = -1$. On the other hand, $\beta_p = \alpha_{i_1} + \dots + \alpha_{i_p}$, and $p < k - 1$, so $(\beta_p, \alpha_{i_k}) = 0$, a contradiction.

So, $\gamma = \beta_p$, then $\delta = \beta_k - \beta_p = \alpha_{i_{p+1}} + \dots + \alpha_{i_k}$. $\square$

Lemma 9.8. *Let $0 = \beta_0, \beta_1, \dots, \beta_k$ is a path-originating sequence.*

Then $(\beta_j, \beta_{j'}) = 1$ if $1 \le j, j' \le k$, $j \ne j'$.

Proof. Using Remark 9.4 and induction on k, it is sufficient to prove that $(\beta_j, \beta_k) = 1$ for $1 \le j < k$.

Denote $\alpha_{i_j} = \beta_j - \beta_{j-1}$ ($1 \le j \le k$). Then $\beta_k = \alpha_{i_1} + \dots + \alpha_{i_k}$. By the definition of a path-originating sequence, if $1 < j < k$, then $(\alpha_{i_{j-1}}, \alpha_{i_j}) = -1$, $(\alpha_{i_j}, \alpha_{i_j}) = 2$, $(\alpha_{i_{j+1}}, \alpha_{i_j}) = -1$, and $(\alpha_{i_{j'}}, \alpha_{i_j}) = 0$ for $j' \ne j-1, j, j+1$. So, $(\beta_k, \alpha_j) = 0$.

By Lemma 2.5, $(\beta_k, \beta_{k-1}) = 1$ since $\beta_k - \beta_{k-1} \in \Delta$. Now, for $1 < j < k$ we have $(\beta_k, \beta_{j-1}) = (\beta_k, \beta_j) - (\beta_k, \alpha_{i_j}) = (\beta_k, \beta_j)$. So, $(\beta_k, \beta_{k-1}) = (\beta_k, \beta_{k-2}) = \dots = (\beta_k, \beta_1) = 1$. $\square$

Lemma 9.9. *Let $0 = \beta_0, \beta_1, \dots, \beta_k$ is a path-originating sequence. Let $\beta_1 = \alpha_{i_1}$.*

Then $\{\beta_1, \dots, \beta_k\}$ is a $\{i_1\}$-cluster.

Proof. Since all differences $\beta_i - \beta_{i-1}$ are different simple roots, the coefficients in front of simple roots in the decomposition of any β_i into a linear combination of simple roots are at most 1.

By Lemma 9.8, $(\beta_j, \beta_{j'}) = 1$ if $1 \le j, j' \le k$, $j \ne j'$. So, $\{\beta_1, \dots, \beta_k\}$ is a $\{i_1\}$-cluster. $\square$

Lemma 9.10. *Let $0 = \beta_0, \beta_1, \dots, \beta_n$ is a path-originating sequence. Let $\beta_1 = \alpha_{i_1}$.*

Then $\{\beta_1, \dots, \beta_n\}, 0, \dots, 0, n, 0, \dots, 0$ (where n occurs at the i_1th position) is an excessive cluster.

Proof. By Remark 9.6, $\alpha_{i_1} \preceq \beta_j$ for $1 \le j \le n$, so $\alpha_{i_1} \in \operatorname{supp} \beta_j$ for $1 \le j \le n$. So, $R_{i_1}(\{\beta_1, \dots, \beta_n\}) = \{\beta_1, \dots, \beta_n\}$.

By Lemma 9.9, $\{\beta_1, \dots, \beta_n\}$ is a $\{i_1\}$-cluster. We have to check that $\{\beta_1, \dots, \beta_n\}, 0, \dots, 0, n, 0, \dots, 0$ is an excessive A-configuration, but since the sequence $0, \dots, 0, n, 0, \dots, 0$ contains only one non-zero entry, the i_1th one, the only requirement in the definition of an excessive A-configuration is that $|R_{i_1}(\{\beta_1, \dots, \beta_n\})| = n$, but this is clear since $R_{i_1}(\{\beta_1, \dots, \beta_n\}) = \{\beta_1, \dots, \beta_n\}$. $\square$

Lemma 9.11. *Let $0 = \beta_0, \beta_1, \dots, \beta_n$ is a path-originating sequence.*

Let $w = \sigma_{\beta_n} \dots \sigma_{\beta_1}$.

Then $\beta_n, \dots, \beta_1$ is an antireduced sorting process for w, and $w = \sigma_{\beta_n} \dots \sigma_{\beta_1}$ is an antireduced expression for w.

Proof. Induction on n. If $n = 0$, then $w = \mathrm{id}$, and the list $\beta_n, \ldots, \beta_1$ is empty, so everything is clear.

Suppose that $n > 1$. Let us check that σ_{β_n} is an antisimple sorting reflection for w. Let us compute $w^{-1}\beta_n$. Note that $w^{-1} = \sigma_{\beta_1} \ldots \sigma_{\beta_n}$

Denote $\alpha_{i_j} = \beta_j - \beta_{j-1}$ $(1 \le j \le n)$. Then $\beta_j = \alpha_{i_1} + \ldots + \alpha_{i_j}$ $(0 \le j \le n)$.

First, $\sigma_{\beta_n}\beta_n = -\beta_n$.

If $n = 1$, we can write $-\beta_n = -\alpha_{i_1}$.

If $n > 1$:

We have $\beta_n = \beta_{n-1} + \alpha_{i_n}$ and $(\beta_{n-1}, \beta_n) = 1$, so $\sigma_{\beta_{n-1}}\beta_n = \alpha_{i_n}$, and $\sigma_{\beta_{n-1}}(-\beta_n) = -\alpha_{i_n}$.

Now, for each j, $n-2 \ge j \ge 1$, we have $(\beta_j, \alpha_{i_n}) = (\alpha_{i_1} + \ldots + \alpha_{i_j}, \alpha_{i_n}) = 0$ by the definition of a path-originating sequence. So, $\sigma_{\beta_j}(-\alpha_{i_n}) = -\alpha_{i_n}$.

Therefore, $w^{-1}\beta_n = -\alpha_{i_n}$.

END If $n > 1$.

Summarizing, for all n we always have $w^{-1}\beta_n = -\alpha_{i_n}$.

Now, $\beta_n \in \Delta^+$ and $\beta_n = w(-\alpha_{i_n})$, so $\beta_n \in w\Delta^-$, and $\beta_n \in \Delta^+ \cap w\Delta^-$. Again, $w^{-1}\beta_n = -\alpha_{i_n}$. By Lemma 3.8, σ_{β_n} is an antisimple sorting reflection for w.

Set $w_1 = \sigma_{\beta_n} w = \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1}$. By the induction hypothesis, $\beta_{n-1}, \ldots, \beta_1$ is an antireduced sorting process for w. Now it follows directly from the definition of a sorting process that $\beta_n, \ldots, \beta_1$ is an antireduced sorting process for w, and $w = \sigma_{\beta_n} \ldots \sigma_{\beta_1}$ is an antireduced expression for w. □

Corollary 9.12. *Let* $0 = \beta_0, \beta_1, \ldots, \beta_n$ *is a path-originating sequence.*

Let $w = \sigma_{\beta_n} \ldots \sigma_{\beta_1}$.

Then $\Delta^+ \cap w\Delta^- = \{\beta_1, \ldots, \beta_n\}$.

Proof. Follows directly from Lemma 3.16 and Lemma 9.11. □

Corollary 9.13. *Let* $0 = \beta_0, \beta_1, \ldots, \beta_n$ *is a path-originating sequence. Let* $\beta_1 = \alpha_{i_1}$.

Let $w = \sigma_{\beta_n} \ldots \sigma_{\beta_1}$.

