DRAFT OF: REPRESENTATIONS OF p-ADIC GROUPS

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Contents

CHAPTER I

Elementary Analysis and Representation Theory

Fix $\mathbf{F}$ a local, non-archimedean field. That is, $\mathbf{F}$ is a finite extension of $\mathbb{Q}_p$ or $\mathbf{F} = \mathbb{F}_q\{\{t\}\}$. We want to do analysis on $X(\mathbf{F})$ where X is an algebraic variety defined over $\mathbf{F}$.

1. l-Spaces

1.1. Definitions and Lemmas.

Definition 1. (1) An l-space is a topological space which is Hausdorff, locally compact and 0-dimensional (i.e. totally disconnected: any point has a basis of open compact neighborhoods).

(2) An l-group is a Hausdorff topological group such that e (= identity) has a basis of neighborhoods which are open compact groups.

It is easy to see that if G is an algebraic group over $\mathbf{F}$ then $G(\mathbf{F})$ is an l-group.

Definition 2. Let V be a representation of an l-group G. A vector $v \in V$ is *smooth* if its stabilizer in G is open.

We will denote the set of smooth points by $V_{\mathrm{sm}} \subset V$.

Proposition 1. (1) *V_{sm} is a G-invariant subspace of V.*

(2) *If V is a topological representation, then V_{sm} is dense in V.*

Proof. Clear. $\square$

We will study *smooth representations*, that is, representations V such that $V_{\mathrm{sm}} = V$. Here are some easy lemmas.

Lemma 1. *Let X be an l-space.*

(1) *If $Y \subset X$ is locally closed (i.e. the intersection of an open and a closed subset), then Y is an l-space.*

(2) *If $K \subset X$ is compact and $K \subset \bigcup_\alpha U_\alpha$ is an open covering, then there exists* disjoint *open compact $V_i \subset X$, $i = 1 \dots k$ such that $V_i \subset U_\alpha$ for some α and $\bigcup V_i \supset K$.*

LEMMA 2. *Let G be an l-group which is countable at infinity (i.e. G is a countable union of compact sets). Suppose that G acts on an l-space X with a finite number of orbits. Then G has an open orbit $X_0 \subset X$ so that $X_0 \approx G \smallsetminus H$ for some closed subgroup $H \subset G$.*

It is obvious that by applying this lemma to $X \smallsetminus X_i$, we can get a stratification $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ such that $X_i \smallsetminus X_{i-1}$ is an orbit.

EXAMPLE. Let $G = \mathrm{GL}(n, F)$, $B =$ the upper triangular matrices. Then we may set $X = G/B$ and consider the action of B on X (on the left). When $n = 2$, $X = \mathbb{P}^1$ and there are two orbits: a single point and the complement of that point.

1.2. Functions and Distributions. If X is an l-space, let $C^\infty(X)$ be the space of locally constant complex-valued functions on X. Let $S(X) \subset C^\infty(X)$ be the space of locally constant, compactly supported functions on X. $S(X)$ will serve as the "test functions" for our analysis on X. Thus, $S^*(X) =$ the set of functionals on $S(X)$ are called *distributions*. Note that as $S(X)$ has no topology, there is obviously no continuity assumed.

PROPOSITION 2 (EXACT SEQUENCE OF AN OPEN SUBSET). *Let $U \subset X$ be open and set $Z = X \smallsetminus U$. Then*

$$0 \to S(U) \to S(X) \to S(Z) \to 0$$

is exact.

PROOF. For the injection at $S(U)$ just extend functions on U by zero to all of X. For the surjection at $S(Z)$ we must explain how to extend functions from a *closed* subset. Since $f \in S(Z)$ is locally constant and compactly supported, we may assume that Z is compact and has a covering by a finite number of open sets U_α with $f|_{U_\alpha} = c_\alpha$ constant. Let V_i be as in Lemma 1 (2). Then we can extend f by defining $f(x) = c_\alpha$ if $x \in V_i \subset U_\alpha$ and zero otherwise. $\square$

COROLLARY. *The sequence of distributions*

$$0 \to S^*(Z) \to S^*(X) \to S^*(U) \to 0$$

is exact.

