1 Introduction

Let G be a simple algebraic group over $\mathbb{C}$ with a simply laced Dynkin diagram, let $B \subset G$ be a Borel subgroup, and let $T \subset B$ be a maximal torus.

Denote the corresponding root system by Δ, the subset of positive roots by Δ^+, and the set of simple roots by Π. Denote the simple roots by $\alpha_1, \ldots, \alpha_r$. Denote the corresponding fundamental weights by $\varpi_1, \ldots, \varpi_r$. Denote the Weil group by W. Denote the reflection correspnding to a root α by σ_α. Denote the length of an element $w \in W$ by $\ell(w)$.

Consider the generalized flag variety G/B. We have two canonical kinds of subvarieties. First, one can associate a divisor to any weight λ, denote this divisor by D_λ. Second, one can assciate a subvariety of codimension k to any Weil group element w of length k, denote this subvariety by X_w.

Note that $X_{\sigma_{\alpha_i}} = D_{\varpi_i}$.

The classes of X_w in Chow group for all $w \in W$ form a basis of the Chow group as of a linear space. In particular, every monomial consisting of divisors D_{ϖ_i} equals a linear combination of some classes of X_w:

$$[D_{\varpi_1}]^{n_1}[D_{\varpi_2}]^{n_2} \ldots [D_{\varpi_r}]^{n_r} = \sum C_{w,n_1,\ldots,n_r}[X_w].$$

We fix the notation $C_{w,n_1,\ldots,n_r}$ in the whole paper. Our goal is to solve the following problem:

Suppose that G is of type E_8. What is the maximal number of divisors that we can multiply (i. e. what is the maximal value of the sum $n_1 + \cdots + n_r$) such that at least one coefficient $C_{w,n_1,\ldots,n_r}$ equals 1?

2 Preliminaries

We choose the scalar multiplication on Δ so that the scalar square of each simple root is 2. The scalar product of two roots α and β is denoted by (α, β). Note that with this choice of scalar multiplication, we can use a simple formula for reflection: usually, we write

$$\sigma_\alpha \beta = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha.$$

But with our chouce of scalar product, we can write

$$\sigma_\alpha \beta = \beta - (\alpha, \beta)\alpha.$$

We use the following Pieri formula:

Proposition 2.1 (add reference!!!). *Let ϖ_i be a simple weight, and let $w \in W$. Then*

$$[D_{\varpi_i}][X_w] = \sum_{\substack{\alpha \in \Delta^+ \\ \ell(\sigma_\alpha w) = \ell(w)+1}} \varpi_i(\alpha)[X_{\sigma_\alpha w}].$$

Note that $\varpi_i(\alpha)$ is precisely the coefficient at α_i in the decomposition of α into a linear combination of simple roots.

We will use the following well-known combinatorial Hall representative lemma and its generalization.

Lemma 2.2 (Hall representative lemma). *Let $A_1, \ldots, A_n$ be several finite sets. Suppose that for each subset $I \subseteq \{1, \ldots, n\}$ one has $|\cup_{i \in I} A_i| \geq |I|$. Then one can choose $a_i \in A_i$ for all i ($1 \leq i \leq n$) so that all elements a_i are different.*

Lemma 2.3 (Geralized Hall representative lemma). *Let $A_1, \ldots, A_r$ be several finite sets, and let $k_1, \ldots, k_r \in \mathbb{N}$. Suppose that for each subset $I \subseteq \{1, \ldots, n\}$ one has*

$$|\cup_{i \in I} A_i| \geq \sum_{i \in I} k_i$$

Then one can choose $a_i \in A_i$ for all i ($1 \leq i \leq n$) so that all elements a_i are different.

Proof. Consider the following collection of sets B_{ij}: $B_{ij} = A_i$, $1 \leq i \leq r$, $1 \leq j \leq k_i$. Let J be a subset of double indices. Let m_i ($1 \leq i \leq r$) be the number of double indices in J that begin with i. Then $m_i \leq k_i$. Also denote the projection of J onto the first coordinate by I. Then $\cup_{(i,j)\in J} B_{ij} = \cup_{i\in I} A_i$, and

$$|\cup_{(i,j)\in J} B_{ij}| = |\cup_{i\in I} A_i| \geq \sum_{i\in I} k_i \geq \sum_{i\in I} m_i = |J|.$$

So, the collection $\{B_{ij}\}$ satisfies the hypothesis of Lemma 2.2. □

The following facts about root systems and Weil groups are well-known and can e found, for example, in [1].

Lemma 2.4. *Let $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, $\alpha \neq -\beta$. Then all possible values of (α, β) are 0, 1, and -1.*

Lemma 2.5. *Let $\alpha, \beta \in \Delta$. Then:*

1. *$\alpha + \beta \in \Delta$ if and only if $(\alpha, \beta) = -1$.*
2. *$\alpha - \beta \in \Delta$ if and only if $(\alpha, \beta) = 1$.*

Corollary 2.6. *For each $\alpha \in \Delta$, the reflection σ_α has the following orbits on Δ:*

1. *$\{\alpha, -\alpha\}$*
2. *$\{\beta\}$ (a fixed point) for each $\beta \in \Delta$, $(\alpha, \beta) = 0$.*
3. *$\{\beta, \gamma\}$ for $\beta, \gamma \in \Delta$, $(\alpha, \beta) = 1$, $(\alpha, \gamma) = -1$, and $\beta = \alpha + \gamma$.*

Lemma 2.7. *If $\alpha, \beta, \gamma \in \Delta$ and $(\alpha, \beta) = 1$, $(\beta, \gamma) = 1$, $(\alpha, \gamma) = 0$,*
then $\delta = \alpha + \gamma - \beta \in \Delta$, and $(\alpha, \delta) = 1$, $(\delta, \gamma) = 1$, $(\delta, \beta) = 0$

Proof. Direct computation of scalar products.
$\alpha - \beta \in \Delta$ by Lemma 2.5.
$(\alpha - \beta, \gamma) = 0 - 1 = -1$
$\delta = \alpha - \beta + \gamma \in \Delta$ by Lemma 2.5.
$(\delta.\alpha) = 2 - 1 + 0 = 1.$
$(\delta, \beta) = 1 - 2 + 1 = 0.$
$(\delta, \gamma) = 0 - 1 + 2 = 1.$ □

Lemma 2.8. *If $\alpha, \beta, \gamma \in \Delta$ and $(\alpha, \beta) = 1$, $(\beta, \gamma) = 1$, $(\alpha, \gamma) = 0$, and there exists a simple root α_i that appears in the decompositions of all three roots α, β, and γ into linear combinations of simple roots with coefficient 1,*
then α_i appears in the decomposition of $\delta = \alpha - \beta + \gamma$ into a linear combination of simple roots also with coefficient 1, and $\delta \in \Delta^+$.

Proof. Direct calculation. □

Lemma 2.9. *If $w \in W$, then $\ell(w) = |\Delta^+ \cap w\Delta^-|$. Moreover, the set $|\Delta^+ \cap w\Delta^-|$ determines w uniquely.*

We will have several examples involving permutation groups. More precisely, there permutation groups will appear as the Weyl groups of groups of type A_r. The Weyl group of a group of type A_r is S_{r+1}. For brevity, we will write $(s_1, s_2, \ldots, s_{r+1})$ instead of

$$\begin{pmatrix} 1 & 2 & \ldots & r+1 \\ s_1 & s_2 & \ldots & s_{r+1} \end{pmatrix}.$$

The transposition interchanging the ith and the jth positions will be denoted by $(i \leftrightarrow j)$.

Example 2.10. The length of an element $(s_1, \ldots, s_{r+1}) \in W$ is the number of inversions, i. e. the number of pairs (i, j) with $i < j$ and $s_i > s_j$.

We use the follwing terminology to compute products of several divisors using Pieri formula.

Definition 2.11. Let $\alpha \in \Delta^+$ and let $w \in W$. We say that the reflection σ_α is:

1. A *sorting reflection* for w if $\ell(\sigma_\alpha w) < \ell(w)$;
2. A *desorting reflection* for w if $\ell(\sigma_\alpha w) > \ell(w)$;
3. An *admissible sorting reflection* for w if $\ell(\sigma_\alpha w) = \ell(w) - 1$;
4. An *antisimple sorting reflection* for w if $\ell(\sigma_\alpha w) = \ell(w) - 1$ and $w^{-1}\alpha \in -\Pi$.

Example 2.12. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \dots, s_{r+1})$, then the sorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ with $i < j$ and $s_i > s_j$, and the desorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ with $i < j$ and $s_i < s_j$. This example motivates the usage of the words "sorting" and "desorting".

We will also need to consider two different kidns of orders on Δ. First, there is the standard order $\prec$ on Δ: we say that $\alpha \prec \beta$ if $\beta - \alpha$ is a sum of positive roots. Additionally, for each $w \in W$ we will need an order we will denote by $\prec_w$: we say that $\alpha \prec_w \beta$ if $w^{-1}\alpha \prec w^{-1}\beta$.

Remark 2.13. *If $\alpha, \beta \in \Delta$ and $(\alpha, \beta) = 1$, then, by Lemma 2.5, α and β are comparable for $\prec$ and for the orders $\prec_w$ for all $w \in W$.*

Definition 2.14. Let v be a linear combination of roots, $v = \sum a_i \alpha_i$. The set of simple roots α_i such that $a_i \neq 0$ is called the *support* of v (notation: $\operatorname{supp} v$).

Lemma 2.15. *Let $w \in W$.*
If $\alpha, \beta, \gamma \in w\Delta^-$ and $(\alpha, \beta) = 1$, $(\beta, \gamma) = 1$, $(\alpha, \gamma) = 0$, and ($\alpha \prec_w \beta$ or $\gamma \prec_w \beta$), then $\delta = \alpha - \beta + \gamma \in w\Delta^-$

Proof. Without loss of generality, $\alpha \prec_w \beta$.
By Lemma 2.5, $\alpha - \beta \in \Delta$. $\alpha \prec_w \beta$, so $\alpha - \beta \in w\Delta^-$.
By Lemma 2.7, $\delta = \alpha - \beta + \gamma \in \Delta$. $\alpha - \beta \in w\Delta^-$ and $\gamma \in w\Delta^-$, so $\delta \in w\Delta^-$. □

3 Sorting

Lemma 3.1. *Let $\alpha \in \Delta^+$, and $\beta \in \Delta$. Suppose that $(\alpha, \beta) = 1$. σ_α interchanges β with another simple root, which we denote by γ.*
Then there are exaclty three possibilities:

(i) $\beta, \gamma \in \Delta^+$.

(ii) $\beta \in \Delta^+$, $\gamma \in \Delta^-$.

(iii) $\beta, \gamma \in \Delta^-$.

Proof. The only remaining case is $\beta \in \Delta^-$, $\gamma \in \Delta^+$. Let us check that this is impossible. Note that $\beta = \alpha + \gamma$. So, if $\alpha \in \Delta^+$, $\gamma \in \Delta^+$, then $\beta = \alpha + \gamma \in \Delta^+$, a contradiction. □

Lemma 3.2. *Let $w \in W$, $\alpha \in \Delta^+$, and $\beta \in \Delta$. Suppose that $(\alpha, \beta) = 1$. σ_α interchanges β with another simple root, which we denote by γ.*
Then there are exaclty three possibilities:

1. $\alpha \in w\Delta^-$, $\beta \in \Delta^+$, $\gamma \in \Delta^-$, $\beta \in w\Delta^-$, $\gamma \in w\Delta^+$.
   Then $\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, *and* $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| > |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

2. *$\alpha \in w\Delta^+$, $\beta \in \Delta^+$, $\gamma \in \Delta^-$, $\beta \in w\Delta^+$, $\gamma \in w\Delta^-$.*

   Then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \varnothing$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$, and $|\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-)| < |\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

3. *Otherwise, $|\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-)| = |\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. More precisely:*

   (a) If $\alpha \in w\Delta^-$, $\beta \in \Delta^+$, $\gamma \in \Delta^+$, $\beta \in w\Delta^-$, and $\gamma \in w\Delta^+$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\gamma\}$,

   (b) If $\alpha \in w\Delta^+$, $\beta \in \Delta^+$, $\gamma \in \Delta^+$, $\beta \in w\Delta^+$, and $\gamma \in w\Delta^-$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\gamma\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$,

   (c) Otherwise, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)$.

Proof. Note that $(\alpha,\gamma) = -1$, and $\beta = \alpha + \gamma$.

Note also that $\beta \in w\Delta^-$ if and only if $\gamma \in \sigma_\alpha w\Delta^-$, and $\gamma \in w\Delta^-$ if and only if $\beta \in \sigma_\alpha w\Delta^-$.

Let us consider the 3 cases from Lemma 3.1:

(i) $\beta, \gamma \in \Delta^+$.

Then $\beta \in \Delta^+ \cap w\Delta^-$ if and only if $\gamma \in \Delta^+ \cap \sigma_\alpha w\Delta^-$, and $\gamma \in \Delta^+ \cap w\Delta^-$ if and only if $\beta \in \Delta^+ \cap \sigma_\alpha w\Delta^-$. Therefore, $|\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-)| = |\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

If $\beta,\gamma \in w\Delta^-$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta,\gamma\}$, and 3c is true.