Then $\Delta^+ \cap w\Delta^-, 0, \ldots, 0, n, 0, \ldots, 0$ *(where n occurs at the i_1th position) is an excessive cluster.*

Proof. Follows directly from Lemma 9.10 and Corollary 9.12. □

Corollary 9.14. *Let* $0 = \beta_0, \beta_1, \ldots, \beta_n$ *is a path-originating sequence. Let* $\beta_1 = \alpha_{i_1}$.

Let $w = \sigma_{\beta_n} \ldots \sigma_{\beta_1}$.

Then $\Delta^+ \cap w\Delta^-, 0, \ldots, 0, n, 0, \ldots, 0$ *(where n occurs at the i_1th position) is an excessively clusterizable A-configuration.*

Proof. Follows directly from the definition of an excessively clusterizable A-configuration for $I = \{i_1\}$ and Corollary 9.12. □

Lemma 9.15. *Let* $w \in W$, $\alpha_{i_1} \in \Pi$, $n = \ell(w)$.

Suppose that $\Delta^+ \cap w\Delta^-, 0, \ldots, 0, n, 0, \ldots, 0$ *(where n occurs at the i_1th position) is an excessive cluster.*

Then it is possible to write $\Delta^+ \cap w\Delta^-$ *as* $\{\beta_1, \ldots, \beta_n\}$, *where* $0 = \beta_0, \beta_1, \ldots, \beta_n$ *is a path-originating sequence, [if* $n > 0$, *then* $\beta_1 = \alpha_{i_1}$ *], and* $\beta_n, \ldots, \beta_1$ *is an antireduced sorting process for w* ($w = \sigma_{\beta_n} \ldots \sigma_{\beta_1}$).

Proof. Induction on n. If $n = 0$, everything is clear. Suppose that $n > 0$.

First, $\Delta^+ \cap w\Delta^-, 0, \ldots, 0, n, 0, \ldots, 0$ (where n occurs at the i_1th position) is an excessive cluster, so, in particular, it is an excessive A-configuration.

Note that there is only one possibility for the set I from the definition of an excessive A-configuration, namely $I = \{i_1\}$, because all other entries in the sequence $0, \ldots, 0, n, 0, \ldots, 0$ are zeros. So, this definition actually says that $R_{\{i_1\}}(w) = \Delta^+ \cap w\Delta^-$.

The definition of an excessive cluster also says that $R_{\{i_1\}}(w)$ is a $\{\alpha_{i_1}\}$-cluster.

In other words, $\Delta^+ \cap w\Delta^-$ is a $\{\alpha_{i_1}\}$-cluster.

By Corollary 3.11, there exists a root, denote it by β_n, $\beta_n \in \Delta^+ \cap w\Delta^-$, such that σ_{β_n} is an antisimple sorting reflection for w.

Denote $w_1 = \sigma_{\beta_n} w$. By Lemma 3.13, $\Delta^+ \cap w_1\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \beta_n$.

Recall that $R_{\{i_1\}}(w) = \Delta^+ \cap w\Delta^-$. In particular, $\alpha_{i_1} \in \operatorname{supp} \beta_n$, and we can use Proposition 5.10. It says that $\Delta^+ \cap w_1\Delta^-, 0, \ldots, 0, n-1, 0, \ldots, 0$ (where n occurs at the i_1th position) is an excessively clusterizable A-configuration.

By the induction hypothesis, it is possible to write $\Delta^+ \cap w_1\Delta^-$ as $\{\beta_1, \ldots, \beta_{n-1}\}$, where $0 = \beta_0, \beta_1, \ldots, \beta_{n-1}$ is a path-originating sequence, [if $n > 1$, then $\beta_1 = \alpha_{i_1}$], and $\beta_{n-1}, \ldots, \beta_1$ is an antireduced sorting process for w_1 ($w_1 = \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1}$).

Then we can write $\Delta^+ \cap w\Delta^- = \{\beta_1, \ldots, \beta_n\}$. It also already follows from the choice of β_n and of the definition of an antireduced sorting process that $\beta_n, \beta_{n-1}, \ldots, \beta_1$ is an antireduced sorting process for w ($w = \sigma_{\beta_n} \ldots \sigma_{\beta_1}$).

Denote $\alpha_{i_j} = \beta_j - \beta_{j-1}$ for $1 \le j < n$.

$\Delta^+ \cap w\Delta^-$ is a $\{\alpha_{i_1}\}$-cluster and $R_{\{i_1\}}(w) = \Delta^+ \cap w\Delta^-$, so there are no orthogonal roots in $\Delta^+ \cap w\Delta^-$. Indeed, if $\delta_1, \delta_2 \in \Delta^+ \cap w\Delta^-$ and $(\delta_1, \delta_2) = 0$, then $\delta_1, \delta_2 \in R_{\{i_1\}}(w)$, and $\alpha_{i_1} \in \operatorname{supp} \delta_1 \cap \operatorname{supp} \delta_2$, a contradiction with the definition of a $\{\alpha_i\}$-cluster.

Therefore, if $1 \le j, j' \le k$, $j \ne j'$, then $(\beta_j, \beta_{j'}) = 1$.

Denote $\gamma = \beta_n - \beta_{n-1}$. By Lemma 2.5, $\gamma \in \Delta$, $(\gamma, \beta_{n-1}) = -1$, and $(\gamma, \beta_n) = 1$. Also, σ_{β_n} is an antisimple sorting reflection for w, so, by Lemma 3.10, β_n is a $\prec_w$-maximal element of $\Delta^+ \cap w\Delta^-$, so $\gamma = \beta_n - \beta_{n-1} \in w\Delta^+$, and $-\gamma = \beta_{n-1} - \beta_n \in w\Delta^-$.

Also, $(\gamma, \beta_n) = 1$, so $(-\gamma, \beta_n) = -1$, and $-\gamma$ cannot be in $\Delta^+ \cap w\Delta^-$, because $\Delta^+ \cap w\Delta^-$ is a $\{\alpha_{i_1}\}$-cluster. So, $-\gamma \notin \Delta^+$, and $\gamma \in \Delta^+$.

Assume that $\gamma \notin \Pi$. Then there exist $\gamma_1, \gamma_2 \in \Delta^+$ such that $\gamma_1 + \gamma_2 = \gamma$. We have $(\gamma, \beta_{n-1}) = -1$, $(\gamma_1, \beta_{n-1}) + (\gamma_2, \beta_{n-1}) = -1$. So, one of the products (γ_1, β_{n-1}) and (γ_2, β_{n-1}) equals -1, and the other equals 0.

Without loss of generality (after a possible interchange of γ_1 and γ_2) we may suppose that $(\gamma_1, \beta_{n-1}) = -1$ and $(\gamma_2, \beta_{n-1}) = 0$. Recall that if $1 \le j, j' \le k$, $j \ne j'$, then $(\beta_j, \beta_{j'}) = 1$. So, γ_2 cannot be equal to one of the roots β_j, $1 \le j \le k$. In other words, $\gamma_2 \notin \Delta^+ \cap w\Delta^-$.

Set $\delta = \beta_{n-1} + \gamma_1$. By Lemma 2.5, $\delta \in \Delta^+$. Then $\delta + \gamma_2 = \beta_{n-1} + \gamma_1 + \gamma_2 = \beta_{n-1} + \gamma = \beta_n$.

Clearly, $\beta_{n-1} \prec \delta$. We already know that $0 = \beta_0, \beta_1, \ldots, \beta_{n-1}$ is a path-originating sequence, so by Remark 9.6, $\beta_1 \prec \beta_2 \prec \ldots \prec \beta_{n-1}$. So, $\beta_j \prec \delta$ for $1 \le j < n$, and $\beta_j \ne \delta$ for $1 \le j < n$. Also, $\beta_n \ne \delta$ since $\beta_n = \delta + \gamma_2$, and $\gamma_2 \in \Delta^+$, so $\gamma_2 \ne 0$.