It will be important for us to distinguish *compactly supported* distributions. For future reference, we record the definition

DEFINITION 3. If X is an l-space, the *support of a distribution* $\mathcal{E} \in S^*(X)$ is $\operatorname{Supp} \mathcal{E} =$ the smallest closed subset S such that $\mathcal{E}|_{X \smallsetminus S} = 0$.

Let us now consider the algebra structure on $S(X)$. Unless X is compact, $S(X)$ has no identity element; 1 is not compactly supported. However, if $L \subset X$ is open and compact, e_L = characteristic function of L is an idempotent in $S(X)$.

DEFINITION 4. (1) An algebra $\mathcal{H}$ is an *idempotented algebra* if for every finite collection of elements of $\mathcal{H}$, $\{f_i\}$, there exists an idempotent $e \in \mathcal{H}$ such that $ef_i = f_ie$ for all i.
(2) A module M of an idempotented algebra $\mathcal{H}$ is called *non-degenerate* or *unital* if $\mathcal{H}M = M$.

It is clear that $S(X)$ is an idempotented algebra: let L be an open compact set containing the support of the f_i's; then $e = e_L$ works.

As a matter of notation, if $\mathcal{H}$ is an idempotented algebra, we will denote by $\mathcal{M}(\mathcal{H})$ the category of non-degenerate $\mathcal{H}$-modules.

1.3. Sheaves. As ususal, spaces of functions on l-spaces may be viewed as cannonical examples of more general objects: sheaves.

NOTATION: Suppose X is an l-space. Let $\mathrm{Sh}(X)$ be the category of sheaves of $\mathbb{C}$-vector spaces on X. If $\mathcal{F} \in \mathrm{Sh}(X)$, set $S(\mathcal{F})$ = the space of compactly supported global sections of $\mathcal{F}$. For an open subset $U \subset X$, define $S(U, \mathcal{F}) = S(\mathcal{F}|_U)$.

THEOREM 1. *If* $\mathcal{F} \in \mathrm{Sh}(X)$ *then* $S(\mathcal{F})$ *is a non-degenerate* $S(X)$*-module and* $\mathcal{F} \mapsto S(\mathcal{F})$ *gives an equivalence of categories between* $\mathrm{Sh}(X)$ *and* $\mathcal{M}(S(X))$.

PROOF. As we are considering only compactly supported sections, the first statement is clear. For the second statement, we define the inverse map as follows: let M be a non-degenerate $S(X)$-module. Define $\mathcal{F}$ by $S(L, \mathcal{F}) = e_L M$ for compact open subsets $L \subset X$, and $S(U, \mathcal{F}) = \lim_{L \subset U} S(L, \mathcal{F})$ for $U \subset X$ an arbitrary open subset. $\square$

Suppose X is an l-space and $\mathcal{F}$ is a sheaf on X. If $\pi \colon X \to Y$ is a continuous map of l-spaces, let $\pi_!(\mathcal{F})$ be the sheaf on Y defined by

$$\pi_!(\mathcal{F})(W) = \{\xi \in S(\pi^{-1}(W), \mathcal{F}) |\, \text{the map } \operatorname{Supp}(\xi) \to W \text{ is proper}\}$$

for $W \subset Y$ open.

PROPOSITION 3. *Let* $\pi \colon X \to Y$ *be a continuous map of* l*-spaces.*
(1) *The functors* π^* *and* $\pi_!$ *are exact.*
(2) Base Change: *Consider the pull-back diagram*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\tau'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, "\tau"] & Y \end{tikzcd}$$

If $\mathcal{F}$ is a sheaf on X, then $\tau^*(\pi_!\mathcal{F}) = \pi'_!(\tau')^*\mathcal{F}$.