If $\beta,\gamma \in w\Delta^+$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 3c is true.

If $\beta \in w\Delta^+$ and $\gamma \in w\Delta^-$, then α must be in $w\Delta^+$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^-$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\gamma\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$, and 3b is true.

If $\beta \in w\Delta^-$ and $\gamma \in w\Delta^+$, then α must be in $w\Delta^-$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^+$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\gamma\}$, and 3a is true.

(ii) $\beta \in \Delta^+$, $\gamma \in \Delta^-$.

If $\beta,\gamma \in w\Delta^-$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\beta\}$, and 3c is true.

If $\beta,\gamma \in w\Delta^+$, then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 3c is true.

If $\beta \in w\Delta^+$ and $\gamma \in w\Delta^-$, then α must be in $w\Delta^+$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^-$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \varnothing$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\gamma\}$, and 2 is true.

If $\beta \in w\Delta^-$ and $\gamma \in w\Delta^+$, then α must be in $w\Delta^-$, otherwise $\beta = \alpha + \gamma$ would be in $w\Delta^+$. So, $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 1 is true.

(iii) $\beta, \gamma \in \Delta^-$.

Then $\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and 3c is true.

□

Lemma 3.3. *Let $w \in W$ and let $\alpha \in \Delta^+$. Then:*

σ_α is a sorting reflection for w if and only if $\alpha \in \Delta^+ \cap w\Delta^-$. Otherwise, σ_α is a desorting reflection for w.

Proof. The reflection σ_α acting on Δ has some fixed points (they are precisely the roots orthogonal to α), and the other roots can be split into pairs (β,γ) such that σ_α interchanges β and γ ($(\alpha,-\alpha)$ is one of such pairs).

Consider a pair (β,γ) such that σ_α interchanges β and γ. Suppose also that $\beta \neq \pm\alpha$. Then, since the Dynkin diagram is simply laced, $(\alpha,\beta) = \pm 1$. Without loss of generality, let us assume that $(\alpha,\beta) = 1$. Then $(\alpha,\gamma) = -1$, and $\beta = \alpha + \gamma$.

Suppose first that $\alpha \in w\Delta^-$. Then, in the classification of Lemma 3.2, case 2 is impossible, since it requires $\alpha \in \Delta^+$. And in both of the other cases, we have $|\{\beta,\gamma\} \cap (\Delta^+ \cap w\Delta^-)| \geq |\{\beta,\gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

END Suppose first that $\alpha \in w\Delta^-$.

Now suppose that $\alpha \in w\Delta^+$. Then, in the classification of Lemma 3.2, case 1 is impossible, since it requires $\alpha \in \Delta^-$. And in both of the other cases, we have $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

END Now suppose that $\alpha \in w\Delta^+$.

END Consider a pair (β, γ)

So, we can conclude that if $\alpha \in w\Delta^-$, then for every pair (β, γ) such that σ_α interchanges β and γ, and $\beta \neq \pm\alpha$, we have $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \geq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. Also, if $\alpha \in w\Delta^-$, then $\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-) = \{\alpha\}$, $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and $|\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-)| > |\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. The summation over all orbits of σ_α in Δ gives us $|(\Delta^+ \cap w\Delta^-)| > |(\Delta^+ \cap \sigma_\alpha w\Delta^-)|$ if $\alpha \in w\Delta^-$.

And we can also conclude that if $\alpha \in w\Delta^+$, then for every pair (β, γ) such that σ_α interchanges β and γ, and $\beta \neq \pm\alpha$, we have $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. Also, if $\alpha \in w\Delta^+$, then $\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-) = \varnothing$, $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\alpha\}$, and $|\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-)| < |\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$. The summation over all orbits of σ_α in Δ gives us $|(\Delta^+ \cap w\Delta^-)| < |(\Delta^+ \cap \sigma_\alpha w\Delta^-)|$ if $\alpha \in w\Delta^+$. □

Lemma 3.4. *Let $w \in W$ and $\alpha \in \Delta^+ \cap w\Delta^-$.*

Then σ_α is an admissible sorting reflection for w if and only if it is impossible to find roots $\beta, \delta \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \delta$.

Proof. Again, note that $\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-) = \{\alpha\}$, $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$, and $|\{\alpha, -\alpha\} \cap (\Delta^+ \cap w\Delta^-)| > |\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

Also note again that if (β, γ) is a pair such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$, then case 2 in Lemma 3.2 is not possible since it requires $\alpha \in w\Delta^+$, and $|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$.

So, the summation over all orbits of σ_α on Δ tells us that $|(\Delta^+ \cap w\Delta^-)| = |(\Delta^+ \cap \sigma_\alpha w\Delta^-)| + 1$ if and only if all inequalities

$|\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)| \leq |\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)|$ for all pairs (β, γ) such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$,

become equalities.

And all these inequalities become equalities if and only if case 1 does not occur for any pair (β, γ) such that σ_α interchanges β and γ and $\beta \neq \pm\alpha$. In other words, $\ell(w) = \ell(\sigma_\alpha w) + 1$ if and only if there are no pairs (β, γ) such that

σ_α interchanges β and γ, $(\alpha, \beta) = 1$, $\beta \in \Delta^+$, $\gamma \in \Delta^-$, $\beta \in w\Delta^-$, $\gamma \in w\Delta^+$.

And if we denote $\delta = -\gamma$, then we see that the non-existance of such pairs is eqivalent to the non-existance of pairs (β, δ) such that

$\alpha = \beta + \delta$, $(\alpha, \beta) = 1$, $\beta \in \Delta^+$, $\delta \in \Delta^+$, $\beta \in w\Delta^-$, $\delta \in w\Delta^-$.

Finally, note that by Lemma 2.5, if $\beta, \delta, \beta + \delta \in \Delta^+$, then automatically $(\beta, \delta) = -1$. □

Example 3.5. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \ldots, s_{r+1})$, then the admissible sorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ such that $i < j$, $s_i > s_j$, and there are no indices k such that $i < j < k$ and $s_i > s_k > s_j$.

Lemma 3.6. *Let $w \in W$ and $\alpha \in \Delta^+ \cap w\Delta^-$. Suppose that σ_α is an admissible sorting reflection. Then the set $\Delta^+ \cap \sigma_\alpha w\Delta^-$ can be obtained from the set $\Delta^+ \cap w\Delta^-$ by the following procedure:*

For each $\beta \in \Delta^+ \cap w\Delta^-$:

1. *If $\beta = \alpha$, don't put anything into $\Delta^+ \cap \sigma_\alpha w\Delta^-$.*
2. *If $(\alpha, \beta) = 1$, $\alpha \prec \beta$, and $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, then put $\beta - \alpha$ into $\Delta^+ \cap \sigma_\alpha w\Delta^-$.*
3. *Otherwise, put β into $\Delta^+ \cap \sigma_\alpha w\Delta^-$.*

Note that this lemma in fact establishes a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \alpha$ and $\Delta^+ \cap \sigma_\alpha w\Delta^-$.

Proof. Let us check that for every orbit of σ_α on Δ, the above procedure gives the correct intersection of this orbit with $\Delta^+ \cap \sigma_\alpha w\Delta^-$. See Corollary 2.6 for the list of orbits.

If the orbit consists of one root, β, then $(\alpha, \beta) = 0$. We apply case 3 of the procedure, and indeed, $\{\beta\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)$ since $\sigma_\alpha\beta = \beta$.

If the orbit is $\alpha, -\alpha$, then we apply case 1 of the procedure. And indeed, it is clear that $\{\alpha, -\alpha\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \varnothing$.

Finally, consider an orbit $\{\beta, \gamma\}$, where $(\alpha, \beta) = 1$, $(\alpha, \gamma) = -1$, and $\beta = \alpha + \gamma$. Lemma 3.2 gives us 5 possibilities in total, among them:

Case 1 is prohibitied by Lemma 3.4 (if case 1 was true, then we would have $\beta \in \Delta^+ \cap w\Delta^-$, $-\gamma \in \Delta^+ \cap w\Delta^-$, and $\alpha = \beta + (-\gamma)$).

Case 2 is impossible since $\alpha \in w\Delta^-$.

If case 3a of Lemma 3.2 holds, then $\alpha, \beta \in \Delta^+ \cap w\Delta^-$. Also, $\gamma \in \Delta^+$, $\gamma = \beta - \alpha$, so $\alpha \prec \beta$. Finally, $\gamma \notin w\Delta^-$, so the conditions of case 2 are satisfied. By Lemma 3.2, $\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta\}$, $\{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-) = \{\gamma\}$, and indeed, case 2 tells us that we should put $\gamma = \beta - \alpha$ into $(\Delta^+ \cap \sigma_\alpha w\Delta^-)$ instead of β.

Case 3b is impossible since $\alpha \in w\Delta^-$.

Finally, suppose that case 3c of Lemma 3.2 holds. Let us check that the conditions of case 2 of the procedure are not satisfied (and the procedure tells us that we should use case 3).

Clearly, the conditions of case 2 of the procedure are not satisfied for γ since $(\alpha, \gamma) = -1$

Assume the contrary, assume that the conditions of case 2 are satisfied for β. $\alpha \in \Delta^+$, $\alpha \in w\Delta^-$, $\beta \in \Delta^+$, $\beta \in w\Delta^-$. Since $\beta \prec \alpha$, $\gamma = \beta - \alpha \in \Delta^+$. Since $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, $\gamma \in w\Delta^+$. So, case 3a of Lemma 3.2 holds, and we have assumed that case 3c of Lemma 3.2 holds. A contradiction.

END Assume the contrary.

So, the procedure tells us that we should use case 3 and put all roots from $\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-)$ into $\Delta^+ \cap \sigma_\alpha w\Delta^-$. And this is correct since by case 3c of Lemma 3.2, $\{\beta, \gamma\} \cap (\Delta^+ \cap w\Delta^-) = \{\beta, \gamma\} \cap (\Delta^+ \cap \sigma_\alpha w\Delta^-)$. ☐

Lemma 3.7. *If $w \in W$, $\alpha \in \Delta^+ \cap w\Delta^-$, and $w^{-1}\alpha \in -\Pi$, then σ_α is an antisimple sorting reflection.*

Proof. The only thing we have to check is that σ_α is an admissible sorting reflection. We use Lemma 3.4. Assume that there are roots $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\alpha = \beta + \gamma$. But then $-w^{-1}\alpha = (-w^{-1}\beta) + (-w^{-1}\gamma)$, $-w^{-1}\alpha \in \Pi$, and $-w^{-1}\beta, -w^{-1}\gamma \in \Delta^+$, a contradiction. ☐

Example 3.8. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \ldots, s_{r+1})$, then the antisimple sorting reflections for w are precisely the transpositions $(i \leftrightarrow j)$ such that $i < j$ and $s_i = s_j + 1$.

Lemma 3.9. *Let $w \in W$. The roots $\alpha \in \Delta^+ \cap w\Delta^-$ such that $w^{-1}\alpha \in -\Pi$ are exaclty the maximal elements of the set $\Delta^+ \cap w\Delta^-$ with respect to the order $\prec_w$.*

Proof. Direction 1.

Let $\alpha \in \Delta^+ \cap w\Delta^-$, $w^{-1}\alpha \in -\Pi$. Assume that $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec_w \beta$. Then, by the definition of $\prec_w$, $w^{-1}\alpha \prec w^{-1}\beta$. But $w^{-1}\beta \in \Delta^-$, $w^{-1}\alpha \in -\Pi$, a contradiction.

Direction 2.

Let α be a maximal element of $\Delta^+ \cap w\Delta^-$ with respect to $\prec_w$. Assume that $w^{-1}\alpha \notin -\Pi$. Then, since $w^{-1}\alpha \in \Delta^-$, it is possible to decompose $w^{-1}\alpha = \beta + \gamma$, where $\beta, \gamma \in \Delta^-$. We have $w\beta + w\gamma = \alpha$. $w\beta$ and $w\gamma$ cannot be both negative, since their sum, α, is positive. At least one of the roots $w\beta$ and $w\gamma$ is positive, let us assume without loss of generality that $w\beta \in \Delta^+$. We have $w\beta \in \Delta^+$, $w\beta \in w\Delta^-$, and $w^{-1}w\beta - w^{-1}\alpha = -\gamma \in \Delta^+$, so $\alpha \prec_w w\beta$, a contradiction with maximality of α. ☐

Corollary 3.10. *For every $w \in W$, $w \neq \mathrm{id}$, there exists at least one $\alpha \in \Delta^+ \cap w\Delta^-$ such that σ_α is an antisimple sorting reflection for w.* ☐

The following lemma illustrates an advantage of antisimple sorting reflections.

Lemma 3.11. *Let $w \in W$. If $\alpha \in \Delta^+ \cap w\Delta^-$ is such that σ_α is an antisimple sorting reflection, then $\Delta^+ \cap \sigma_\alpha w\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \alpha$.*

Proof. We use Lemma 3.6. We have to check that case 2 never occurs.

Assume that case 2 occurs for some $\beta \in \Delta^+ \cap w\Delta^-$. This means that $\gamma = \beta - \alpha \in w\Delta^+$, $w^{-1}\gamma = w^{-1}\beta - w^{-1}\alpha \in \Delta^+$, and $\alpha \prec_w \beta$. But then α is not a maximal element of $\Delta^+ \cap w\Delta^-$ with respect to $\prec_w$, a contradiction with Lemma 3.9. $\square$

To use Chevalley-Pieri formula, we will use the following terminology.