Therefore, $\delta \notin \Delta^+ \cap w\Delta^-$.

On the other hand, $\delta, \gamma_2 \in \Delta^+$. So, $\delta, \gamma_2 \notin w\Delta^-$ and $\delta, \gamma_2 \in w\Delta^+$. But then $\beta_n = \delta + \gamma_2 \in w\Delta^+$, $\beta_n \notin w\Delta^-$, $\beta_n \notin \Delta^+ \cap w\Delta^-$, a contradiction.

END Assume that $\gamma \notin \Pi$.

Therefore, $\gamma \in \Pi$.

If $n = 1$, then:

$\beta_{n-1} = \beta_0 = 0$, and we see that $\beta_1 = \gamma \in \Pi$. We also know that $\alpha_{i_1} \in \operatorname{supp} \beta_1$, so in fact $\alpha_{i_1} = \beta_1 = \gamma$.

END If $n = 1$.

Denote $\alpha_{i_n} = \gamma$. The previous argument shows that there is no conflict of notation for $n = 1$.

If $n = 1$, then it is already clear that β_0, β_1 is a path-originating sequence. Let us check that if $n > 1$, then $\beta_0, \ldots, \beta_n$ is also a path-originating sequence.

We already know that $\beta_0, \ldots, \beta_{n-1}$ is a path-originating sequence, so the products $(\alpha_j, \alpha_{j'})$ for $1 \le j, j' < n$ are the same as they should be in the definition of a path-originating sequence. So we have to check that $(\alpha_{i_n}, \alpha_{i_{n-1}}) = -1$ and $(\alpha_{i_n}, \alpha_{i_j}) = 0$ for $1 \le j < n-1$.

First, note that $(\beta_{n-1}, \alpha_{i_n}) = -1$ by Lemma 2.5. If $1 \le j < n-1$, then $(\beta_j, \alpha_{i_n}) = (\beta_j, \beta_n) - (\beta_j, \beta_{n-1}) = 1 - 1 = 0$. If $j = 0$, then $\beta_j = 0$, and also $(\beta_j, \alpha_{i_n}) = 0$.

So, if $0 \le j < n-1$, then $(\beta_j, \alpha_{i_n}) = 0$.

Now, $(\alpha_{i_n}, \alpha_{i_{n-1}}) = (\alpha_{i_n}, \beta_{n-1}) - (\alpha_{i_n}, \beta_{n-2}) = -1$, and if $1 \le j < n-1$, then $(\alpha_{i_n}, \alpha_{i_j}) = (\alpha_{i_n}, \beta_j) - (\alpha_{i_n}, \beta_{j-1}) = 0$.

So, $\beta_0, \ldots, \beta_n$ is a path-originating sequence. □

Corollary 9.16. *Let $w \in W$, $\alpha_{i_1} \in \Pi$, $n = \ell(w)$.*

Suppose that $\Delta^+ \cap w\Delta^-, 0, \ldots, 0, n, 0, \ldots, 0$ (where n occurs at the i_1th position) is an excessively clusterizable A-configuration.

Then it is possible to write $\Delta^+ \cap w\Delta^-$ as $\{\beta_1, \dots, \beta_n\}$, where $0 = \beta_0, \beta_1, \dots, \beta_n$ is a path-originating sequence, [if $n > 0$, then $\beta_1 = \alpha_{i_1}$], and $\beta_n, \dots, \beta_1$ is an antireduced sorting process for w ($w = \sigma_{\beta_n} \dots \sigma_{\beta_1}$).

Proof. There is only one possibility for the set I from the definition of an excessively clusterizable A-configuration, namely $I = \{i_1\}$, because all other entries in the sequence $0, \dots, 0, n, 0, \dots, 0$ are zeros. So, this definition actually requires $\Delta^+ \cap w\Delta^-, 0, \dots, 0, n, 0, \dots, 0$ to be an excessive cluster. The claim now follows from Lemma 9.15. □

Proposition 9.17. *Let $w \in W$, let $\alpha_{i_1} \in \Pi$, and let $n = \ell(w)$.*
The following conditions are equivalent:

1. *$C_{w,0,\dots,0,n,0,\dots,0} = 1$ (where n occurs at the i_1th position)*
2. *$\Delta^+ \cap w\Delta^-, 0, \dots, 0, n, 0, \dots, 0$ (where n occurs at the i_1th position) is an excessive cluster.*
3. *$\Delta^+ \cap w\Delta^-, 0, \dots, 0, n, 0, \dots, 0$ (where n occurs at the i_1th position) is an excessively clusterizable A-configuration.*
4. *There exists a path-originating sequence $0, \beta_1, \dots, \beta_n$ such that $\alpha_{i_1} = \beta_1$, and $\Delta^+ \cap w\Delta^- = \{\beta_1, \dots, \beta_n\}$.*
5. *There exists a path-originating sequence $0, \beta_1, \dots, \beta_n$ such that $\alpha_{i_1} = \beta_1$ and $w = \sigma_{\beta_n} \dots \sigma_{\beta_1}$.*

The sequence $0, \beta_1, \dots, \beta_n$ in conditions 4 and 5 is actually the same and unique. Moreover, $\beta_n, \dots, \beta_1$ is an antireduced sorting process for w, and and $w = \sigma_{\beta_n} \dots \sigma_{\beta_1}$ is an antireduced expression.

Proof. 1 ⇔ 3 follows from Theorem 8.1.
2 ⇒ 4 follows from Lemma 9.15.
2 ⇒ 5 follows from Lemma 9.15.
3 ⇒ 4 follows from Corollary 9.16.
3 ⇒ 5 follows from Corollary 9.16.
4 ⇒ 2 follows from Lemma 9.10.
5 ⇒ 3 follows from Corollary 9.14.
Uniqueness in 4 follows from Remark 9.6.
By Corollary 9.12, if 5 holds for some path-originating sequence, then 4 also holds for the same sequence, so the path-originating sequence in 5 is also unique and is the same in 4 and 5.
Finally, the "moreover" part follows from Lemma 9.11. □

Corollary 9.18. *Let $i \in \{1, \dots, r\}$. Then the maximal number n such that D_i^n is a multiplicity-free monomial equals the length of the longest simple path in the Dynkin diagram that starts at the ith vertex.* □

10 Powers of many divisors

In this section, we are going to give an upper bound on the length of $w \in W$ such that there exist numbers $n_1, \dots, n_r$ such that $C_{w,n_1,\dots,n_r} = 1$. We are going to talk about simply excessively clusterizable A-configurations most of the time, and then we will use Proposition 5.23.

Lemma 10.1. *Let $w \in W$. Let $\Delta^+ \cap w\Delta^-, n_1, \dots, n_r$ be a simply excessively clusterizable A-configuration.*
Let $i_1 \in \{1, \dots, r\}$ be an index such that:
denote $k_{i_1} = n_{i_1}$ and $k_j = 0$ if $j \ne i_1$
then, in terms of this notation:
$k_{i_1} > 0$ and
$|R_{\{\alpha_{i_1}\}}(w)| = k_{i_1}$ (note that this implies that $((\Delta^+ \cap w\Delta^-) \setminus R_{\{i_1\}}(w)), n_1 - k_1, \dots, n_r - k_r$ is an A-configuration) and

$R_{\{\alpha_{i_1}\}}(w), k_1, \ldots, k_r$ *is a simple excessive cluster and*

$((\Delta^+ \cap w\Delta^-) \setminus R_{\{\alpha_{i_1}\}}(w)), n_1 - k_1, \ldots, n_r - k_r$ *is simply excessively clusterizable.*

Then there is a path $i_1, \ldots, i_{k_{i_1}}$ *of length* k_{i_1} *in the Dynkin diagram, which is simple (i. e. no vertices reappear) and is such that:*

there exists a root $\alpha \in \Delta^+ \cap w\Delta^-$ *such that* $\alpha_{i_j} \in \operatorname{supp} \alpha$ *for all* j $(1 \le j \le k_{i_1})$*.*

Proof. It is clear from the definitions of a simply excessively clusterizable A-configuration and an excessively clusterizable A-configuration that $\Delta^+ \cap w\Delta^-, n_1, \ldots, n_r$ is an excessively clusterizable A-configuration.