PROOF. Omitted. □

PROPOSITION 4. *Let $j\colon U \to X$ and $i\colon Z \to X$ be the natural embeddings where U is an open subset of X and $Z = X \setminus U$. Let $\mathcal{F}$ be a sheaf on X. Then*

(1) $0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_!(i^*\mathcal{F}) \to 0$
(2) $0 \to S(\mathcal{F}, U) \to S(\mathcal{F}, X) \to S(\mathcal{F}, Z) \to 0$ *and*
(3) $0 \to S^*(\mathcal{F}, Z) \to S^*(\mathcal{F}, X) \to S^*(\mathcal{F}, U) \to 0$

are exact.

PROOF. (1) is a well-known general statement about these functors. For (2) and (3), the proof is the same as the special case $\mathcal{F} = S(X)$ which was given in section 1.2. □

DEFINITION 5. Let G be an l-group acting on an l-space X. Let

$$p\colon G \times X \to X \qquad \text{be the projection, and}$$
$$a\colon G \times X \to X \qquad \text{be the action.}$$

An *equivariant sheaf* on X is a pair $(\mathcal{F}, \rho)$ where $\mathcal{F} \in \mathrm{Sh}(X)$ and ρ is an isomorphism $\rho\colon p^*(\mathcal{F}) \cong a^*(\mathcal{F})$ which is compatible with the group structure on G.

IMPORTANT EXAMPLE. If $X = \{x\}$ is a point, then an element of $\mathrm{Sh}(X)$ is just a vector space V. In this case, $p = a\colon G \to x$ is trivial so $a^*(V) = p^*(V) = \{$locally constant functions on G with values in V $\}$. If V and ρ define an *equivariant* sheaf on $X = \{x\}$ and $g \in G$, then by considering stalks at g, we get a map $\rho(g)\colon V \to V$. This clearly defines a representation of G on V.

FACT. *This representation is smooth.*

PROOF. Consider the stalk at the identity. As G is totally disconnected, each germ may be represented by the constant section on a sufficiently small open neighborhood which we can take to be an open compact subgroup. It follows that each $v \in V$ has an open stabilizer. □

COROLLARY. *There is a bijective correspondence between equivariant sheaves on a point and smooth representations of G.*

HISTORICAL NOTE/GEOMETRIC INTUITION. Bernstein and Zelevinsky originally gave the following equivalent defintion for $\mathrm{Sh}(X)$.

DEFINITION 6. An l-sheaf on X is

(1) A family of vector spaces $\mathcal{F}_x$, $x \in X$.
(2) A family of sections, called *regular sections*, $\phi\colon x \to v_x \in \mathcal{F}_x$.

such that

i: A section φ which is locally regular is regular.
ii: any vector $v_x \in \mathcal{F}_x$ extends to a regular section.
iii: $\varphi(x) = 0$ implies $\varphi = 0$ in a neighborhood of x.

2. The Hecke Algebra

We are interested in studying $\mathcal{M}(G)$ = the category of smooth representations of an l-group G.

2.1. The Hecke Algebra. Recall that, if X is an l-space, the support of a distribution $\mathcal{E} \in S^*(X)$ is the smallest closed subset $S = \operatorname{Supp}\mathcal{E}$ so that $\mathcal{E}|_{X \smallsetminus S} = 0$. Let us consider the case $\operatorname{Supp}\mathcal{E} = L$, a compact open set. Then $\mathcal{E}$ defines a functional on the space of all locally constant functions, $C^\infty(X)$, by

$$< \mathcal{E}, f > \stackrel{\text{def}}{=} < \mathcal{E}, e_L f > .$$

Moreover, it is clear that, for $X = G$ an l-group, the set $S^*(G)_c$ of compactly supported distributions is an algebra under convolution, denoted $\mathcal{E} * \mathcal{E}'$. Let G act on this algebra by left translation. A distribution on an l-group is called *locally constant* if it is invariant by some open subgroup of G.