Definition 3.12. Let $w \in W$, $n = \ell(w)$. We say that *a process of sorting of* w is a sequence of roots $\beta_1, \ldots, \beta_n$ such that:

1. $w = \sigma_{\beta_1}\sigma_{\beta_2} \ldots \sigma_{\beta_n}$.
2. Denote $w_i = \sigma_{\beta_i} \ldots \sigma_{\beta_1} w = \sigma_{\beta_{i+1}} \ldots \sigma_{\beta_n}$ $(0 \le i \le n)$. Then for each i, $0 \le i < n$, $\sigma_{\beta_{i+1}}$ has to be an admissible sorting reflection for w_i. In other words, $\ell(w_i)$ has to be $n - i$ for $0 \le i \le n$.

We say that the ith *step* $(1 \le i \le n)$ of the sorting process is the reflection σ_{β_i}, and that the *current* element of W after the ith step of the process (before the $(i+1)$st step of the process) is $w_i = \sigma_{\beta_i} \ldots \sigma_{\beta_1} w = \sigma_{\beta_{i+1}} \ldots \sigma_{\beta_n}$.

We say that the sorting process is *antireduced*, and the equality $w = \sigma_{\beta_1}\sigma_{\beta_2} \ldots \sigma_{\beta_n}$ is an *antireduced expression for* w, if $\sigma\beta_i$ is an antisimple reflection for w_{i-1} for all i, $1 \le i \le n$.

If we only know for some i, $1 \le i \le n$, that $\sigma\beta_i$ is an antisimple reflection for w_{i-1}, we will say that *the ith step of the sorting process is antisimple.*

Definition 3.13. Let $w \in W$, $n = \ell(w)$. Similarly, we say that *a sorting process prefix of* w is a sequence of roots $\beta_1, \ldots, \beta_k$ $(k \le n)$ such that:

> Denote $w_i = \sigma_{\beta_i} \ldots \sigma_{\beta_1} w$ $(0 \le i \le k)$. Then for each i, $0 \le i < k$, $\sigma_{\beta_{i+1}}$ has to be an admissible sorting reflection for w_i. In other words, $\ell(w_i)$ has to be $n - i$ for $0 \le i \le k$.

We say that the sorting process prefix is *antireduced*, if $\sigma\beta_i$ is an antisimple reflection for w_{i-1} for all i, $1 \le i \le k$.

Lemma 3.14. *If $\beta_1, \ldots, \beta_n$ is an antireduced sorting process for $w \in W$, then $\{\beta_1, \ldots, \beta_n\} = \Delta^+ \cap w\Delta^-$.*

Moreover, if $\beta_1, \ldots, \beta_n$ is a sorting process for $w \in W$ such that the first k steps are antisimple, and $w_k = \sigma_{\beta_k} \ldots \sigma_{\beta_1} w = \sigma_{\beta_{k+1}} \ldots \sigma_{\beta_n}$, then $\{\beta_{k+1}, \ldots, \beta_n\} = \Delta^+ \cap w_k\Delta^-$.

Proof. This follows directly from Lemma 3.11 and the definition of an antisimple sorting process. $\square$

Corollary 3.15. *If $\beta_1, \ldots, \beta_n$ is an antireduced sorting process for $w \in W$, then there are no coinciding roots among $\beta_1, \ldots, \beta_n$.*

Example 3.16. If $G = SL_{r+1}$, then $W = S_{r+1}$. If $w = (s_1, \ldots, s_{r+1})$, and we have a sorting process of w, then the sequence of the current elements of W is a sequence of $(r+1)$-tuples ("arrays") of numbers, where each next $(r+1)$-tuple is obtained from the previous one by interchanging two numbers so that this interchange is an admissible sorting reflection (see Example 3.5). In the end, our $(r+1)$-tuple has to become $(1, 2, \ldots, r+1)$.

Such a sorting process is antireduced if at each step we actually interchange a number i with $i+1$, and $i+1$ has to be located to the left of i immediately before this interchange.

(Remark about relation to programming, we will not need it later: An antireduced sorting process is *not* what is called "bubble sorting" in programming. Bubble sorting can be obtained from a certain *reduced* expression for w (but not from any reduced expression, only from a certain one)).

Definition 3.17. Given a set of positive roots $A \subseteq \Delta^+$ we call a function $f\colon A \to \Pi$ a *distribution of simple roots* on A if $f(\alpha) \in \operatorname{supp}\alpha$ for each $\alpha \in A$

For a given simple root α_i, the number of roots $\alpha \in A$ such that $f(\alpha) = \alpha_i$ is called the *D-multiplicity* of α_i in the distribution.

If we have a distribution with $f(\alpha) = \alpha_i$, we say that the distribution *assigns* the simple root α_i to α.

Definition 3.18. Given a list of positive roots $\beta_1, \ldots, \beta_n$, i. e. order matters, multiple occurrences allowed, we call a function $f\colon \{1, \ldots, n\} \to \Pi$ a *distribution of simple roots* on $\beta_1, \ldots, \beta_n$ if $f(k) \in \operatorname{supp} \beta_k$ for each k, $1 \le k \le n$.

Sometimes we will treat this function as a list (an n-tuple) of its values: $f(1), \ldots, f(k)$. This is convnient, for example, if we want to remove some roots from the list $\beta_1, \ldots, \beta_n$, and at the same time remove the corresponding simple roots from the list $f(1), \ldots, f(k)$.

For a given simple root α_i, the number of indices k, $1 \le k \le n$ such that $f(k) = \alpha_i$ is called the *D-multiplicity* of α_i in the distribution.

If we have a distribution with $f(k) = \alpha_i$, we say that the distribution *assigns* the simple root α_i to the kth root in the list, β_k.

If we need to know the D-multiplicities of all simple roots in a distribution, we briefly say "a distribution with D-multiplicities $n_1, \ldots, n_r$" instead of "a distribution with D-multiplicities $n_1, \ldots, n_r$ of simple roots $\alpha_1, \ldots, \alpha_r$, respectively".

Definition 3.19. We call a tuple $w, n_1, \ldots, n_r$, where $w \in W$, $n_i \in \mathbb{Z}_{\ge 0}$, $n_1 + \ldots + n_r = \ell(w)$, a *configuration of D-multiplicities.*

Definition 3.20. Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. We say that a simple root α_i is *involved* into this configuration if $n_i > 0$.

Definition 3.21. Let $w \in W$. We say that a *labeled sorting process* of w is a sorting process $\beta_1, \ldots, \beta_n$ of w with the following addiotional information:

We have a simple root distribution on the list $\beta_1, \ldots, \beta_n$.

This distribution will be called the *distribution of labels*, or the *list of labels*, of the labeled sorting process. The simple root it assgns to β_k will be called the *label* at β_k.

In other words, when, at a certain (kth) step of the sorting process, we perform an admissible sorting reflection along a root (β_k), we assign to this step a label, which is a simple root from $\operatorname{supp} \beta_k$.

Note that the distribution of labels is actually a function from $\{1, \ldots, n\}$ to Π (i. e. just an n-tuple of simple roots), so it makes sense, for example, to speak about "two different labeled sorting processes with the same distribution of labels".

Instead of "a labeled sorting process of w with distribution of labels that has D-multiplicities $n_1, \ldots, n_r$ of simple roots", we briefly say "*a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels*".

Definition 3.22. Let $w \in W$. Let $\beta_1, \ldots, \beta_n$ be a labeled sorting process of w with distribution of labels f.

Since $f(k) \in \operatorname{supp} \beta_i$, $f(k)$ is present in the decomposition of β_i into a linear combination of simple roots. Let a_i be the coefficient in front of $f(k)$ in this linear combination.

The *X-multiplicity* of the sorting process (not to be confused with the D-multiplicity of a simple root in a list of simple roots) is the product $a_1 \ldots a_n$.

Definition 3.23. Let $w \in W$. We say that a *labeled sorting process prefix* of w is a sorting process prefix $\beta_1, \ldots, \beta_k$ of w with the following addiotional information:

We have a simple root distribution on the list $\beta_1, \ldots, \beta_k$.

Instead of "a labeled sorting process prefix of w with distribution of labels that has D-multiplicities $m_1, \ldots, m_r$ of simple roots", we briefly say "*a labeled sorting process prefix of w with D-multiplicities $m_1, \ldots, m_r$ of labels*".

Lemma 3.24. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

$C_{w,n_1,\ldots,n_r}$, the coefficient in front of $[X_w]$ in the decomposition of $[D_1]^{n_1} \ldots [D_r]^{n_r}$ into a linear combination of Schubert classes, can be computed as follows.

Choose any function $f\colon \{1, \ldots, \ell(w)\} \to \Pi$ that takes each value α_j exactly n_j times for all j, $1 \le j \le r$.

Then $C_{w,n_1,\ldots,n_r}$ is the number of [labeled sorting processes of w with the distribution of labels f], counting their X-multiplicities.

Proof. Induction on $\ell(w)$. For $\ell(w) = 0$, this is clear.

If $w \neq \mathrm{id}$, denote by $\gamma_1, \ldots, \gamma_m$ all of the roots from $\Delta^+ \cap w\Delta^-$ such that σ_{γ_j} is an admissible reflection for w. Also denote by g_j the coefficient in front of $f(1)$ in the decomposition of γ_j into a linear combination of simple roots. (Note that g_j may be 0.)

Then the set of all labeled sorting processes of w with distribution of labels f is split into the disjoint union of m subsets: the sorting processes starting with γ_1, …, the sorting processes starting with γ_m.

If we remove the first root (let it be γ_j) and its label $f(1)$ from a labeled sorting process of w, we will get a sorting process of $\sigma_{\gamma_j} w$ with list of labels $f(2), \ldots, f(\ell(w))$. And the X-multiplicity of this sorting process of w equals g_j times the X-multiplicity of this sorting process of $\sigma_{\gamma_j} w$.

So, using the induction hypothesis, it suffices to prove that

$$C_{w,n_1,\ldots,n_r} = \sum_{j=1}^{m} g_j C_{\sigma_{\gamma_j} w, n_1,\ldots,n_{i_1}-1,n_r}.$$

By the definition of $C_{v,n_1,\ldots,n_{i_1}-1,n_r}$, we have

$$[D_{\varpi_1}]^{n_1} \ldots [D_{\varpi_{i_1}}]^{n_{i_1}-1} \ldots [D_{\varpi_r}]^{n_r} = \sum_{v\in W:\ell(v)=\ell(w)-1} C_{v,n_1,\ldots,n_{i_1}-1,\ldots,n_r}[X_v].$$

Proposition 2.1 applied to each $[X_v]$ occurring on the right gives:

$$[D_{\varpi_i}][X_v] = \sum_{\substack{\alpha\in\Delta^+\\ \ell(\sigma_\alpha v)=\ell(v)+1}} \varpi_i(\alpha)[X_{\sigma_\alpha v}].$$

$[X_w]$ appears on the right-hand side if and only if $\sigma_\alpha v = w$ for some $\alpha \in \Delta^+$, i. e. $v = \sigma_\alpha w$ for some $\alpha \in \Delta^+$. Since $\ell(v) = \ell(w) - 1$, the equlaity $v = \sigma_\alpha w$ implies that σ_α is an admissible reflection for w, and $\alpha = \gamma_j$ for some j. The coefficient in front of this $[X_w]$ in the Pieri formula is $\varpi_i(\gamma_j) = g_j$.

Now let us take the linear combination of all Pieri formulas we wrote for all $[X_v]$s with coefficients $C_{v,n_1,\ldots,n_{i_1}-1,\ldots,n_r}$.

On the left, we will get $[D_{\varpi_1}]^{n_1} \ldots [D_{\varpi_{i_1}}]^{n_{i_1}} \ldots [D_{\varpi_r}]^{n_r}$.

On the right, we will get a linear combination of Schubert classes with some coefficients, and the coefficient in front of $[X_w]$ will be $\sum_j g_j C_{\sigma_{\gamma_j} w,n_1,\ldots,n_{i_1}-1,n_r}$. But this coefficient also equals $C_{w,n_1,\ldots,n_r}$. □

Corollary 3.25. *Given $w \in W$, the number of labeled sorting processes with a distribution of labels f counting the X-multiplicities of processes, depends only on the D-multiplicities of simple roots in the distribution f, but not on the distribution f itself itself.* □

4 Criterium of sortability

Lemma 4.1. *For each $w \in W$, there exists at least one antireduced sorting process.*

Proof. Induction on $\ell(w)$. Trivial for $w = \mathrm{id}$.

By Corollary 3.10, there exists a root $\beta_1 \in \Delta^+ \cap w\Delta^-$ such that σ_{β_1} is an antisimple reflection for w.

Let us try to begin the sorting process with β_1. Set $w_1 = \sigma_{\beta_1} w$. $\ell(w_1) = \ell(w) - 1$. There exists an antireduced sorting process for w_1, denote it by $\beta_2, \ldots, \beta_n$. Then $\beta_1, \beta_2, \ldots, \beta_n$ is an antireduced sorting process for w, because the products $\beta_{k+1} \ldots \beta_n$ occurring in the definitions of antireduced sorting processes for w and for w_1 are exactly the same (with the addition of w itself to the sorting process of w, but we have checked explicitly that σ_{β_1} is an antisimple reflection for w). □

For each $A \subseteq \Delta^+$, for each $I \subseteq \{1, \ldots, r\}$, denote by $R_I(A)$ the set of all roots $\alpha \in A$ such that $\operatorname{supp}\alpha$ contains at least one simple root α_i with $i \in I$.