By Theorem 8.1, $C_{w,n_1,\ldots,n_r} = 1$.

Consider the following list of labels of length $\ell(w)$:

$$\underbrace{\alpha_1, \ldots, \alpha_1}_{n_1 \text{ times}}, \ldots, \underbrace{\alpha_{i_1-1}, \ldots, \alpha_{i_1-1}}_{n_{i_1-1} \text{ times}}, \underbrace{\alpha_{i_1+1}, \ldots, \alpha_{i_1+1}}_{n_{i_1-1} \text{ times}}, \ldots, \underbrace{\alpha_r, \ldots, \alpha_r}_{n_r \text{ times}}, \underbrace{\alpha_{i_1}, \ldots, \alpha_{i_1}}_{n_{i_1} \text{ times}}$$

By Lemma 3.26, there exists a labeled sorting process $\beta_1, \ldots, \beta_{\ell(w)}$ of w with this list of labels. Denote $w' = \sigma_{\beta_{\ell(w)-n_{i_1}}} \ldots \sigma_{\beta_1} w$. Then

$\beta_1, \ldots, \beta_{\ell(w)-n_{i_1}}$ is a labeled sorting process prefix of w with D-multiplicities $n_1 - k_1, \ldots, n_r - k_r$ of labels

and

$\beta_{\ell(w)-n_{i_1}+1} \ldots \beta_{\ell(w)}$ is a labeled sorting process of w with D-multiplicities $k_1, \ldots, k_r$ of labels.

So, by Lemma 3.26, $C_{w',k_1,\ldots,k_r} > 0$, and by Lemma 6.5, $C_{w,n_1,\ldots,n_r} \ge C_{w',k_1,\ldots,k_r}$.

Therefore, $C_{w',k_1,\ldots,k_r} = 1$.

By Proposition 9.17, there exists a path-originating sequence $0, \gamma_1, \ldots, \gamma_{n_{i_1}}$ such that $\alpha_{i_1} = \gamma_1$ and $\Delta^+ \cap w'\Delta^- = \{\gamma_1, \ldots, \gamma_{n_{i_1}}\}$.

Choose indices i_j $(2 \le j \le k_{i_1})$ so that $\alpha_{i_j} = \gamma_j - \gamma_{j-1}$. By Remark 9.5, the vertices numbered $i_1, \ldots, i_{n_{i_1}}$ form a simple path in the Dynkin diagram. Moreover, $\gamma_{n_{i_1}} = \alpha_{i_1} + \ldots + \alpha_{n_{i_1}}$.

So, for all j $(1 \le j \le n_{i_1})$, $\alpha_{i_j} \in \operatorname{supp} \gamma_{n_{i_1}}$, and $\gamma_{n_{i_1}} \in \Delta^+ \cap w'\Delta^-$.

Now for each m, $0 \le m \le \ell(w) - n_{i_1}$, denote $w_m = \sigma_{\beta_m} \ldots \sigma_{\beta_1} w$. In particular, $w' = w_{\ell(w)-n_{i_1}}$.

For each m, $1 \le m \le \ell(w) - n_{i_1}$, Lemma 3.7 establishes a bijection $\psi_m \colon (\Delta^+ \cap w_{m-1}\Delta^-) \setminus \{\beta_m\} \to \Delta^+ \cap w_m\Delta^-$. It also follows from Lemma 3.7 that $\delta \preceq \psi_m^{-1}(\delta)$ for all $\delta \in \Delta^+ \cap w_m\Delta^-$.

Set $\alpha = \psi_1^{-1}(\ldots \psi_{\ell(w)-n_{i_1}}^{-1}(\gamma_{n_{i_1}}) \ldots)$. Then $\gamma_{n_{i_1}} \preceq \alpha$. Therefore, for all j $(1 \le j \le n_{i_1})$, $\alpha_{i_j} \in \operatorname{supp} \alpha$. □

Definition 10.2. Let $\alpha_{i_1}, \ldots, \alpha_{i_k} \in \Pi$ be different simple roots, let $n_1, \ldots, n_k \in \mathbb{N}$.

We say that a *multipath* with *beginnings* $i_1, \ldots, i_k$ and with *lengths* $n_1, \ldots, n_k$ in the Dynkin diagram is a sequence of simple paths

$j_{1,1}, \ldots, j_{1,n_1}$,

...

$j_{k,1}, \ldots, j_{k,n_k}$

such that: $i_m = j_{m,1}$ for all m $(1 \le m \le k)$ and if $m < m'$, then i_m does not occur among $j_{m',1}, \ldots, j_{m',n_{m'}}$.

Definition 10.3. Let $j_{1,1}, \ldots, j_{1,n_1}; \ldots; j_{k,1}, \ldots, j_{k,n_k}$ be a multipath. We say that it *avoids vertices* $i_1, \ldots, i_m$ of the Dynkin diagram if

for each m', $1 \le m' \le m$, $i_{m'}$ does not occur among $j_{1,1}, \ldots, j_{1,n_1}; \ldots; j_{k,1}, \ldots, j_{k,n_k}$.

Definition 10.4. Let $j_{1,1}, \ldots, j_{1,n_1}; \ldots; j_{k,1}, \ldots, j_{k,n_k}$ be a multipath. We say that its *total length* is $n_1 + \ldots + n_k$.

MAYBE SHOULD GO TO SECTION 3?

Lemma 10.5. *Let* $w \in W$*,* $\alpha_i \in \Pi$*.*

Denote $k = |R_{\{i\}}(w)|$*.*

Then there exists an antireduced sorting process prefix $\beta_1, \ldots, \beta_k$ *for* w *such that* $R_{\{i\}}(w) = \{\beta_1, \ldots, \beta_k\}$*.*

Proof. Induction on k. If $k = 0$, everything is clear. Suppose $k > 0$.

$k > 0$, so there exists $\alpha \in R_{\{i\}}(w)$, in other words, there exists $\alpha \in \Delta^+ \cap w\Delta^-$ such that $\alpha_i \in \operatorname{supp} \alpha$.

By Corollary 3.12, there exists $\beta_1 \in \Delta^+ \cap w\Delta^-$ such that $\alpha_i \in \operatorname{supp} \beta_1$ and σ_{β_1} is an antisimple sorting reflection for w. Then $\beta_1 \in R_{\{i\}}(w)$.

Denote $w_1 = \sigma_{\beta_1} w$. By Lemma 3.13, $\Delta^+ \cap w_1\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \{\beta_1\}$. $\beta_1 \in R_{\{i\}}(w)$, so $R_{\{i\}}(w_1) = R_{\{i\}}(w) \setminus \{\beta_1\}$ and $|R_{\{i\}}(w_1)| = k - 1$.

By the induction hypothesis, there exists an antireduced sorting process prefix $\beta_2, \ldots, \beta_k$ for w_1 such that $R_{\{i\}}(w_1) = \{\beta_2, \ldots, \beta_k\}$.