Definition 7. The algebra of locally constant, compactly supported distributions on an l-group G, $\mathcal{H}(G) \subset S^*(G)_c$ is called the *Hecke Algebra*.

If Γ is a compact subgroup, then normal Haar measure on Γ, e_Γ, is in $S^*(G)_c$; if Γ is open and compact, then $e_\Gamma \in \mathcal{H}(G)$.[1] Moreover, if $g \in G$, the delta distribution at g, $\mathcal{E}_g \in S^*(G)_c$ (but not in $\mathcal{H}(G)$; it is not locally constant). These satisfy the relations $e_\Gamma * e_\Gamma = e_\Gamma$, $e_\Gamma * \mathcal{E}_g = \mathcal{E}_g * e_\Gamma = e_\Gamma$ if $g \in \Gamma$.

The various spaces of functions and distributions on an l-group G that we have defined are summarized in Table 1. Note that all of them except $\mathcal{H}(G)$ make sense for an arbitrary l-space.

Proposition 5. (1) *Multiplication by Haar measure gives an isomorphism* $S(G) \to \mathcal{H}(G)$.
(2) *Any* $h \in \mathcal{H}(G)$ *is locally constant with respect to right translation.*

This proposition follows from the fact that compact open subgroups form a basis of neighborhoods of the identity and the following obvious but important lemma.

[1] We have previously used the notation e_L for the characteristic function of L. Our use of e_Γ for Haar measure reflects the usual identification of functions and distributions on a compact group.

Functions		Distributions	
		$S^*(G)$	All – dual of $S(G)$
		$\cup$	
Locally Constant	$C^\infty(G)$	$S^*(G)_c$	Compactly Supported
	$\cup$	$\cup$	
Locally Constant Compactly Supported	$S(G)$	$\mathcal{H}(G)$	Compactly Supported Locally Constant
		$\parallel$	
		Hecke Algebra	

TABLE I.1. Spaces asssociated with an l-group G.

LEMMA 3. *Suppose K is a compact open subgroup of G and h is a K-invariant distribution with compact support. Then there exist $g_1, \ldots, g_k \in G$ and $a_1, \ldots, a_k \in \mathbb{C}$ such that*

$$h = \sum_{i=1}^{k} a_i(e_K * \mathcal{E}_{g_i}).$$

The point is that K-invariant distributions are supported on translates of K.

Next, we turn to the most important aspect of the Hecke algebra: its relation to the representation theory of G. If (π, V) is a smooth representation of G, we can give V the structure of an $\mathcal{H}(G)$-module (in fact, of an $S^*(G)_c$-module) as follows. For fixed v, $\pi(g)v$ may be considered as a locally constant function on G with values in V, and thus as an element of $C^\infty(G) \otimes V$. Therefore, it makes sense to define

$$\pi(\mathcal{E})v = < \mathcal{E}, \pi(g)v >$$

for $\mathcal{E} \in \mathcal{H}(G)$. We will sometimes write $\mathcal{E}v$ for $\pi(\mathcal{E})v$.

THEOREM 2. *Let G be an l-group.*

(1) *$\mathcal{H}(G)$ is an idempotented algebra.*
(2) *If V is a smooth G-module, then the corresponding $\mathcal{H}(G)$-module is non-degenerate.*
(3) *This gives an equivalence of categories*

$$\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H}(G))$$

between smooth representations of G and non-degenerate $\mathcal{H}(G)$-modules.

PROOF. (1) follows from proposition **??**, but it is easy to give a direct proof: Clearly, if K is a compact open subgroup of G, e_K is an idempotent in $\mathcal{H}(G)$. In fact, e_K is the unit for the algebra $\mathcal{H}_K = e_K\mathcal{H}(G)e_K$. As $\mathcal{H}(G) = \bigcup \mathcal{H}_K$, this proves part (1) of the proposition. For (2), suppose $v \in V$ is smooth. Then it is invariant by some K; hence $e_K v = v$. For (3) we need a lemma.