Lemma 4.2. *Let $A \subseteq \Delta^+$, and let $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}$ be such that $n_1 + \ldots + n_r = |A|$.*

Denote by J the set of indices i ($1 \leq i \leq r$) such that $n_i > 0$.

The following conditions are equivalent:

1. *For each $I \subseteq J$, $|R_I(A)| \geq \sum_{i\in I} n_i$.*

2. *There exists a simple root distribution on A with D-multiplicities $n_1, \ldots, n_r$.*

3. *For each $I \subseteq \{1, \ldots, r\}$, $|R_I(A)| \geq \sum_{i\in I} n_i$.*

Proof. Note that for each $I \subseteq \{1, \ldots, r\}$, by definition of $R_I(A)$,

$$R_I(A) = \bigcup_{i\in I} R_{\{i\}}(A).$$

$1 \Rightarrow 2$

Condition 1 is equivalent to the hypothesis of generalized Hall representative lemma (Lemma 2.3) appied to the $|J|$ sets: $R_{\{j\}}(A)$ for each $j \in J$.

And Lemma 2.3 says that for each $j \in J$, we can choose n_j elements of $R_{\{j\}}(A)$, i. e. n_j roots $\alpha \in A$ such that $\alpha_j \in \operatorname{supp} \alpha$, and all chosen roots (for different values of j) are different. In total, we chose $\sum_{j\in J} n_j$ roots, and, by the definition of J, $\sum_{j\in J} n_j = n_1 + \ldots + n_r = |A|$. So, each root from A was chosen exactly once, and we can set $f(\alpha) = \alpha_j$ if α was chosen as an element of $R_{\{j\}}(A)$. This is a simple root distribution on A, and it clearly has D-multiplicities $n_1, \ldots, n_r$ of simple roots.

$2 \Rightarrow 3$

Let f be a simple root distribution. Then for each i, $1 \leq i \leq r$, $f^{-1}(\alpha_i) \subseteq R_{\{i\}}(A)$ and $n_i = |f^{-1}(\alpha_i)|$. So, for each $I \subseteq \{1, \ldots, r\}$,

$$\bigcup_{i\in I} f^{-1}(\alpha_i) \subseteq R_I(A).$$

and

$$\sum_{i\in I} n_i = \left|\bigcup_{i\in I} f^{-1}(\alpha_i)\right|$$

Therefore, $\sum_{i\in I} n_i \leq |R_I(A)|$.

$3 \Rightarrow 1$

Follows directly. □

For each $w \in W$, for each $I \subseteq \{1, \ldots, r\}$, we briefly write $R_I(w) = R_I(\Delta^+ \cap w\Delta^-)$.

Corollary 4.3. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

Denote by J the set of indices of involved roots, i. e. of indices i ($1 \leq i \leq r$) such that $n_i > 0$.

The following conditions are equivalent:

1. *For each $I \subseteq J$, $|R_I(w)| \geq \sum_{i\in I} n_i$.*

2. *There exists a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$.*

3. *For each $I \subseteq \{1, \ldots, r\}$, $|R_I(w)| \geq \sum_{i\in I} n_i$.* □

Proposition 4.4. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

Then the following conditions are equivalent:

1. *There exists a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.*

2. *There exists a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$.*

If these conditions are satisified, then there actually exists an antireduced *labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.*

Proof. $1 \Rightarrow 2$. Induction on $\ell(w)$. Suppose that there exists a labeled sorting process of w with D-multiplicities $n_1, \ldots, n_r$ of labels.

It has to start with some admissible sorting reflection, and all admissible sorting reflections are reflections along some of the roots from $\Delta^+ \cap w\Delta^-$. Suppose that the sorting process starts with $\beta \in \Delta^+ \cap w\Delta^-$, and the label assigned to the first step of the sorting process is α_i. Denote $w_1 = \sigma_\beta w$.

The rest of the labeled sorting process of w actually gives us a labeled sorting process of w_1 with D-multiplicities $n_1, \dots, n_i - 1, \dots, n_r$ of labels.

Recall that Lemma 3.6 establishes a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \beta$ and $\Delta^+ \cap w_1\Delta^-$. Denote this bijection by $\psi\colon (\Delta^+ \cap w\Delta^-) \setminus \beta \to \Delta^+ \cap w_1\Delta^-$

Lemma 3.6 says that either $\psi(\gamma) = \gamma$, or $\psi(\gamma) = \gamma - \beta$. In both cases, $\psi(\gamma) \preceq \gamma$.

By the induction hypothesis, there exists a simple root distribution on $\Delta^+ \cap w_1\Delta^-$ with D-multiplicities $n_1, \dots, n_i - 1, \dots, n_r$ of simple roots. Denote this distribution by $f_1\colon \Delta^+ \cap w_1\Delta^- \to \Pi$.

For each $\gamma \in (\Delta^+ \cap w\Delta^-) \setminus \beta$, since $\psi(\gamma) \preceq \gamma$ and $f_1(\psi(\gamma)) \in \operatorname{supp} \psi(\gamma)$, we have $f_1(\psi(\alpha)) \in \operatorname{supp} \gamma$. Also, $\alpha_i \in \operatorname{supp} \beta$. So, we can define the following simple root distribution f on $\Delta^+ \cap w\Delta^-$: $f(\beta = \alpha_i$, and $f(\gamma) = f_1(\psi(\gamma))$ for $\gamma \neq \beta$.

$2 \Rightarrow 1$

We are going to construct an antireduced labeled sorting process, then the last claim in the problem statement will be simultaneously proved.

By Lemma 4.1, there exists an antireduced sorting process of w. Denote the roots occurring in this sorting process by $\beta_1, \dots, \beta_{\ell(w)}$ (in this order). By Lemma 3.14, the set of roots occurring in this antireduced sorting process is exactly $\Delta^+ \cap w\Delta^-$, i. e. $\Delta^+ \cap w\Delta^- = \{\beta_1, \dots, \beta_{\ell(w)}\}$

We also know that there exists a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$, denote it by $f\colon \Delta^+ \cap w\Delta^- \to \Pi$. Let us assign label $f(\beta_k)$ to the kth step of the sorting process, and we will get an antireduced labeled sorting process with D-multiplicities $n_1, \dots, n_r$ of labels. □

Corollary 4.5. *Let $w \in W$. Suppose we have a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$.*

Then there exists a labeled antireduced sorting process for w such that if at a certain step we make a reflection along $\alpha \in \Delta^+ \cap w\Delta^-$ (we make it only once, see Corollary 3.15), we assign the simple root $f(\alpha)$ to it.

In other words, since all roots occurring in an antireduced sorting process are different, to define a function on the set of occurring roots is equivalent to define a function on $\{1, \dots, \ell(w)\}$. And the claim is that we can make the latter function, the distribution of labels of the labeled sorting process, the same as the former function, an arbitrary simple root distribution on $\Delta^+ \cap w\Delta^-$.

Proof. The proof exactly repeats the argument $2 \Rightarrow 1$ in the proof of Proposition 4.4. □

Corollary 4.6. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.*

Then the following conditions are equivalent:

1. *There exists a labeled sorting process of w with D-multiplicities $n_1, \dots, n_r$ of labels.*
2. *For each $I \subseteq \{1, \dots, r\}$, $|R_I(w)| \geq \sum_{i \in I} n_i$.*

If these conditions are satisified, then there actually exists an antireduced *labeled sorting process of w with D-multiplicities $n_1, \dots, n_r$ of labels.*

Proof. The claim follows from Corollary 4.3 and Proposition 4.4. □

Definition 4.7. Let $w \in W$, and let $\alpha \in \Delta^+ \cap w\Delta^-$. A simple root distribution f on $\Delta^+ \cap w\Delta^-$ is called *α-compatible* if σ_α is an admissible sorting reflection for w, and the distribution has the following additional property:

If $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec \beta$, $(\alpha, \beta) = 1$, and $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, then $f(\beta) \notin \operatorname{supp} \alpha$.

Lemma 4.8. *Let $w \in W$, $\alpha \in \Delta^+ \cap w\Delta^-$. Let f be simple root distribution on $\Delta^+ \cap w\Delta^-$.*

The following conditions are equivalent:

1. *f is α-compatible*
2. *For each $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$ and $(\alpha, \beta) = 1$, we have $f(\beta) \notin \operatorname{supp} \alpha$.*

Proof. $1 \Rightarrow 2$

Assume that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$, $(\alpha, \beta) = 1$, and $f(\beta) \in \operatorname{supp} \alpha$. Set $\gamma = \alpha - \beta$ ($\gamma \in \Delta$ by Lemma 2.5). $\alpha \prec_w \beta$, so $\gamma \in w\Delta^-$.

If $\gamma \in \Delta^+$, then σ_α cannot be an admissible reflection for w by Lemma 3.4. If $\gamma \in \Delta^-$, then $\beta \prec \alpha$, and $-\gamma = \beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so we have a contradiction with the definition of α-compatibility.

$2 \Rightarrow 1$

Admissibility of σ_α: assume the contrary. By Lemma 3.4, there exist $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta + \gamma = \alpha$. By Lemma 2.5, $(\beta, \gamma) = -1$, so $(\alpha, \beta) = 1$. $-\gamma = \beta - \alpha \in w\Delta^+$, so $\alpha \prec_w \beta$. Also, $\gamma = \alpha - \beta \in \Delta^+$, so $\beta \prec \alpha$, and $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$. $f(\beta) \in \operatorname{supp} \beta$, so $f(\beta) \in \operatorname{supp} \alpha$, a contradiction.

Now suppose that $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec \beta$, $(\alpha, \beta) = 1$, and $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$.

$\alpha \prec \beta$ and $(\alpha, \beta) = 1$, so $\beta - \alpha \in \Delta^+$.

$\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so $\beta - \alpha \notin w\Delta^-$, so $\beta - \alpha \in w\Delta^+$, and $\alpha \prec_w \beta$.

Condition 2 in the Lemma statement says that $f(\beta) \notin \operatorname{supp} \alpha$, so the definition of α-compatibility holds. $\square$

Lemma 4.9. *Let $w \in W$, $\alpha \in \Delta^+ \cap w\Delta^-$. Let f be simple root distribution on $\Delta^+ \cap w\Delta^-$.*

The following conditions are equivalent:

1. *f is α-compatible*
2. *There are no roots $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$, $(\alpha, \beta) = 1$, $f(\beta) \in \operatorname{supp} \alpha$, and $f(\alpha) \in \operatorname{supp} \beta$.*

Proof. $1 \Rightarrow 2$

Assume that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $\alpha \prec_w \beta$, $(\alpha, \beta) = 1$, $f(\beta) \in \operatorname{supp} \alpha$, and $f(\alpha) \in \operatorname{supp} \beta$. Set $\gamma = \alpha - \beta$ ($\gamma \in \Delta$ by Lemma 2.5). $\alpha \prec_w \beta$, so $\gamma \in w\Delta^-$.

If $\gamma \in \Delta^+$, then σ_α cannot be an admissible reflection for w by Lemma 3.4. If $\gamma \in \Delta^-$, then $\beta \prec \alpha$, and $-\gamma = \beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so we have a contradiction with the definition of α-compatibility.

$2 \Rightarrow 1$

Admissibility of σ_α: assume the contrary. By Lemma 3.4, there exist $\beta, \gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta + \gamma = \alpha$.

$\alpha, \beta, \gamma \in \Delta^+$, so $\operatorname{supp} \alpha = \operatorname{supp} \beta \cup \operatorname{supp} \gamma$.

$f(\alpha) \in \operatorname{supp} \alpha$, so we may assume without loss of generality (after a possible interchange of β and γ) that $f(\alpha) \in \operatorname{supp} \beta$.

By Lemma 2.5, $(\beta, \gamma) = -1$, so $(\alpha, \beta) = 1$. $-\gamma = \beta - \alpha \in w\Delta^+$, so $\alpha \prec_w \beta$. Also, $\gamma = \alpha - \beta \in \Delta^+$, so $\beta \prec \alpha$, and $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$. $f(\beta) \in \operatorname{supp} \beta$, so $f(\beta) \in \operatorname{supp} \alpha$, a contradiction.

Now suppose that $\beta \in \Delta^+ \cap w\Delta^-$, $\alpha \prec \beta$, $(\alpha, \beta) = 1$, and $\beta - \alpha \notin \Delta^+ \cap w\Delta^-$.

$\alpha \prec \beta$ and $(\alpha, \beta) = 1$, so $\beta - \alpha \in \Delta^+$.

$\beta - \alpha \notin \Delta^+ \cap w\Delta^-$, so $\beta - \alpha \notin w\Delta^-$, so $\beta - \alpha \in w\Delta^+$, and $\alpha \prec_w \beta$.

$\alpha \prec \beta$, so $\operatorname{supp} \alpha \subseteq \operatorname{supp} \beta$. $f(\alpha) \in \operatorname{supp} \alpha$, so $f(\alpha) \in \operatorname{supp} \beta$.