Then $\beta_1, \ldots, \beta_k$ is an antireduced sorting process prefix for w, and $R_{\{i\}}(w) = \{\beta_1, \ldots, \beta_k\}$. $\square$

Lemma 10.6. *Let $w \in W$. Let $\Delta^+ \cap w\Delta^-, n_1, \ldots, n_r$ be a simply excessively clusterizable A-configuration.*

Let $I \subseteq \Pi$ be a subset such that $R_I(w) = \varnothing$.

Then there exists a number s and a sequence of indices $i_1, \ldots, i_s$ $(1 \le i_m \le r)$ such that:

all of them are different, and

a $(1 \le a \le r)$ is present among $i_1, \ldots, i_s$ if and only if $n_a > 0$, and

there exists a multipath with beginnings $i_1, \ldots, i_s$ and with lengths $n_{i_1}, \ldots, n_{i_s}$ that avoids I.

Proof. Induction on $\ell(w)$. If $w = \mathrm{id}$, we can take the empty sequence of indices i_m. Suppose that $w \ne \mathrm{id}$.

If $w \ne \mathrm{id}$, then by the definition of a simply excessively clusterizable configuration, $i_1 \in \{1, \ldots, r\}$ be an index such that:

denote $k_{i_1} = n_{i_1}$ and $k_j = 0$ if $j \ne i_1$

then, in terms of this notation:

$k_{i_1} > 0$ and

$|R_{\{\alpha_{i_1}\}}(w)| = k_{i_1}$ (note that this implies that $((\Delta^+ \cap w\Delta^-) \setminus R_{\{i_1\}}(w)), n_1 - k_1, \ldots, n_r - k_r$ is an A-configuration) and

$R_{\{\alpha_{i_1}\}}(w), k_1, \ldots, k_r$ is a simple excessive cluster and

$((\Delta^+ \cap w\Delta^-) \setminus R_{\{\alpha_{i_1}\}}(w)), n_1 - k_1, \ldots, n_r - k_r$ is simply excessively clusterizable.

By Lemma 10.1, there exists a simple path $j_{1,1}, \ldots, j_{1,n_{i_1}}$ of length n_{i_1} in the Dynkin diagram such that $i_1 = j_{1,1}$ and there exists a root $\alpha \in \Delta^+ \cap w\Delta^-$ such that $\alpha_{j_{1,m}} \in \operatorname{supp} \alpha$ for all m $(1 \le m \le n_{i_1})$.

Assume that $j_{1,1}, \ldots, j_{1,n_{i_1}}$ does not avoid I. Then there exists m $(1 \le m \le n_{i_1})$ such that $\alpha_{j_{1,m}} \in I$. Then $\operatorname{supp} \alpha \cap I \ne \varnothing$, so $\alpha \in R_I(w)$, and $R_I(w) \ne \varnothing$, a contradiction. So, $j_{1,1}, \ldots, j_{1,n_{i_1}}$ avoids I.

By Lemma 10.5, there exists an antireduced sorting process prefix $\beta_1, \ldots, \beta_{n_{i_1}}$ for w such that $R_{\{\alpha_{i_1}\}}(w) = \{\beta_1, \ldots, \beta_{n_{i_1}}\}$. Set $w' = \sigma_{\beta_{n_{i_1}}} \ldots \sigma_{\beta_1} w$. By Lemma 3.16, $\Delta^+ \cap w'\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus R_{\{\alpha_{i_1}\}}(w)$.

$R_I(w) = \varnothing$, so $R_{I \cup \{\alpha_{i_1}\}}(w) = R_{\{\alpha_{i_1}\}}(w)$, and $R_{I \cup \{\alpha_{i_1}\}}((\Delta^+ \cap w\Delta^-) \setminus R_{\{\alpha_{i_1}\}}(w)) = R_{\{\alpha_{i_1}\}}(w)) \setminus R_{\{\alpha_{i_1}\}}(w)) = \varnothing$. In other words, $R_{I \cup \{\alpha_{i_1}\}}(w') = \varnothing$.

By the induction hypothesis,

there exists a number s' and a sequence of indices $i'_1, \ldots, i'_{s'}$ $(1 \le i'_m \le r)$ such that:

all of them are different, and

a $(1 \le a \le r)$ is present among $i'_1, \ldots, i'_{s'}$ if and only if $n_a - k_a > 0$, and

there exists a multipath with beginnings $i'_1, \ldots, i'_{s'}$ and with lengths $n_{i'_1} - k_{i'_1}, \ldots, n_{i'_{s'}} - k_{i'_{s'}}$ that avoids $I \cup \{\alpha_{i_1}\}$.

Let us reformulate this conclusion of the induction hypothesis using the fact that $k_{i_1} = n_{i_1}$ and $k_m = 0$ if $m \ne i_1$. Denote also $s = s' + 1$ and $i_m = i'_{m-1}$ $(2 \le m \le s)$. We get the following:

We have a sequence of indices $i_2, \ldots, i_s$ $(1 \le i'_m \le r)$ such that:

all of them are different, and

a $(1 \le a \le r)$ is present among $i_2, \ldots, i_s$ if and only if $n_a > 0$ and $a \ne i_1$, and

there exists a multipath with beginnings $i_2, \ldots, i_s$ and with lengths $n_{i_2}, \ldots, n_{i_s}$ that avoids $I \cup \{\alpha_{i_1}\}$.

Denote this multipath by $j_{2,1}, \ldots, j_{2,n_{i_2}}; \ldots; j_{s,1}, \ldots, j_{s,n_{i_s}}$.

We can say the following about the sequence of indices $i_1, \ldots, i_s$:

all of them are different, and

a $(1 \le a \le r)$ is present among $i_1, \ldots, i_s$ if and only if $n_a > 0$.

Consider the sequences of paths $j_{1,1}, \ldots, j_{1,n_{i_1}}; j_{2,1}, \ldots, j_{2,n_{i_2}}; \ldots; j_{s,1}, \ldots, j_{s,n_{i_s}}$.

The only thing we have to check to conclude that this is a multipath is that if $1 < m$, then i_1 does not occur among $j_{m,1}, \ldots, j_{m,n_m}$. But this is true since $j_{2,1}, \ldots, j_{2,n_{i_2}}; \ldots; j_{s,1}, \ldots, j_{s,n_{i_s}}$ avoids $I \cup \{\alpha_{i_1}\}$.

So, $j_{1,1}, \ldots, j_{1,n_{i_1}}; j_{2,1}, \ldots, j_{2,n_{i_2}}; \ldots; j_{s,1}, \ldots, j_{s,n_{i_s}}$ is a multipath. It avoids I since $j_{1,1}, \ldots, j_{1,n_{i_1}}$ avoids I and $j_{2,1}, \ldots, j_{2,n_{i_2}}; \ldots; j_{s,1}, \ldots, j_{s,n_{i_s}}$ avoids $I \cup \{\alpha_{i_1}\}$.

Its beginnings are $i_1, i_2, \ldots, i_s$ and its lengths are $n_{i_1}, \ldots, n_{i_s}$. ☐

Lemma 10.7. *Let $w \in W$. Let $\Delta^+ \cap w\Delta^-, n_1, \ldots, n_r$ be a simply excessively clusterizable A-configuration.*

Denote by J the set of simple roots α_j such that $n_j > 0$, denote $s = |J|$.

Then there exists a sequence of indices $i_1, \ldots, i_s$ such that $J = \{i_1, \ldots, i_s\}$

and

a multipath with beginnings $i_1, \ldots, i_s$ and with lengths $n_{i_1}, \ldots, n_{i_s}$.

Proof. Follows directly from Lemma 10.6 ☐

Proposition 10.8. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities such that $C_{w,n_1,\ldots,n_r} = 1$.*

Denote by J the set of involved roots.

Then there exists a multipath in the Dynkin diagram whose total length is $\ell(w)$ and whose beginnings are contained in J.