LEMMA 4. *Let $\mathcal{H}$ be any idempotented algebra, $a\colon \mathcal{H} \to \mathcal{H}$ an operator commuting with the right action of $\mathcal{H}$. Then, for any non-degenerate $\mathcal{H}$-module M, there exists an operator $A\colon M \to M$ such that for all morphisms $\varphi\colon M \to N$ there is a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ M & \xrightarrow{\varphi} & N \end{array}$$

and furthermore, for $h \in \mathcal{H}$, $A \circ h = a(h)$ as operators on M.

PROOF. Since M is non-degenerate, each element of M has the form hm for some $h \in \mathcal{H}$. Thus, we may define $A(hm) = a(h)m$. □

To complete the proof of the proposition we must show, given an $\mathcal{H}(G)$-module M, how to define a G-module. First observe that if $\mathcal{E} \in S^*(G)_c$, then $h \mapsto \mathcal{E} * h$ is an operator on $\mathcal{H}(G)$ commuting with convolution on the right. By the lemma, this extends to an operator $\mathcal{E}\colon M \to M$. Specializing $\mathcal{E} = \mathcal{E}_g$, this gives M the structure of a G-module. □

2.2. Applications. Let $\mathcal{M}$ be an abelian category, P an object in M (we will write $P \in \mathrm{Ob}(\mathcal{M})$.) Recall that P is *projective* if the functor

$$\mathcal{M} \to \text{Abelain Groups}$$

given by

$$X \mapsto \mathrm{Hom}(P, X)$$

is exact.

THEOREM 3. *If $\mathcal{H}$ is an idempotented algebra, then the category $\mathcal{M}(\mathcal{H})$ has enough projectives.*

PROOF. Let $e \in \mathcal{H}$ be any idempotent. Consider the $\mathcal{H}$-module $P_e = \mathcal{H}e$. This is projective since $\mathrm{Hom}_{\mathcal{H}}(P_e, X) = eX$ is clearly exact. Note that the direct sum of any collection of the P_e is also projective.

If $M \in \mathrm{Ob}\,\mathcal{M}(\mathcal{H})$ and $\xi \in M$, then it follows from non-degeneracy that there exists an idempotent e so that $e\xi = \xi$. Hence, ξ is in the image of the map $P_e \to M$ given by $he \mapsto h\xi$. Taking the direct sum over all $\xi \in M$ of the associated P_e, we see that M is a quotient of a projective object. □

In order to prove that $\mathcal{M}(\mathcal{H})$ has enough *injectives*, we would like to just take duals. However, it is easy to see that the dual of a smooth representation may not be smooth. Instead, we use

Definition 8. If (π, V) is a smooth representation, the *contragredient representation* $(\tilde{\pi}, \tilde{V})$ is given by $\tilde{V} =$ the smooth part of V^* and $\tilde{\pi} = \pi^*(g^{-1})|_{\tilde{V}}$.

Proposition 6. (1) *For all compact open subgroups $K \subset G$, $\tilde{\pi}(e_K)\tilde{V} = (\pi(e_K)V)^*$.*
(2) $\mathrm{Hom}_G(V, \tilde{W}) = \mathrm{Hom}_G(W, \tilde{V})$
(3) $V \hookrightarrow \tilde{\tilde{V}}$

Proof. (1)Let V^K be the vectors fixed by K. Then $\pi(e_K)V = V^K$ so (1) reduces to $\tilde{V}^K = (V^K)^*$, which is obvious from the definition of smooth. For (2),

$$\begin{aligned} \mathrm{Hom}_G(V, \tilde{W}) &= \mathrm{Hom}_G(V, W^*) \\ &= \mathrm{Hom}_G(V \otimes W, \mathbb{C}) \\ &= \mathrm{Hom}_G(W, V^*) \\ &= \mathrm{Hom}_G(W, \tilde{V}) \end{aligned}$$

where the first and last equalities follow since the image of a smooth module is always smooth. (3) Follows from (1). □