Condition 2 in the Lemma statement says that $f(\beta) \notin \operatorname{supp} \alpha$, so the definition of α-compatibility holds. $\square$

Corollary 4.10. *Let $w \in W$, and let $\alpha \in \Delta^+ \cap w\Delta^-$ be such that $w^{-1}\alpha \in -\Pi$.*

Then every simple root distribution on $\Delta^+ \cap w\Delta^-$ is α-compatible.

Proof. Since $w^{-1}\alpha \in -\Pi$, there are no roots $\beta \in w\Delta^-$ such that $\alpha \prec_w \beta$. $\square$

Lemma 4.11. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities, and let $\alpha \in \Delta^+ \cap w\Delta^-$.*

Suppose that there exists an α-compatible distribution f of simple roots on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots. Suppose that $f(\alpha) = \alpha_i$

Then there exists a labeled sorting process for w that starts with α, the label at this α is $f(\alpha)$, and the whole list of labels is $\alpha_i, \alpha_1, \ldots, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_i, \ldots, \alpha_r, \ldots, \alpha_r$, where, after (excluding) the first α_i, [each α_j is written n_j times, except for α_i, which is written $n_i - 1$ times].

In particular, there exists [a labeled sorting process for w with D-multiplicities $n_1, \ldots, n_i, \ldots, n_r$ of labels] that starts with α, and the label at this α is $f(\alpha)$.

Proof. We start our sorting process with α. Set $w_1 = \sigma_\alpha w$.

By Lemma 3.6 establishes a bijection between $(\Delta^+ \cap w\Delta^-) \setminus \beta$ and $\Delta^+ \cap w_1\Delta^-$. Denote this bijection by $\psi\colon (\Delta^+ \cap w\Delta^-) \setminus \beta \to \Delta^+ \cap w_1\Delta^-$

The definition of α-compatibility says, in terms of Lemma 3.6, that if case 2 of the procedure in Lemma 3.6 holds for some $\beta \in \Delta^+ \cap w\Delta^-$, then $f(\beta) \notin \operatorname{supp} \alpha$. Since $f(\beta) \in \operatorname{supp} \alpha$ for such β, then also $f(\beta) \in \operatorname{supp}(\beta - \alpha) = \operatorname{supp}(\psi(\beta))$.

And if case 3 holds in the procedure in Lemma 3.6 for some $\beta \in \Delta^+ \cap w\Delta^-$, then $\psi(\beta) = \beta$, so clearly, $f(\beta) \in \operatorname{supp}(\psi(\beta))$.

So, $f(\beta) \in \operatorname{supp}(\psi(\beta))$ for all $\beta \in (\Delta^+ \cap w\Delta^-) \setminus \alpha$, and we can set $f_1\colon \Delta^+ \cap w_1\Delta^-$, $f_1(\gamma) = f(\psi^{-1}(\gamma))$. Then $f_1(\gamma) \in \operatorname{supp} \gamma$, so f_1 is a simple root distribution on $\Delta^+ \cap w_1\Delta^-$ with with D-multiplicities $n_1, \ldots, n_i - 1, \ldots, n_r$ of simple roots.

By Proposition 4.4, there exists a labeled sorting process of w_1 with D-multiplicities $n_1, \ldots, n_i - 1, \ldots, n_r$ of labels.

By Corollary 3.25, there exists a labeled sorting process of w_1 with the list of labels $\alpha_1, \ldots, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_i, \ldots, \alpha_r, \ldots, \alpha_r$, where each α_j is written n_j times, except for α_i, which is written $n_i - 1$ times.

We write α with label α_i in front of this sorting process, and we get the claim. □

5 Clusters

Definition 5.1. Let $A \subseteq \Pi$ be a set of simple roots.

A subset $B \subseteq \Delta^+$ is called a *cluster with set of essential roots* A (or, briefly, an A-cluster) if the following conditions hold:

1. If $\alpha, \beta \in B$, $\alpha \neq \beta$, then (α, β) can be equal to 1 or 0, but not -1.
2. If $\alpha, \beta \in B$ and $(\alpha, \beta) = 0$, then $\operatorname{supp} \alpha \cap \operatorname{supp} \beta \cap A = \varnothing$. In other words, $\operatorname{supp} \alpha$ and $\operatorname{supp} \beta$ don't have essential roots in common.

6 Criterion of unique sortability

6.1 Basic sufficient conditions for non-unique sortability

Lemma 6.1. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exists a root $\alpha \in \Delta^+ \cap w\Delta^-$ such that the [coefficient in front of $f(\alpha)$ in the decomposition of α into a linear combination of simple roots] is at least 2,

then $C_{w,n_1,\ldots,n_r} \geq 2$.

Proof. By Corollary 4.5, there exists an antireduced labeled sorting process such that when we perform a reflection along a root $\beta \in \Delta^+ \cap w\Delta^-$, the label at this root is $f(\beta)$. Denote the corresponding distribution of simple roots $\{1, \ldots, \ell(w)\} \to \Pi$ by f_1. The D-multiplicities of labels of this sorting process are $n_1, \ldots, n_r$.

In particular, when we perform the reflection σ_α, the label is $f(\alpha)$. By the definition of X-multiplicity, that means that the X-multiplicity of this sorting process is at least 2 (more precisely, it is a positive integer divisible by 2).

By Lemma 3.24, $C_{w,n_1,\ldots,n_r}$ is the number of [labeled sorting processes of w with the distribution of labels f_1], counting their X-multiplicities, so, it is at least 2 since we have a labeled sorting process with distribution of labels f_1 and X-multiplicity at least 2. □

Lemma 6.2. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = -1$ and $f(\alpha) = f(\beta)$,

then $C_{w,n_1,\ldots,n_r} \geq 2$

Proof. $\alpha, \beta \in \Delta$, $(\alpha, \beta) = -1$, so $\alpha + \beta \in \Delta$.

$\alpha, \beta \in \Delta^+$, so $\alpha + \beta \in \Delta^+$.

$\alpha, \beta \in w\Delta^-$, so $w^{-1}\alpha, w^{-1}\beta \in \Delta^-$, so $w^{-1}(\alpha + \beta) = w^{-1}\alpha + w^{-1}\beta \in \Delta^-$, so $\alpha + \beta \in w\Delta^-$.

Therefore, $\alpha + \beta \in \Delta^+ \cap w\Delta^-$.

Denote $f(\alpha) = f(\beta) = \alpha_i$, $f(\alpha + \beta) = \alpha_j$.

Clearly, $\operatorname{supp}(\alpha+\beta) = \operatorname{supp}\alpha \cup \operatorname{supp}\beta$. $\alpha_j \in \operatorname{supp}(\alpha+\beta)$, so α_j is in at least one of $(\operatorname{supp}\alpha, \operatorname{supp}\beta)$.

Without loss of generality, suppose that $\alpha_j \in \operatorname{supp}\alpha$.

Consider the following new simple root distribution g on $\Delta^+ \cap w\Delta^-$:

$g(\alpha + \beta) = \alpha_i$, $g(\alpha) = \alpha_j$, and $g(\gamma) = f(\gamma)$ for all other $\gamma \in \Delta^+ \cap w\Delta^-$.

$\alpha_i \in \operatorname{supp}\alpha$ and $\alpha_i \in \operatorname{supp}\beta$, so the coefficient in front of $g(\alpha + \beta) = \alpha_i$ in the decomposition of $\alpha + \beta$ into a linear combination of simple roots is at least 2.

The claim follows from Lemma 6.1. $\square$

Lemma 6.3. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

If there exist two simple root distributions $f, g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $\alpha \neq \beta$ such that f is α-compatible, g is β-compatible, and $f(\alpha) = g(\beta)$,

then $C_{w,n_1,\ldots,n_r} \geq 2$

Proof. Denote $\alpha_i = g(\alpha) = f(\beta)$. Denote by L the following list of simple roots (i. e. a function $\{1, \ldots, \ell(w)\} \to \Pi$): $\alpha_i, \alpha_1, \ldots, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_i, \ldots, \alpha_r, \ldots, \alpha_r$, where, after (excluding) the first α_i, [each α_j is written n_j times, except for α_i, which is written $n_i - 1$ times].

By Lemma 4.11, there exists a labeled sorting process for w that starts with α, the label at this α is $f(\alpha)$, and the whole list of labels is L.

And there is another labeled sorting process for w that starts with β, the label at this β is $g(\beta)$, and the whole list of labels is L.

By Lemma 3.24, $C_{w,n_1,\ldots,n_r} \geq 2$. $\square$

Corollary 6.4. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$, $\alpha \neq \beta$ such that f is both α-compatible and β-compatible and $f(\alpha) = f(\beta)$,

then $C_{w,n_1,\ldots,n_r} \geq 2$

Proof. This is the previous lemma with $f = g$. $\square$

Lemma 6.5. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities, let $0 \leq k \leq \ell(w)$.*

Let $\beta_1, \ldots, \beta_k$ be a labeled sorting process prefix of w with D-multiplicities $m_1, \ldots, m_r$ of labels. Suppose that $m_i \leq n_i$. Denote $w_k = \sigma_{\beta_k} \ldots \sigma_{\beta_1} w$.

Then $C_{w,n_1,\ldots,n_r} \geq C_{w_k,n_1-m_1,\ldots,n_r-m_r}$.

In particular, if $C_{w_k,n_1-m_1,\ldots,n_r-m_r} \geq 2$, then $C_{w,n_1,\ldots,n_r} \geq 2$.

Proof. Denote the list of labels of the labeled sorting process prefix $\beta_1, \ldots, \beta_k$ by L.

Fix a function $\{k+1, \ldots, \ell(w)\} \to \Pi$ with D-multiplicities $n_1 - m_1, \ldots, n_r - m_r$ of simple roots. For example, fix the following list of simple roots: $\alpha_1, \ldots, \alpha_1, \ldots, \alpha_r, \ldots, \alpha_r$, where α_i is repeated $n_i - m_i$ times. Denote this list by L'.

For each labeled sorting process of w_k with distribution of labels L, do the following. Denote this sorting process by $\beta_{k+1}, \ldots, \beta_{\ell(w)}$. Write $\beta_1, \ldots, \beta_k$ in front of $\beta_{k+1}, \ldots, \beta_{\ell(w)}$, and assign the original labels to these $\beta_1, \ldots, \beta_k$. We get a labeled sotring process of w with list of labels L, L'. The D-multiplicities of labels in L, L' are $n_1, \ldots, n_r$. And the X-multiplicity of this sorting process of w is divisible by the X-multiplicity of the sorting process of w_k.

Note that we will get different labeled sorting process of w for different labeled sorting processes of w_k.

By Lemma 3.24, $C_{w_k,n_1-m_1,\ldots,n_r-m_r}$ is the numer of labeled sorting processes of w_k with list of labels L', counting their X-multiplicities, and $C_{w,n_1,\ldots,n_r}$ is the number of labeled sorting processes of w with list of labels L, L', counting their X-multiplicities. So, $C_{w,n_1,\ldots,n_r} \geq C_{w_k,n_1-m_1,\ldots,n_r-m_r}$. $\square$

Corollary 6.6. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities, let $0 \le k \le \ell(w)$.*

Let f be a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots.

Let $\beta_1, \ldots, \beta_k$ be an antireduced labeled sorting process prefix with the label $f(\beta_i)$ at each β_i (this is well-defined by Lemma 3.11). Denote $w_k = \sigma_{\beta_k} \ldots \sigma_{\beta_1} w$.

Denote by g the restriction of f onto $\Delta^+ \cap w_k\Delta^-$, and denote by $p_1, \ldots, p_r$ the D-multiplicities of simple roots in g.

Then $C_{w,n_1,\ldots,n_r} \ge C_{w_k,p_1,\ldots,p_r}$.

In particular, if $C_{w_k,p_1,\ldots,p_r} \ge 2$, then $C_{w,n_1,\ldots,n_r} \ge 2$.

Proof. Clearly, if we denote the D-multiplicities of simple roots of the distribution of labels on $\beta_1, \ldots, b_k$ by $m_1, \ldots, m_r$, then $p_i = n_i - m_i$.

The claim now follows from Lemma 6.5. □

Lemma 6.7. *Let $w \in W$. Suppose that $\Delta^+ \cap w\Delta^-$ contains exactly one root α such that $w^{-1}\alpha \in -\Pi$.*

Then for every $\beta \in \Delta^+ \cap w\Delta^-$, $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$.

Proof. Fix $\beta \in \Delta^+ \cap w\Delta^-$. Denote $w^{-1}\alpha = -\alpha_i$ and $w^{-1}\beta = -\sum_j a_j\alpha_j$.

Clearly, $\operatorname{supp} w(a_i\alpha_i) = \operatorname{supp} \alpha$. Since α is the only root in $\Delta^\cap w\Delta^-$ such that $w^{-1}\alpha \in -Pi$, for all other roots α_j with $j \ne i$ we have $w(-\alpha_j) \notin \Delta^+ \cap w\Delta^-$. Clearly, $w(-\alpha_j) \in w\Delta^-$, so $w(-\alpha_j) \notin \Delta^+$ if $i \ne j$, and $w(-\alpha_j) \in \Delta^-$ if $i \ne j$.