Proof. By Theorem 8.1, $\Delta^+ \cap w\Delta^-, n_1, \ldots, n_r$ is an excessively clusterizable A-configuration.

By Proposition 5.23, there exist numbers $m_1, \ldots, m_r$ such that:

$m_1 + \ldots + m_r = n_1 + \ldots + n_r$, and

if $m_i > 0$, then $\alpha_i \in J$, and

$\Delta^+ \cap w\Delta^-, m_1, \ldots, m_r$ is a simply excessively clusterizable A-configuration.

The claim follows from Lemma 10.7. ☐

Corollary 10.9. *Let $D_1^{n_1} \ldots D_r^{n_r}$ be a multiplicity-free monomial.*

Denote by J the set of roots α_i such that $n_i > 0$.

Then there exists a multipath in the Dynkin diagram whose total length is $n_1 + \ldots + n_r$ and whose beginnings are contained in J.

Proof. As we figured out in Introduction, if $D_1^{n_1} \ldots D_r^{n_r}$ is a multiplicity-free monomial, then there exists $w \in W$ such that $C_{w,n_1,\ldots,n_r} = 1$.

The claim follows from Proposition 10.8. ☐

Lemma 10.10. *Let $w \in W$. Let $I \subseteq \Pi$ be a subset such that $R_I(w) = \Delta^+ \cap w\Delta^-$.*

Let $0 = \beta_0, \beta_1, \ldots, \beta_n$ be a path-originating sequence. Denote $\alpha_{i_k} = \beta_k - \beta_{k-1}$.

Suppose that $\alpha_{i_k} \notin I$ for all k ($1 \le k \le n$).

Denote $w' = \sigma_{\beta_n} \ldots \sigma_{\beta_1} w$

Then:

σ_{β_n} is an admissible sorting reflection for w,

and $R_I(w') = (\sigma_{\beta_n} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$.

and $\Delta^+ \cap w'\Delta^- = \{\beta_1, \ldots, \beta_n\} \cup (\sigma_{\beta_n} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$,

and this union is disjoint,

and for every $\alpha_j \in I$, for every $\gamma \in R_I(w)$:

the coefficient in front of α_j in the decomposition of γ into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_n} \ldots \sigma_{\beta_1}\gamma$ into a linear combination of simple roots.

Proof. Induction on n. If $n = 0$, everything is clear. Suppose that $n > 0$.

Suppose that we already know the induction hypothesis for $n - 1$. Note that σ_{β_n} is an admissible sorting reflection for w' if and only if σ_{β_n} is an admissible desorting reflection for $\sigma_{\beta_n} w' = \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1} w$. So, let us check that σ_{β_n} is an admissible desorting reflection for $\sigma_{\beta_n} w' = \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1} w$.

By the induction hypothesis, $\Delta^+ \cap (\sigma_{\beta_n} w')\Delta^- = \{\beta_1, \ldots, \beta_{n-1}\} \cup (\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-) = \{\beta_1, \ldots, \beta_{n-1}\} \cup R_I(\sigma_{\beta_n} w')$.

By Remark 9.2, $\beta_n = \alpha_{i_1} + \ldots + \alpha_{i_n}$. We have $\alpha_{i_k} \notin I$ for all k $(1 \le k \le n)$, so $\operatorname{supp} \beta_n \cap I = \varnothing$, and $\beta_n \notin R_I(\sigma_{\beta_n} w')$. Also, $\beta_n \ne \beta_k$ for $k < n$, so $\beta_n \notin \Delta^+ \cap (\sigma_{\beta_n} w')\Delta^-$.

But $\beta_n \in \Delta^+$, so $\beta_n \in \Delta^+ \cap (\sigma_{\beta_n} w')\Delta^+$.

If $\beta_n = \gamma + \delta$, where $\gamma, \delta \in \Delta^+$, then, by Lemma 9.7, without loss of generality, there exists k $(1 \le k < n)$ such that $\gamma = \beta_k$ and $\delta = \alpha_{i_{k+1}} + \ldots + \alpha_{i_n}$. So, $\gamma \in \Delta^+ \cap (\sigma_{\beta_n} w')\Delta^-$. By Lemma 3.6, σ_{β_n} is an admissible desorting reflection for $\sigma_{\beta_n} w' = \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1} w$. So, σ_{β_n} is an admissible sorting reflection for w'.

Recall that $\operatorname{supp} \beta_n \cap I = \varnothing$. By Lemma 6.36, $\sigma_{\beta_n} R_I(w') = R_I(\sigma_{\beta_n} w')$, so $R_I(w') = \sigma_{\beta_n}^{-1} R_I(\sigma_{\beta_n} w') = \sigma_{\beta_n} R_I(\sigma_{\beta_n} w')$.

By the induction hypothesis, $R_I(\sigma_{\beta_n} w') = (\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$, so $R_I(w') = (\sigma_{\beta_n} \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$.

Finally, Lemma 3.7 establishes a bijection between $(\Delta^+ \cap w'\Delta^-) \setminus \beta_n$ and $\Delta^+ \cap (\sigma_{\beta_n} w')\Delta^-$. Denote this bijection by $\psi \colon (\Delta^+ \cap w'\Delta^-) \setminus \beta_n \to \Delta^+ \cap (\sigma_{\beta_n} w')\Delta^-$.

By the induction hypothesis, $R_I(\sigma_{\beta_n} w') = (\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$.

and $\Delta^+ \cap \sigma_{\beta_n} w'\Delta^- = \{\beta_1, \ldots, \beta_{n-1}\} \cup (\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$.

By Lemma 6.36, $\psi^{-1}(R_I(\sigma_{\beta_n} w')) = R_I(w')$. So, $\psi^{-1}((\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)) = R_I(w')$.

It from Lemma 3.7 that $\psi^{-1}(\beta_k)$ $(1 \le k \le n-1)$ is either $\beta_n + \beta_k$, or β_k. But $(\beta_n, \beta_k) = 1$ by Lemma 9.8, so $\beta_n + \beta_k \notin \Delta$ by Lemma 2.5. So, $\psi^{-1}(\beta_k) = \beta_k$.

Therefore, $(\Delta^+ \cap w'\Delta^-) \setminus \beta_n = \{\beta_1, \ldots, \beta_{n-1}\} \cup R_I(w')$.

We already know that $R_I(w') = (\sigma_{\beta_n} \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$ and that σ_{β_n} is an admissible sorting reflection for w'. So, $\sigma_{\beta_n} \in \Delta^+ \cap w'\Delta^-$.

Therefore, $\Delta^+ \cap w'\Delta^- = \{\beta_1, \ldots, \beta_{n-1}\} \cup (\sigma_{\beta_n} \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$.

By Remark 9.2, $\beta_k = \alpha_{i_1} + \ldots + \alpha_{i_k}$ for $1 \le k \le n$. And $\alpha_{i_j} \notin I$ for $1 \le j \le k$ by Lemma hypothesis. So, $\operatorname{supp} \beta_k \cap I = \varnothing$, and $\beta_k \notin R_I(w') = (\sigma_{\beta_n} \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$, and the union $\Delta^+ \cap w'\Delta^- = \{\beta_1, \ldots, \beta_{n-1}\} \cup (\sigma_{\beta_n} \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$ is disjoint.

By the induction hypothesis,

for every $\alpha_j \in I$, for every $\gamma \in R_I(w) = \Delta^+ \cap w\Delta^-$:

the coefficient in front of α_j in the decomposition of γ into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1} \gamma$ into a linear combination of simple roots.

We already know that $R_I(\sigma_{\beta_n} w') = (\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$, so $\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1} \gamma R_I(\sigma_{\beta_n} w')$.

By Lemma 6.36 again, the coefficient in front of α_j in the decomposition of $\sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1} \gamma$ into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_n} \ldots \sigma_{\beta_1} \gamma$ into a linear combination of simple roots.