Lemma 5. *If P is a projective object, then $\tilde{P}$ is an injective object.*

Proof. We must show that $X \mapsto \mathrm{Hom}(X, \tilde{P})$ is exact. But $\mathrm{Hom}(X, \tilde{P}) = \mathrm{Hom}(P, \tilde{X})$ by the proposition, so this is clear. □

Definition 9. A smooth representation (π, V) of G is called *admissible* if for every open compact subgroup K, the space V^K is finite dimensional.

It is straightforward to prove that

Proposition 7. *V is admissible if and only if $V \to \tilde{\tilde{V}}$ is an ismorphism.*

In Chapter 2 we will prove that every irreducible representation is admissible. *This is a non-trivial fact.* Using it, we can show

Theorem 4. *$\mathcal{M}(\mathcal{H})$ has enough injectives.*

Proof. Fix X. We may assume that X is irreducible. The proposition implies that $\tilde{X}$ is also irreducible. As we have enough projectives, there is an epimorphism $P \twoheadrightarrow \tilde{X}$. Since $\tilde{X}$ is irreducible, the corresponding map $X \cong \tilde{\tilde{X}} \to \tilde{P}$ is injective. This proves the theorem since by lemma 5 $\tilde{P}$ is injective. □

3. Some Functors

The way to make an advance in representation theory is to find a way to construct representations. Practically our only tool is the induction functor which we will discuss after reviewing adjoint functors. Finally, we discuss the Jacquet functor which will be very important for us.

3.1. Adjoint Functors. In this section, we review some standard facts about adjoint functors. All functors are covariant. Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories, $\mathfrak{F}\colon \mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{G}\colon \mathfrak{B} \to \mathfrak{A}$ functors. Recall that $\mathfrak{G}$ is *left adjoint* to $\mathfrak{F}$ if there is an equivalence of functors

$$\mathrm{Hom}_{\mathfrak{B}}(-, \mathfrak{F}-) \cong \mathrm{Hom}_{\mathfrak{A}}(\mathfrak{G}-, -).$$

In this case, we also say the $\mathfrak{F}$ is right adjoint to $\mathfrak{G}$. If it exists, the left (right) adjoint is unique.

Associated to a pair of adjoint functors, as above, and objects $X \in \mathrm{Ob}\,\mathfrak{A}$ and $Y \in \mathrm{Ob}\,\mathfrak{B}$, we cannonically associate two *adjunction* morphisms:

$$\alpha\colon \mathfrak{G}\mathfrak{F}X \to X$$
$$\beta\colon Y \to \mathfrak{F}\mathfrak{G}Y.$$

The morphism α is the image of the identity in $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{F}X, \mathfrak{F}X)$ under the given equivalence, and similarly for β.

It is obvious that the adjunction maps satisfy the condition that the compositions

$$\mathfrak{F}X \xrightarrow{\beta_{\mathfrak{F}X}} \mathfrak{F}\mathfrak{G}\mathfrak{F}X \xrightarrow{\mathfrak{F}\alpha} \mathfrak{F}X$$
$$\mathfrak{G}Y \xrightarrow{\mathfrak{G}\beta} \mathfrak{G}\mathfrak{F}\mathfrak{G}Y \xrightarrow{\alpha_{\mathfrak{G}Y}} \mathfrak{G}Y$$

are canonically the identity. Conversly, given maps which are functorial in X and Y and satisfy these conditions we can recover the adjointness of $\mathfrak{F}$ and $\mathfrak{G}$.