Therefore, all coefficients in the decomposition of $w(-\sum_{j\ne i} a_j\alpha_j)$ into a linear combination of simple roots are nonpositive.

We also know that $\beta \in \Delta^+$, so all coefficients in its decomposition into a linear combination of simple roots are nonnegative.

Since all coefficients in the decomposition of $w(-\sum_{j\ne i} a_j\alpha_j)$ into a linear combination of simple roots are nonpositive, the (nonnegative) coefficients in the decomposition of β into a linear combination of simple roots are smaller than or equal to the correspnding (also nonnegative) coefficients in the decomposition of $w(a_i\alpha_i)$ into a linear combination of simple roots.

So, $\operatorname{supp} \beta \subseteq \operatorname{supp} w(a_i\alpha_i) = \operatorname{supp} \alpha$. □

Lemma 6.8. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities. Suppose that $\Delta^+ \cap w\Delta^-$ contains exactly one root α such that $w^{-1}\alpha \in -\Pi$.*

Suppose that there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$ and $f(\alpha) = f(\beta)$.

Then at least one of the following statements is true:

1. *$C_{w,n_1,\ldots,n_r} \ge 2$.*
2. *There exists a (possibly different) simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with (the same) D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exist $\beta', \beta'' \in \Delta^+ \cap w\Delta^-$ such that $\alpha \ne \beta'$, $\alpha \ne \beta''$, $(\beta', \beta'') = 0$ and $g(\beta') = g(\beta'') = f(\alpha)$.*

Proof. First, until the end of the proof, call a root $\gamma \in \Delta^+ \cap w\Delta^-$ *red* if $\gamma \ne \alpha$ and there exists a simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $g(\alpha) = g(\gamma) = f(\alpha)$.

Clearly, β is a red root.

Without loss of generality (after a possible change of f) we may assume that β is a [maximal in the sense of $\prec_w$] element of the set of {red roots γ such that $(\gamma, \alpha) = 0$}.

Suppose first that there exists a red root γ such that $(\gamma, \alpha) = -1$.

This means that there exists a simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $g(\alpha) = g(\gamma) = f(\alpha)$. By Lemma 6.2 (applied to the distribution g), $C_{w,n_1,\ldots,n_r} \ge 2$.

Now we suppose until the end of the proof that if γ is a red root, then $(\gamma, \alpha) = 0$ or $(\gamma, \alpha) = 1$.

Similarly, note that if there exists a red root γ such that the coefficient in front of $f(\alpha)$ in the decomposition of γ into a linear combination of simple roots is at least 2, then $C_{w,n_1,\ldots,n_r} \ge 2$ by lemma 6.1.

So, we also suppose until the end of the proof that if γ is a red root, then the coefficient in front of $f(\alpha)$ in the decomposition of γ into a linear combination of simple roots is 1.

Also, if the coefficient in front of $f(\alpha)$ in the decomposition of α into a linear combination of simple roots is at least 2, then $C_{w,n_1,\ldots,n_r} \geq 2$ by lemma 6.1.

So, we also suppose until the end of the proof that the coefficient in front of $f(\alpha)$ in the decomposition of α into a linear combination of simple roots is 1.

1. Consider the case when f is a β-compatible distribution.

   By Lemma 4.10, f is also an α-compatible distribution. By Corollary 6.4, $C_{w,n_1,\ldots,n_r} \geq 2$.

   END Consider the case when f is a β-compatible distribution.

2. Now consider the case that f is not a β-compatible distribution.

   By Lemma 4.9, this means that there exists a root $\delta \in \Delta^+ \cap w\Delta^-$ such that $\beta \prec_w \delta$, $(\beta, \delta) = 1$, $f(\delta) \in \operatorname{supp} \beta$, and $f(\beta) \in \operatorname{supp} \delta$.

   $(\beta, \delta) = 1$, so $\delta \neq \alpha$ since $(\beta, \alpha) = 0$.

   Since $f(\beta) \in \operatorname{supp} \delta$, $f(\delta) \in \operatorname{supp} \beta$, we can consider a new simple root distribution h on $\Delta^+ \cap w\Delta^-$: $h(\beta) = f(\delta)$, $h(\delta) = f(\beta)$, and $h(\epsilon) = \epsilon$ for all $\epsilon \in \Delta^+ \cap w\Delta^-$, $\epsilon \neq \beta$, $\epsilon \neq \delta$. Clearly, h has D-multiplicities $n_1, \ldots, n_r$ of simple roots as well as f. Note also that $h(\delta) = f(\beta) = f(\alpha) = h(\alpha)$. Therefore, δ is a red root, and there are only two possibilities for (δ, α): $(\delta, \alpha) = 0$ and $(\delta, \alpha) = 1$.

   In fact, $(\delta, \alpha) = 0$ is also impossible, because $\beta \prec_w \delta$, and β is a maximal with respect to $\prec_w$ element of the set of red roots orthogonal to α.

   So, $(\delta, \alpha) = 1$.

   By Lemma 2.7, $\alpha - \delta + \beta \in \Delta$. Denote $\beta' = \alpha - \delta + \beta$. Lemma 2.7 also says that $(\beta', \delta) = 0$. It also says that $(\beta', \alpha) = 1$, so $\alpha \neq \beta'$.

   We are now supposing that the coefficients in front of $f(\alpha)$ in the decompositions of α and of all red roots into linear combinations of simple roots are all 1. By Lemma 2.8, $\beta' \in \Delta^+$, and the coefficient in front of $f(\alpha)$ in the decomposition of β' into a linear combination of simple roots is 1. In particular, $f(\alpha) \in \operatorname{supp} \beta'$.

   $\beta \prec_w \delta$, so, by Lemma 2.15, $\beta' \in w\Delta^-$.

   By Lemma 6.7, $\operatorname{supp} \beta' \subseteq \operatorname{supp} \alpha$, so $f(\beta') \in \operatorname{supp} \alpha$.

   Set $\beta'' = \delta$ and define a new simple root distribution g on $\Delta^+ \cap w\Delta^-$ as follows:

   $g(\alpha) = f(\beta')$.

   $g(\beta') = g(\beta'') = f(\alpha) = f(\beta)$.

   $g(\beta) = f(\beta'')$.

   Clearly, g has D-multiplicities $n_1, \ldots, n_r$ of simple roots as well as f.

   END consider the case that f is not a β-compatible distribution.

□

Lemma 6.9. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities.*

If there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that there exist roots $\delta', \delta'' \in \Delta^+ \cap w\Delta^-$ such that $(\delta', \delta'') = 0$ and $f(\delta') = f(\delta'')$,

then $C_{w,n_1,\ldots,n_r} \geq 2$

Proof. We are going to construct two different labeled sorting processes with the same list of labels.

Both sorting processes will begin in the same way and proceed in the same way, while possible.

Set $w_0 = w$.

We perform the following *antisimple* reflections while we don't say we want to stop. We will denote the current element of W after i reflections by w_i.

While we perform these reflections, we will sometimes need to modify the distribution f. In rigorous terms, we will have several simple root distributions $f_0 = f, f_1, \ldots, f_k$ $(0 \le k < \ell(w))$ such that when we perform the ith reflection (and it will be the ith reflection in both of the sorting processes we will construct), and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$ (recall that we are doing antisimple reflections, see Lemma 3.11), we assign (in both processes) the label $f_i(\gamma)$ to it. And when we modify our distribution later, i. e. when we define f_j with $j > i$, we don't change its value that was already assigned to a step of the sorting process, i. e. $f_j(\gamma)$ will be the same as $f_i(\gamma)$.

Also, all distributions f_i will have the same D-multiplicities of simple roots as f.

In the end, when we stop after k steps, it will be true that when we performed the ith reflection and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$, the label assigned to this reflection was $f_k(\gamma)$.

Also, while we perform this reflections, we will sometimes need to modify the values of δ' and δ''. Again, in rigorous terms, we will have two sequences of roots, $\delta'_0 = \delta', \delta'_1 \ldots, \delta'_k$ and $\delta''_0 = \delta'', \delta''_1 \ldots, \delta''_k$ such that $(\delta'_i, \delta''_i) = 0$, $f_i(\delta'_i) = f_i(\delta''_i) = f(\delta')$, and $\delta'_i, \delta''_i \in \Delta^+ \cap w_i\Delta^-$. In particular, this means that $|\Delta^+ \cap w_i\Delta^-| = \ell(w_i) \ge 2$, and this means that at a certain point we will have to stop explicitly, we cannot exhaust the whole $|\Delta^+ \cap w\Delta^-|$.

For each $i \in \mathbb{N}$, starting from $i = 1$.

1. If there exists $\gamma \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\gamma \in -\Pi$, $\gamma \ne \delta'_{i-1}$, $\gamma \ne \delta''_{i-1}$,

   then:

   Set $f_i = f_{i-1}$, $\delta'_i = \delta'_{i-1}$, $\delta''_i = \delta''_{i-1}$

   We are only performing antisimple reflections now, so by Lemma 3.11, $\Delta^+ \cap w_{i-1}\Delta^- \subseteq \Delta^+ \cap w\Delta^-$, and $\gamma \in \Delta^+ \cap w\Delta^-$, and f is defined on γ.

   we say that the ith step of both sorting processes will be $\beta_i = \gamma$ with label $f_i(\gamma)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

   $\beta_i \ne \delta'_i$, $\beta_i \ne \delta''_i$, so $\delta'_i, \delta''_i \in \Delta^+ \cap w_i\Delta^-$.

   And we CONTINUE with the next step of the sorting process (with the next value of i).

2. Otherwise, if ($w_{i-1}^{-1}\delta'_{i-1} \in -\Pi$ and $w_{i-1}^{-1}\delta''_{i-1} \in -\Pi$), then we say that we WANT TO STOP.

3. Otherwise, there is only one $\gamma \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\gamma \in -\Pi$, and this γ is either δ'_{i-1} or δ''_{i-1}.

   Without loss of generality, suppose that $\gamma = \delta'_{i-1}$.

   Restrict f_{i-1} onto $\Delta^+ \cap w_{i-1}\Delta^-$, and denote the result by g_{i-1}. Temporarily (until the end of this step of the sorting process) denote the D-multiplicities of simple roots in g_{i-1} by $m_1, \ldots, m_r$.

   Let us apply Lemma 6.8 to w_{i-1}, to the distribution g_{i-1}, and to δ'_{i-1} and δ''_{i-1}.

   Lemma 6.8 may tell us $C_{w_{i-1}, m_1, \ldots, m_r} \ge 2$. Then by Corollary 6.6, $C_{w, n_1, \ldots, n_r} \ge 2$. Stop everything, we are done.

   Otherwise, Lemma 6.8 gives us a new simple root distribution, which we denote by g_i, on $\Delta^+ \cap w_{i-1}\Delta^-$ and a new pair of roots, which we denote by δ'_i and δ''_i, such that:

   the D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \ldots, m_r$.

   $\delta'_i, \delta''_i \in \Delta^+ \cap w_{i-1}\Delta^-$,

   $(\delta'_i, \delta''_i) = 0$,

   $g_i(\delta'_i) = g_i(\delta''_i) = g_{i-1}(\delta'_{i-1}) = f(\delta')$

   $\delta'_i \ne \gamma$, $\delta''_i \ne \gamma$.

   Expand this new distribution g_i to the whole $\Delta^+ \cap w\Delta^-$ using f_{i-1}. In rigorous terms, define the following new distribution f_i on $\Delta^+ \cap w\Delta^-$: $f_i(\alpha) = g_i(\alpha)$ if $\alpha \in \Delta^+ \cap w_{i-1}\Delta^-$, and $f_i(\alpha) = f_{i-1}(\alpha)$ otherwise.

The D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \dots, m_r$, so the D-multiplicities of simple roots in f_i are the same as the D-multiplicities of simple roots in f_{i-1}, they are $n_1, \dots, n_r$.

Now we again say that the ith step of both sorting processes will be $\beta_i = \gamma$ with label $f_i(\gamma)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

Again, $\beta_i \neq \delta'_i$, $\beta_i \neq \delta''_i$, so $\delta'_i, \delta''_i \in \Delta^+ \cap w_i\Delta^-$.

And we CONTINUE with the next step of the sorting process (with the next value of i).

END For each $i \in \mathbb{N}$, starting from $i = 1$.

After a certain number (denote it by k) of steps, we will stop. At this point we will have a simple root distribution f_k on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots, a sequence $\beta_1, \dots, \beta_k$ of elements of $\Delta^+ \cap w\Delta^-$, a sequence $w_0 = w, w_1, \dots, w_k$ of elements of W such that

[σ_{β_i} is an antisimple sorting reflection for w_{i-1}, and $w_i = \sigma_{\beta_i} w_i$],

and two roots $\delta'_k, \delta''_k \in \Delta^+ \cap w_k\Delta^-$ such that $(\delta'_k, \delta''_k) = 0$, $f_k(\delta'_k) = f_k(\delta''_k) = f(\delta')$, and $w^{-1}\delta'_k, w^{-1}\delta''_k \in -\Pi$.

Again restrict f_k onto $\Delta^+ \cap w_k\Delta^-$, and denote the result by g_k. Denote the D-multiplicities of simple roots in g_k by $m_1, \dots, m_r$.