So,

the coefficient in front of α_j in the decomposition of γ into a linear combination of simple roots

=

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_n} \ldots \sigma_{\beta_1} \gamma$ into a linear combination of simple roots. □

Lemma 10.11. *Let $w, n_1, \ldots, n_r$ be a simply excessively clusterizable configuration of D-multiplicities. Let $I \subseteq \Pi$ be the set of involved roots.*

Let $0 = \beta_0, \beta_1, \ldots, \beta_p$, where $p > 0$, be a path-originating sequence. Denote $\alpha_{i_k} = \beta_k - \beta_{k-1}$. Suppose that $\alpha_{i_k} \notin I$ for all k $(1 \le k \le p)$.

Denote $w' = \sigma_{\beta_n} \ldots \sigma_{\beta_1} w$.

Set $m_{i_1} = p$ and $m_j = n_j$ if $j \ne i_1$.

Then $w', m_1, \ldots, m_r$ is a simply excessively clusterizable configuration of D-multiplicities.

Proof. By Lemma 5.21, $R_I(w) = \Delta^+ \cap w\Delta^-$.

Clearly, $n_{i_1} = 0$, so $m_1 + \ldots + m_r = n_1 + \ldots + n_r + p$.

By Lemma 10.10, $\Delta^+ \cap w'\Delta^-$ is the disjoint union of $\{\beta_1, \ldots, \beta_p\}$ and $(\sigma_{\beta_p} \ldots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$.

So, $\ell(w') = p + \ell(w)$, and $w', m_1, \dots, m_r$ is a configuration of D-multiplicities.

By Lemma 10.10, for every $j \in I$, for every $\gamma \in R_I(w)$:

the coefficient in front of α_j in the decomposition of γ into a linear combination of simple roots

$=$

the coefficient in front of α_j in the decomposition of $\sigma_{\beta_p} \dots \sigma_{\beta_1}\gamma$ into a linear combination of simple roots.

By Lemma 5.25, $(\sigma_{\beta_p} \dots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-), n_1, \dots, n_r$ is a simply excessively clusterizable configuration.

By Lemma 9.10, $\{\beta_1, \dots, \beta_p\}, 0, \dots, 0, p, 0, \dots, 0$ (where p occurs at the i_1th position) is an excessive cluster. It follows directly from the definitions of a simple excessive cluster and of a simply excessively clusterizable A-configuration that $\{\beta_1, \dots, \beta_p\}, 0, \dots, 0, p, 0, \dots, 0$ (where p occurs at the i_1th position) is a simply excessively clusterizable A-configuration.

By Lemma 10.10, the sets $\{\beta_1, \dots, \beta_p\}$ and $(\sigma_{\beta_p} \dots \sigma_{\beta_1})(\Delta^+ \cap w\Delta^-)$ are disjoint.

By Lemma 5.20, $w', m_1, \dots, m_r$ is a simply excessively clusterizable configuration of D-multiplicities.

□

Lemma 10.12. *Let*

$j_{1,1}, \dots, j_{1,n_1}$,

$\dots$

$j_{k,1}, \dots, j_{k,n_k}$

be a multipath.

Set $\beta_{p,q} = \alpha_{j_{p,1}} + \dots + \alpha_{j_{p,q}}$ *for* $1 \le p \le k$ *and* $0 \le q \le n_p$.

Set $w = \sigma_{\beta_{k,n_k}} ... \sigma_{\beta_{k,1}} \dots \sigma_{\beta_{1,n_1}} ... \sigma_{\beta_{1,1}}$.

Also set $m_{j_{p,1}} = n_p$ *and set* $m_i = 0$ *if* $i \notin \{j_{1,1}, \dots, j_{k,1}\}$.

Then $w, m_1, \dots, m_r$ *is a simply excessively clusterizable configuration of D-multiplicities.*

Proof. Induction on k. If $k = 0$, everything is clear, suppose that $k > 0$.

Set $w' = \sigma_{\beta_{k-1,n_{k-1}}} ... \sigma_{\beta_{k-1,1}} \dots \sigma_{\beta_{1,n_1}} ... \sigma_{\beta_{1,1}}$. Set $m'_{j_{k,1}} = 0$, set $m'_i = m_i$ for $i \ne j_{k,1}$.

By the induction hypothesis, $w', m'_1, \dots, m'_r$ is a simply excessively clusterizable configuration of D-multiplicities.

Denote $I = \{j_{1,1}, \dots, j_{k-1,1}\}$. Then I is set of indices i such that $m'_i > 0$.

By the definition of a multipath, $j_{k,i} \notin I$ for $1 \le i \le n_k$. Clearly, $0 = \beta_{k,0}, \beta_{k_1}, \dots, \beta_{k,n_k}$ is a path-originating sequence.

By Lemma 10.11, $w, m_1, \dots, m_r$ is a simply excessively clusterizable configuration of D-multiplicities.

□

Proposition 10.13. *Let* $l \in \mathbb{Z}_{\ge 0}$, *let* $J \subseteq \Pi$.

If there exists a multipath in the Dynkin diagram whose total length is l *and whose beginnings are contained in* J,

then there exists a configuration of D-multiplicities $w, n_1, \dots, n_r$ *with* $\ell(w) = l$ *such that* $C_{w,n_1,\dots,n_r} = 1$ *and such that the set of involved roots is contained in* J.

More precisely, if the lengths of the multipath are $m_1, \dots, m_k$ *and the beginnings are* $i_1, \dots, i_k$, *then the numbers* n_j *are defined as follows:* $n_{i_p} = m_p$ *for* $1 \le p \le k$ *and* $n_j = 0$ *if* $j \notin \{m_1, \dots, m_k\}$. *The set of involved roots is* $\{\alpha_{i_1}, \dots, \alpha_{i_k}\}$.

In this case, $D_1^{r_1} \dots D_r^{n_r}$ *is a multiplicity-free monomial.*

Proof. Let us keep the notation k, $m_1, \dots, m_k$, and $n_1, \dots, n_r$ from the "more precisely" part of the problem statement.

By Lemma 10.12, there exists $w \in W$ such that $w, n_1, \dots, n_r$ is a simply excessively clusterizable configuration of D-multiplicities. By Lemma 5.19, $w, n_1, \dots, n_r$ is an excessively clusterizable configuration of D-multiplicities. By Theorem 8.1, $C_{w,n_1,\dots,n_r} = 1$.

The last claim follows from the discussion in the end of Introduction. □

Theorem 10.14. *Let* $I \subseteq \Pi$.

the maximal degree of a multiplicity free monomial $D_1^{n_1} \dots D_r^{n_r}$, where $n_i = 0$ if $\alpha_i \notin I$, (i. e. the maximal value of the sum $n_1 + \dots + n_r$ over all r-tuples $n_1, \dots, n_r$ of nonnegative integers such that $D_1^{n_1} \dots D_r^{n_r}$ is a multiplicity-free monomial and $n_i = 0$ for each $\alpha_i \in \Pi \setminus I$)

equals

the maximal total length of a multipath in the Dynkin diagram whose beginnings are contained in I.

Proof. Follows directly from Corollary 10.9 and Proposition 10.13. □

11 Numerical estimates

Lemma 11.1. *If the Dynkin diagram has type A_r, then the maximal degree of a multiplicity free monomial $D_1^{n_1} \dots D_r^{n_r}$ is $r(r+1)/2$*

Proof. It is easy to construct a multipath of total length $r(r+1)/2 = r + \dots + 1$:

$1, \dots, r$;

$2, \dots, r$;

$\dots$

r.

Set $n_1, \dots, n_r = r, \dots, 1$. By Proposition 10.13, $D_1^{n_1} \dots D_r^{n_r}$ is a multiplicity-free monomial.