By Corollary 4.10, g_k is both δ'_k-compatible and δ''_k-compatible. By Corollary 6.4, $C_{w_k, m_1, \dots, m_r} \geq 2$. By Corollary 6.6, $C_{w, n_1, \dots, n_r} \geq 2$. □

6.2 Uniqueness and non-uniqueness of sortability in case of excessive configuration

Definition 6.10. Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.

Denote by J the set of indices of involved roots, i. e. of indices i $(1 \leq i \leq r)$ such that $n_i > 0$.

We say that it is *excessive* if:

$|R_J(w)| \geq \sum_{i \in J} n_i$

and

For each $I \subset J$, $I \neq J$, one has: $|R_I(w)| > \sum_{i \in I} n_i$.

Lemma 6.11. *Let $w, n_1, \dots, n_r$ be an excessive configuration of D-multiplicities. Then for each $\alpha \in \Delta^+ \cap w\Delta^-$, there exists an involved simple root $\alpha_i \in \operatorname{supp} \alpha$.*

Proof. By Corollary 4.3, there exists a simple roots distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots.

α_i is an involved root, so $n_i > 0$, and there is a root $\alpha \in \Delta^+ \cap w\Delta^-$ such that $f(\alpha) = \alpha_i$. Then $\alpha_i \in \operatorname{supp} \alpha$. □

Definition 6.12. Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.

We say that it is a *free-first-choice configuration* if for each $\alpha \in \Delta^+ \cap w\Delta^-$ and for each $\alpha_i \in \operatorname{supp} \alpha$ such that $n_i > 0$ there exists a simple root distribution f on $\Delta^+ \cap w\Delta-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that $f(\alpha) = \alpha_i$

Lemma 6.13. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities. If it is excessive, then it is a free-first-choice configuration.*

Proof. Fix $\alpha \in \Delta^+ \cap w\Delta^-$ and an involved root $\alpha_i \in \operatorname{supp} \alpha$. Set $A = (\Delta^+ \cap w\Delta^-) \setminus \alpha$.

Denote by J the set of indices j $(1 \leq j \leq r)$ such that $n_j > 0$. Note that $i \in J$.

Set $m_j = n_j$ for $j \neq i$ and $m_i = n_i - 1$. Since $n_i > 0$, $m_j \geq 0$ for all j $(1 \leq j \leq r)$.

Let $I \subseteq J$. Clearly, $\sum_{j \in I} n_j \geq \sum_{j \in I} m_j$ and $|R_I(A)| \geq |R_I(w)| - 1$.

If $I \neq J$, then $|R_I(A)| \geq |R_I(w)| - 1 > (\sum_{j \in I} n_j) - 1 \geq (\sum_{j \in I} m_j) - 1$. Since all number here are integers, $|R_I(A)| \geq \sum_{j \in I} m_j$.

If $I = J$, then $\sum_{j \in I} m_j = (\sum_{j \in J} n_j) - 1$, and $|R_I(A)| \geq |R_I(w)| - 1 \geq (\sum_{j \in I} n_j) - 1 = \sum_{j \in I} m_j$.

So, for all $I \subseteq J$ we have $|R_I(A)| \geq \sum_{j \in I} m_j$.

Denote by J' the set of indices $j \in \{1, \dots, r\}$ such that $m_j > 0$. Clearly, $J' \subseteq J$. So, for all $I \subseteq J'$ we also have $|R_I(A)| \geq \sum_{j \in I} m_j$. By Lemma 4.2, there exists a simple root distribution g on A with D-multiplicities $m_1, \dots, m_r$.

Set $f(\alpha) = \alpha_i$ and $f(\beta) = g(\beta)$ for $\beta \in A$. This is a distribution of simple roots on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$. □

Definition 6.14. Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities.

We say that this configuration *has large essential coordinates* if there exists $\alpha \in \Delta^+ \cap w\Delta^-$ and $\alpha_i \in \Pi$ such that $n_i > 0$ and the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is at least 2.

We say that this configuration *has small essential coordinates* if it does not have large essential coordinates.

Lemma 6.15. *Let $w, n_1, \dots, n_r$ be a free-first-choice configuration of D-multiplicities. If it has large essential coordinates,*

then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. Since the configuration has large essential coordinates, there exists $\alpha \in \Delta^+ \cap w\Delta^-$ and $\alpha_i \in \Pi$ such that $n_i > 0$ and the coefficient in front of α_i in the decomposition of α into a linear combination of simple roots is at least 2.

By the definition of a free-first-choice configuration, there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that $f(\alpha) = \alpha_i$. The claim follows from Lemma 6.1. □

Lemma 6.16. *Let $w, n_1, \dots, n_r$ be a free-first-choice configuration of D-multiplicities.*

If there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = -1$ and an involved *simple root $\alpha_i \in \operatorname{supp}\alpha \cap \operatorname{supp}\beta$,*

then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. $\alpha, \beta \in \Delta$, $(\alpha, \beta) = -1$, so $\alpha + \beta \in \Delta$.

$\alpha, \beta \in \Delta^+$, so $\alpha + \beta \in \Delta^+$.

$\alpha, \beta \in w\Delta^-$, so $w^{-1}\alpha, w^{-1}\beta \in \Delta^-$, so $w^{-1}(\alpha + \beta) = w^{-1}\alpha + w^{-1}\beta \in \Delta^-$, so $\alpha + \beta \in w\Delta^-$.

Therefore, $\gamma = \alpha + \beta \in \Delta^+ \cap w\Delta^-$.

Since $\alpha_i \in \operatorname{supp}\alpha$ and $\alpha_i \in \operatorname{supp}\beta$, the coefficient in front of α_i in the decomposition of $\gamma = \alpha + \beta$ into a linear combination of simple roots is at least 2.

α_i is an involved root, so the configuration $w, n_1, \dots, n_r$ has large essential coordiantes. The claim follows from Lemma 6.16. □

Definition 6.17. Let $w \in W$. We call a simple root distribution f on $\Delta^+ \cap w\Delta^-$ *flexible* if there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$, $f(\beta) \in \operatorname{supp}\alpha$, and $f(\alpha) \in \operatorname{supp}\beta$.

Lemma 6.18. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities. If there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \dots, n_r$ of simple roots and roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $w^{-1}\alpha, w^{-1}\beta \in -\Pi$, $f(\alpha) \in \operatorname{supp}\beta$, and $f(\beta) \in \operatorname{supp}\alpha$,*

then $C_{w,n_1,\dots,n_r} \geq 2$.

Proof. By Lemma 4.10, f is α-compatible.

Consider another simple roots distribution g on $\Delta^+ \cap w\Delta^-$: $g(\alpha) = f(\beta)$, $g(\beta) = f(\alpha)$, and $g(\gamma) = f(\gamma)$ for all other $\gamma \in \Delta^+ \cap w\Delta^-$. Since $f(\alpha) \in \operatorname{supp}\beta$ and $f(\beta) \in \operatorname{supp}\alpha$, this is really a simple root distribution. Clearly, it also has D-multiplicities $n_1, \dots, n_r$ of simple roots.

By Lemma 4.10, g is β-compatible.

By Lemma 6.3, $C_{w,n_1,\dots,n_r} \geq 2$. □

Lemma 6.19. *Let $w, n_1, \dots, n_r$ be a configuration of D-multiplicities that has small essential coordiantes. Suppose that $\Delta^+ \cap w\Delta^-$ contains exactly one root α such that $w^{-1}\alpha \in -\Pi$.*

Suppose that there exists a simple root distribution $f\colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \dots, n_r$ of simple roots such that there exists $\beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$ and $f(\alpha) \in \operatorname{supp}\beta$.

Then at least one of the following statements is true:

1. $C_{w,n_1,\ldots,n_r} \geq 2$.

2. There exists a simple root distribution $g\colon \Delta^+ \cap w\Delta^- \to \Pi$ *whose restriction to* $\Delta^+ \cap (\sigma_\alpha w)\Delta^- = (\Delta^+ \cap w\Delta^-) \setminus \alpha$ *is flexible.*

Proof. The proof is very similar to the proof of Lemma 6.8.

First, until the end of the proof, call a root $\gamma \in \Delta^+ \cap w\Delta^-$ *red* if $\gamma \neq \alpha$ and $f(\alpha) \in \operatorname{supp} \gamma$.

Clearly, β is a red root.

Without loss of generality we may assume that β is a [maximal in the sense of $\prec_w$] element of the set of {red roots γ such that $(\gamma, \alpha) = 0$}.

Denote $\alpha_i = f(\alpha)$. Since f is a simple root distribution with D-multiplicities $n_1, \ldots, n_r$ of simple roots and $f(\alpha) = \alpha_i$, $n_i > 0$.

Assume that there exists a red root γ such that $(\gamma, \alpha) = -1$.

This means that $f(\gamma) = f(\alpha)$, in particluar, $f(\alpha) \in \operatorname{supp} \alpha$, $f(\alpha) \in \operatorname{supp} \gamma$.

By Lemma 2.5, $\alpha + \gamma \in \Delta$.

$\alpha, \gamma \in \Delta^+$, so $\alpha + \gamma \in \Delta^+$.

$\alpha, \gamma \in w\Delta^-$, so $\alpha + \gamma \in w\Delta^-$.

Therefore, $\alpha + \gamma \in \Delta^+ \cap w\Delta^-$.

$f(\alpha) \in \operatorname{supp} \alpha$, $f(\alpha) \in \operatorname{supp} \gamma$, so, the coefficient in front of $f(\alpha)$ in the decomposition of $\alpha + \gamma$ into the linear combination of simple roots is at least 2. We know that $n_i > 0$, so $w, n_1, \ldots, n_r$ is actually a configuration that has large essential coordinates. A contradiction.

Therefore, if γ is a red root, then $(\gamma, \alpha) = 0$ or $(\gamma, \alpha) = 1$.

By Lemma 6.7, $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$, so $f(\beta) \in \operatorname{supp} \alpha$. We also know that $f(\alpha) \in \operatorname{supp} \beta$.

Consider another simple roots distribution h on $\Delta^+ \cap w\Delta^-$: $h(\alpha) = f(\beta)$, $h(\beta) = f(\alpha)$, and $h(\gamma) = f(\gamma)$ for all other $\gamma \in \Delta^+ \cap w\Delta^-$. Since $f(\alpha) \in \operatorname{supp} \beta$ and $f(\beta) \in \operatorname{supp} \alpha$, this is really a simple root distribution. Clearly, it also has D-multiplicities $n_1, \ldots, n_r$ of simple roots.

1. Consider the case when h is a β-compatible distribution.

   By Lemma 4.10, f is an α-compatible distribution. By Lemma 6.3, $C_{w,n_1,\ldots,n_r} \geq 2$.

   END Consider the case when h is a β-compatible distribution.

2. Now consider the case that h is not a β-compatible distribution.

   By Lemma 4.9, this means that there exists a root $\gamma \in \Delta^+ \cap w\Delta^-$ such that $\beta \prec_w \gamma$, $(\beta, \gamma) = 1$, $h(\gamma) \in \operatorname{supp} \beta$, and $h(\beta) \in \operatorname{supp} \gamma$.

   $(\beta, \gamma) = 1$, so $\gamma \neq \alpha$ since $(\beta, \alpha) = 0$.

   $\gamma \neq \alpha$, $\gamma \neq \beta$, so $f(\gamma) = h(\gamma) \in \operatorname{supp} \beta$, and $h(\beta) = f(\alpha) \in \operatorname{supp} \gamma$.

   $f(\alpha) \in \operatorname{supp} \gamma$, so γ is a red root, and $(\gamma, \alpha) ne - 1$.

   $(\gamma, \alpha) = 0$ is also impossible since $\beta \prec_w \gamma$, and we would have a contradiciton with the minimality of β with respect to $\prec_w$ in the set of red roots orthogonal to α.

   So, $(\gamma, \alpha) = 1$. Recall that $(\alpha, \beta) = 0$.

   Set $\delta = \alpha - \gamma + \beta$. By Lemma 2.7, $\delta \in \Delta$ and $(\delta, \gamma) = 0$.

   By Lemma 2.7, $\alpha - \delta + \beta \in \Delta$. Lemma 2.7 also says that $(\gamma, \delta) = 0$. It also says that $(\beta, \delta) = 1$, $(\delta, \alpha) = 1$, so $\alpha \neq \delta$.

   Since $w, n_1, \ldots, n_r$ has small essential coordinates, and $n_i > 0$ that the coefficients in front of $f(\alpha) = \alpha_i$ in the decompositions of α and of all red roots into linear combinations of simple roots are all 1. By Lemma 2.8, $\delta \in \Delta^+$, and the coefficient in front of α_i in the decomposition of δ into a linear combination of simple roots is 1. In particular, $f(\alpha) \in \operatorname{supp} \delta$.

   $\beta \prec_w \gamma$, so, by Lemma 2.15, $\delta \in w\Delta^-$. Therefore, $\delta \in \Delta^+ \cap w\Delta^-$.

   Now let us check that $f(\gamma) \in \operatorname{supp} \delta$ or $f(\delta) \in \operatorname{supp} \gamma$.

   Assume the contrary: $f(\gamma) \notin \operatorname{supp} \delta$ and $f(\delta) \notin \operatorname{supp} \gamma$. Recall that $f(\gamma) \in \operatorname{supp} \beta$. Recall also that $(\delta, \beta) = 1$. So, $\beta - \delta \in \Delta$, and either $\beta - \delta \in \Delta^-$, or $\beta - \delta \in \Delta^+$.

But if $\beta - \delta \in \Delta^-$, then $\beta \prec \delta$, so $\operatorname{supp}\beta \subseteq \operatorname{supp}\delta$, and it is impossible to have $f(\gamma) \in \operatorname{supp}\beta$ and $f(\gamma) \notin \operatorname{supp}\delta$, a contradiciton.

So, $\beta - \delta \in \Delta^+$. Then $\alpha \prec \gamma = \beta - \delta + \alpha$, so $\operatorname{supp}\alpha \subseteq \operatorname{supp}\gamma$. By Lemma 6.7, $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$, so $\operatorname{supp}\beta \subseteq \operatorname{supp}\gamma$.

Also, $\beta - \delta \in \Delta^+$, so $\delta \prec \beta$, and $\operatorname{supp}\delta \subseteq \operatorname{supp}\beta$. We know that $f(\delta) \in \operatorname{supp}\delta$, so $f(\delta) \in \operatorname{supp}\beta$. We know that $\operatorname{supp}\beta \subseteq \operatorname{supp}\gamma$, so $f(\delta) \in \operatorname{supp}\gamma$, a contradiciton.

Therefore, $f(\gamma) \in \operatorname{supp}\delta$ or $f(\delta) \in \operatorname{supp}\gamma$.

Let us consider 3 cases:

(a) $f(\gamma) \in \operatorname{supp}\delta$ and $f(\delta) \in \operatorname{supp}\gamma$. Set $g = f$. Then $g(\delta) \in \operatorname{supp}\gamma$, $g(\gamma) \in \operatorname{supp}\delta$.

(b) $f(\gamma) \in \operatorname{supp}\delta$, but $f(\delta) \notin \operatorname{supp}\gamma$. Recall that $f(\alpha) \in \operatorname{supp}\delta$. By Lemma 6.7, $\operatorname{supp}\delta \subseteq \operatorname{supp}\alpha$, so $f(\delta) \in \operatorname{supp}\alpha$.
Set $g(\alpha) = f(\delta)$, $g(\delta) = f(\alpha)$, and $g(\epsilon) = f(\epsilon)$ for all other $\epsilon \in \Delta^+ \cap w\Delta^-$. This is a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots.
Recall also that $f(\alpha) \in \operatorname{supp}\gamma$.
Summarizing, $g(\delta) = f(\alpha) \in \operatorname{supp}\gamma$, $g(\gamma) = f(\gamma) \in \operatorname{supp}\delta$.

(c) $f(\delta) \in \operatorname{supp}\gamma$, but $f(\gamma) \notin \operatorname{supp}\delta$. Similarly to the previous case:
Recall that $f(\alpha) \in \operatorname{supp}\gamma$. By Lemma 6.7, $\operatorname{supp}\gamma \subseteq \operatorname{supp}\alpha$, so $f(\gamma) \in \operatorname{supp}\alpha$.
Set $g(\alpha) = f(\gamma)$, $g(\gamma) = f(\alpha)$, and $g(\epsilon) = f(\epsilon)$ for all other $\epsilon \in \Delta^+ \cap w\Delta^-$. This is a simple root distribution on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots.
Recall also that $f(\alpha) \in \operatorname{supp}\delta$.
Summarizing, $g(\gamma) = f(\alpha) \in \operatorname{supp}\delta$, $g(\delta) = f(\delta) \in \operatorname{supp}\gamma$.

END consider 3 cases.

So, we have constructed a simple root distribution g on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $g(\delta) \in \operatorname{supp}\gamma$, $g(\gamma) \in \operatorname{supp}\delta$.

Recall that $\alpha \neq \delta$, $\alpha \neq \gamma$, and $(\gamma, \delta) = 0$ so the restriction of g to $(\Delta^+ \cap w\Delta^-) \setminus \alpha$ is flexible.

END consider the case that h is not a β-compatible distribution.

□

Lemma 6.20. *Let $w, n_1, \ldots, n_r$ be a configuration of D-multiplicities that has small essential coordiantes.*

If there exists a flexible simple root distribution $f \colon \Delta^+ \cap w\Delta^- \to \Pi$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots,

then $C_{w,n_1,\ldots,n_r} \geq 2$

Proof. The proof is very similar to the proof of Lemma 6.9

Set $w_0 = w$.

We perform the following *antisimple* reflections while we don't say we want to stop. This way we construct a labeled antisimple sorting process prefix. Again, we will denote the current element of W after i reflections by w_i.

Again, we will have several simple root distributions $f_0 = f, f_1, \ldots, f_k$ ($0 \leq k < \ell(w)$) such that when we perform the ith reflection (and it will be the ith reflection in both of the sorting processes we will construct), and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$ (recall that we are doing antisimple reflections, see Lemma 3.11), we assign the label $f_i(\gamma)$ to it. And when we modify our distribution later, i. e. when we define f_j with $j > i$, we don't change its value that was already assigned to a step of the sorting process, i. e. $f_j(\gamma)$ will be the same as $f_i(\gamma)$.

Also, all distributions f_i will have the same D-multiplicities of simple roots as f.

In the end, when we stop after k steps, it will be true that when we performed the ith reflection and this reflection is σ_γ for some $\gamma \in \Delta^+ \cap w\Delta^-$, the label assigned to this reflection was $f_k(\gamma)$.

We will also maintain the following fact: the restriction of f_i onto $\Delta^+ \cap w_i\Delta^-$ ($i \geq 0$) is flexible.

For each $i \in \mathbb{N}$, starting from $i = 1$.

1. If there exist two roots $\gamma, \gamma' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\gamma, w_{i-1}^{-1}\gamma' \in -\Pi$, $f_{i-1}(\gamma) \in \operatorname{supp}\gamma'$, $f_{i-1}(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$ then we say that we WANT TO STOP.
2. Otherwise, if there exist three *different* roots $\alpha, \gamma, \gamma' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\alpha \in -Pi$, $f_{i-1}(\gamma) \in \operatorname{supp}\gamma'$, $f_{i-1}(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$, then:

   Set $f_i = f_{i-1}$

   we say that the ith step of the sorting process prefix will be $\beta_i = \alpha$ with label $f_i(\alpha)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

   $\Delta^+ \cap w_i\Delta^-$ still contains γ and γ', so the restriction of f_i to $\Delta^+ \cap w_i\Delta^-$ is flexible.

   And we CONTINUE with the next step of the sorting process (with the next value of i).
3. Otherwise:

   We know that (we are maintaining the fact that) the restriction of f_{i-1} to $\Delta^+ \cap w_{i-1}\Delta^-$ is flexible. So, there exist $\gamma, \gamma' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $f_{i-1}(\gamma) \in \operatorname{supp}\gamma'$, $f_{i-1}(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$.

   By Lemma 3.10, there exists $\alpha \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\alpha \in -Pi$.

   All three roots α, γ, γ' cannot be different, this would be case 2. But $\gamma \neq \gamma'$ since $(\gamma, \gamma') = 0$. So, $\alpha = \gamma$ or $\alpha = \gamma'$, without loss of generality let us suppose that $\alpha = \gamma$.

   Note that $w_{i-1}^{-1}\gamma' \notin -\Pi$, otherwise this would be case 1.

   Also, we cannot have another root $\alpha' \in \Delta^+ \cap w_{i-1}\Delta^-$, different from $\alpha = \gamma$, such that $w_{i-1}^{-1}\alpha' \in -\Pi$, this would also be case 2. In other words, there exists exaclty one root $\alpha' \in \Delta^+ \cap w_{i-1}\Delta^-$ such that $w_{i-1}^{-1}\alpha' \in -\Pi$, and this root is α.

   Restrict f_{i-1} onto $\Delta^+ \cap w_{i-1}\Delta^-$, and denote the result by g_{i-1}. Temporarily (until the end of this step of the sorting process) denote the D-multiplicities of simple roots in g_{i-1} by $m_1, \ldots, m_r$.

   We are going to apply Lemma 6.19 to w_{i-1}. The only condition we have to check is that the configuration $w_{i-1}, m_1, \ldots, m_r$ has small essential coordinates. But we are doing only antisimple reflections, so $\Delta^+ \cap w_{i-1}\Delta^- \subseteq \Delta^+ \cap w\Delta^-$. Also, $n_j \geq m_j$ by the definition of m_j. So, if for some $\delta \in \Delta^+ \cap w_{i-1}\Delta^-$, the coefficient in front of some α_j in the decomposition of δ into a linear combination of simple roots is at least 2, and $m_j > 0$, then $n_j > 0$, and $\delta \in \Delta^+ \cap w\Delta^-$. But this is impossible since $w, n_1, \ldots, n_r$ is a configuration with small essential coordiantes.

   So, the configuration $w_{i-1}, m_1, \ldots, m_r$ has small essential coordinates, and we can use Lemma 6.19.

   Lemma 6.19 may tell us $C_{w_{i-1}, m_1, \ldots, m_r} \geq 2$. Then by Corollary 6.6, $C_{w, n_1, \ldots, n_r} \geq 2$. Stop everything, we are done.

   Otherwise, Lemma 6.8 gives us a new simple root distribution, which we denote by g_i, on $\Delta^+ \cap w_{i-1}\Delta^-$ such that:

   the D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \ldots, m_r$,

   and the restriction of g_i to $(\Delta^+ \cap w_{i-1}\Delta^-) \setminus \alpha$ is flexible.

   Expand this new distribution g_i to the whole $\Delta^+ \cap w\Delta^-$ using f_{i-1}. In rigorous terms, define the following new distribution f_i on $\Delta^+ \cap w\Delta^-$: $f_i(\delta) = g_i(\delta)$ if $\delta \in \Delta^+ \cap w_{i-1}\Delta^-$, and $f_i(\delta) = f_{i-1}(\delta)$ otherwise.

   The D-multiplicities of simple roots in g_i are the same as the D-multiplicities of simple roots in g_{i-1}, they are $m_1, \ldots, m_r$, so the D-multiplicities of simple roots in f_i are the same as the D-multiplicities of simple roots in f_{i-1}, they are $n_1, \ldots, n_r$.

   Now we again say that the ith step of both sorting processes will be $\beta_i = \alpha$ with label $f_i(\alpha)$, we perform the reflection σ_{β_i}, we set $w_i = \sigma_{\beta_i} w_{i-1}$.

   The restriction of f_i to $\Delta^+ \cap w_i\Delta^-$ is the same as the restriction of g_i to $(\Delta^+ \cap w_{i-1}\Delta^-) \setminus \alpha$, it is flexible.

   And we CONTINUE with the next step of the sorting process (with the next value of i).

END For each $i \in \mathbb{N}$, starting from $i = 1$.

After a certain number (denote it by k) of steps, we will stop. At this point we will have a simple root distribution f_k on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots, a sequence $\beta_1, \ldots, \beta_k$ of elements of $\Delta^+ \cap w\Delta^-$, a sequence $w_0 = w, w_1, \ldots, w_k$ of elements of W such that σ_{β_i} is an antisimple sorting reflection for w_{i-1}, and $w_i = \sigma_{\beta_i} w_i$, and two roots $\gamma, \gamma' \in \Delta^+ \cap w_k\Delta^-$ such that $w_k^{-1}\gamma, w_k^{-1}\gamma' \in -\Pi$, $f_k(\gamma) \in \operatorname{supp}\gamma'$, $f_k(\gamma') \in \operatorname{supp}\gamma$, and $(\gamma, \gamma') = 0$.

Again restrict f_k onto $\Delta^+ \cap w_k\Delta^-$, and denote the result by g_k. We know (we were maintaining the fact that) g_k is flexible. Denote the D-multiplicities of simple roots in g_k by $m_1, \ldots, m_r$.

By Lemma 6.18, $C_{w_k, m_1, \ldots, m_r} \geq 2$. By Corollary 6.6, $C_{w, n_1, \ldots, n_r} \geq 2$. □

Lemma 6.21. *Let $w, n_1, \ldots, n_r$ be an excessive configuration of D-multiplicities.*
If there exist roots $\alpha, \beta \in \Delta^+ \cap w\Delta^-$ such that $(\alpha, \beta) = 0$ and $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$,
then $C_{w, n_1, \ldots, n_r} \geq 2$.

Proof. If the configuration has large essential coordinates, $C_{w, n_1, \ldots, n_r} \geq 2$ by lemma 6.15.

Suppose that the configuration has small essential coordinates. By Lemma 6.11, there exists a simple root $\alpha_i \in \operatorname{supp}\beta$ involved in $w, n_1, \ldots, n_r$.

By Lemma 6.13, $w, n_1, \ldots, n_r$ is a free-first-choice configuration, so there exists a simple root distribution f on $\Delta^+ \cap w\Delta^-$ with D-multiplicities $n_1, \ldots, n_r$ of simple roots such that $f(\alpha) = \alpha_i$.

So, $f(\alpha) \in \operatorname{supp}\beta$. Also, $f(\beta) \in \operatorname{supp}\alpha$ since $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$. So, f is a flexible distribution.

By Lemma 6.20, $C_{w, n_1, \ldots, n_r} \geq 2$. □

References

[1] Humphreys, Lie algebras.