On the other hand, $r(r+1)/2$ is the maximal length of any element of the Weyl group of type A_r at all, so all monomials of higher degrees equal 0 in the Chow ring. □

Recall that we are working only with simply laced Dynkin diagrams.

Lemma 11.2. *If there exists a simple path in the Dynkin diagram that passes through all vertices, then this Dynkin diagram is of type A_r.*

Proof. Let $i_1, \dots, i_k$ be a path. We can identify the Dynkin diagram we have (denote it by Ξ) with A_r by sending $i_j \mapsto j$. Then the edge between i_j and i_{j+1} is mapped to the edge between j and $j+1$.

Dynkin diagrams have no loops, so there are no other edges in Ξ. There are no other edges in A_r either, so this is an isomorphism of Dynkin diagrams. □

Lemma 11.3. *The maximal total length of a multipath is always $\leq r(r+1)/2$. An equality is possible only of the diagram is of type A_r.*

Proof. Let

$j_{1,1}, \dots, j_{1,m_1}$,

$\dots$

$j_{k,1}, \dots, j_{k,m_k}$

be a multipath. Its total length is $m_1 + \dots + m_k$. By definition, for each i, $1 < i \leq k$, the vertices $j_{1,1}, \dots, j_{i-1,1}$ do not appear among $j_{i,1}, \dots, j_{i,k_i}$. So, $m_i \leq r - (i-1)$.

So, $m_1 + \dots + m_k \leq r + (r-1) + \dots + r - k + 1 \leq r + (r-1) + \dots + 1 = r(r+1)/2$.

This inequality become an equality $m_1 + \dots + m_k = r(r+1)/2$ only if $k = r$ and $m_i = r - (i-1)$ for all i ($1 \leq i \leq k$).

In particular, if $m_1 + \dots + m_k = r(r+1)/2$, then $m_1 = r$. By Lemma 11.2, this is possible only if the Dynkin diagram is of type A_r. □

Proposition 11.4. *If the Dynkin diagram has type D_r ($r \geq 4$), then the maximal degree of a multiplicity free monomial $D_1^{n_1} \dots D_r^{n_r}$ is $r(r+1)/2 - 1$.*

Proof. By Theorem 10.14, it suffices to prove that the maximal total length of a multipath in the Dynkin diagram of type D_r is $r(r+1)/2 - 1$.

Suppose we have a multipath

$j_{1,1}, \dots, j_{1,m_1}$,

$\dots$

$j_{k,1}, \dots, j_{k,m_k}$

Its total length is $m_1 + \ldots + m_k$.
If $k = 1$, then the total length of the multipath is at most $r - 1 < r(r + 1)/2 - 1$
If $k \geq 2$, then, by the definition of a multipath,
$j_{2,1}, \ldots, j_{2,m_2}$,
$\ldots$
$j_{k,1}, \ldots, j_{k,m_k}$
is a multipath that avoids vertex $j_{1,1}$. In other words, if we denote the original Dynkin diagram by Ξ, then this is a multipath in the Dynkin diagram $\Xi \setminus \{j_{1,1}\}$. Its total length is $m_2 + \ldots + m_k$. By Lemma 11.3, $m_2 + \ldots + m_k \leq (r - 1)r/2$.

$j_{1,1}, \ldots, j_{1,m_1}$ is a simple path in the whole Dynkin diagram of type D_r, so by Lemma 11.2, its length is at most $r - 1$, in other words, $m_1 \leq r - 1$.

Therefore, $m_1 + \ldots + m_r \leq r - 1 + (r - 1)r/2 = r + r(r - 1)/2 - 1 = r(r + 1)/2 - 1$.

An example of multipath of total length $r(r + 1)/2 - 1$ can be constructed as follows:
$r, r - 2, r - 3, \ldots, 1$;
$r - 1, r - 2, \ldots, 1$;
$r - 2, \ldots, 1$;
$\ldots$,
1.
The total length is indeed $(r-1)+(r-1)+(r-2)+\ldots+1 = (r+\ldots+1)-1 = r(r+1)/2-1$. □

Theorem 11.5. *If the Dynkin diagram has type E_r $(6 \leq r \leq 8)$, then the maximal degree of a multiplicity free monomial $D_1^{n_1} \ldots D_r^{n_r}$ is $r(r + 1)/2 - 2$.*
In other words, this maximal degree
for E_6 is 19,
for E_7 is 26,
for E_8 is 34.

Proof. Similar to type D.

By Theorem 10.14, it suffices to prove that the maximal total length of a multipath in the Dynkin diagram of type E_r is $r(r + 1)/2 - 2$.

Suppose we have a multipath
$j_{1,1}, \ldots, j_{1,m_1}$,
$\ldots$
$j_{k,1}, \ldots, j_{k,m_k}$
Its total length is $m_1 + \ldots + m_k$.
If $k = 1$, then the total length of the multipath is at most $r - 1 < 8 < 19$.
If $k \geq 2$, then, by the definition of a multipath,
$j_{2,1}, \ldots, j_{2,m_2}$,
$\ldots$
$j_{k,1}, \ldots, j_{k,m_k}$
is a multipath that avoids vertex $j_{1,1}$. In other words, if we denote the original Dynkin diagram by Ξ, then this is a multipath in the Dynkin diagram $\Xi \setminus \{j_{1,1}\}$. Its total length is $m_2 + \ldots + m_k$.

Let us consider 2 cases:

Case 1. $j_{1,1} = 2$.

Then $\Xi \setminus \{j_{1,1}\}$ is a diagram of type A_{r-1}. By Lemma 11.1, $m_2 + \ldots + m_k \leq (r - 1)r/2$. A direct observation of Dynkin diagrams of types E_6, E_7, and E_8 shows that the maximal length of a path in Ξ starting at 2 is always $r - 2$, so $m_1 \leq r - 2$, and $m_1 + \ldots + m_r \leq r - 2 + (r - 1)r/2 = r + r(r - 1)/2 - 2 = r(r + 1)/2 - 2$.

Case 2. $j_{1,1} \neq 2$.

Then a direct observation of Dynkin diagrams of types E_6, E_7, and E_8 shows that $\Xi \setminus \{j_{1,1}\}$ is not of type A_{r-1}. (More precisely, it can be
either of types D or E, if $j_{1,1}$ is 1 or r,
or not connected if $j_{1,1} \neq 1$ and $j_{1,1} \neq r$.)
By Lemma 11.3, $m_2 + \ldots + m_k < (r - 1)r/2$, and $m_2 + \ldots + m_k \leq (r - 1)r/2 - 1$.

$j_{1,1},\ldots,j_{1,m_1}$ is a simple path in the whole Dynkin diagram of type D_r, so by Lemma 11.2, its length is at most $r-1$, in other words, $m_1 \le r-1$. Therefore, $m_1+\ldots+m_r \le r-1+(r-1)r/2-1 = r+r(r-1)/2-2 = r(r+1)/2-2$.

It is easy to construct a multipath of total length $r(r+1)/2-2$:

$2,4,5,\ldots,r$;

$1,3,4,5,\ldots,r$;

$3,4,5,\ldots,r$;

$\ldots$;

r.

The total length is indeed $(r-2)+(r-1)+(r-2)+\ldots+1 = (r+\ldots+1)-2 = r(r+1)/2-2$. □

Lemma 11.6. *If the Dynkin diagram is the disjoint union of several subdiagrams $\Xi_1,\ldots,\Xi_n$, and for each i,*

the maximal total length of a multipath in Ξ_i

is m_i,

then the maximal total length of a multipath in the whole Dynkin diagram is $m_1+\ldots+m_n$.

Proof. Follows directly from the definition of a multipath. □

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