Proposition 8. *Suppose that $\mathfrak{F}\colon \mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{G}\colon \mathfrak{B} \to \mathfrak{A}$ are functors between* abelian *categories. Suppose $\mathfrak{G}$ is left adjoint to $\mathfrak{F}$. Then*

(1) *$\mathfrak{G}$ is right exact and $\mathfrak{F}$ is left exact.*
(2) *If $\mathfrak{G}$ is exact then $\mathfrak{F}$ maps injective objects to injective ones.*
(3) *If $\mathfrak{F}$ is exact then $\mathfrak{G}$ maps projective objects to projective ones. The converse also holds in the case that $\mathfrak{B}$ has enough projectives.*

The next theorem is known as the *adjoint functor theorem.*

Theorem 5. *Suppose that $\mathfrak{A}$ is a "reasonably small" abelian category and that $\mathfrak{F}\colon \mathfrak{A} \to \mathfrak{B}$ is a functor. Then $\mathfrak{F}$ has a left adjoint if and only if $\mathfrak{F}$ preserves limits.*

We will not pursue precisely what "reasonably small" means. Suffice it to say that we will not have to worry about this condition.

3.2. Induction. If H is a closed subgroup of G, we may restrict G-modules to H. This gives a functor $\mathrm{Res} = \mathrm{Res}_H^G\colon \mathcal{M}(G) \to \mathcal{M}(H)$. As usual, there is a version of Frobenius reciprocity. We omit the (standard) proof.

CLAIM. Res *has a right adjoint:* $\mathrm{Ind} = \mathrm{Ind}_H^G\colon \mathcal{M}(H) \to \mathcal{M}(G)$, *defined as follows: Let* (ρ, V) *be a smooth* H*-module. Consider* $L(V) = \{f\colon G \to V | f(hg) = \rho(h)f(g)\}$ *with the action* $\pi(g)f(x) = f(xg)$. *The induced module is the smooth part of* $L(V)$.

It is convenient to have another model for Ind. Recall that an H-module is equivalent to an H-equivariant sheaf on a point (see section 1.3). Let $Y = H\backslash G$ be the homogeneous space. It is not hard to see that we have an equivalence of categories between H-equivariant sheaves on a point and G-equivariant sheaves on Y. Thus, to define a map from H-modules to G-modules, it is enough to define a map from G-equivariant sheaves on Y to G-equivariant sheaves on a point.

FACT. *If* $p\colon Y \to \{point\}$ *is the trivial map, and* $\mathcal{F}$ *is a* G *equivariant sheaf on* Y, *then* $p_!\mathcal{F}$ *is a* G*-equivariant sheaf on the point.*

PROOF. Use Base Change from section 1.3. $\square$

REMARKS. 1. This claim would be false if we used p_* instead of $p_!$. 2. It can be checked that this functor is equivalent to Ind.

There is another functor $\mathrm{ind}\colon \mathcal{M}(H) \to \mathcal{M}(G)$ given by

$$\mathrm{ind}(V) = \{f \in L(V) | f \text{ has compact support modulo } H\}.$$

PROPOSITION 9. *These functors have the following properties.*

1. $\mathrm{ind}_H^G \subset \mathrm{Ind}_H^G$
2. *Both are exact.*
3. *If* $H\backslash G$ *is compact,* $\mathrm{Ind} = \mathrm{ind}$.
4. *If* $H\backslash G$ *is compact, induction maps admissible representations to admissible representations.*

PROOF. (1) and (3) are obvious. For (2), use the second description of Ind and recall from Proposition 3 that $p_!$ is exact. The result for ind follows from the result for Ind. (4) Let V be an admissible representation of H and fix $K \subset G$ a compact open subgroup. Let $\{Hg_iK\}$ be a system of coset representatives for $H\backslash G/K$. By our assumption, this is a finite set. It is clear that an element, f, of $L(V)^K$ is determined by its values on the g_i. Moreover, the image of g_i under f must lie in the subset of V fixed by $H \cap g_iKg_i^{-1}$ which is finite dimensional since we are assuming that V is admissible. Therefore, there can be only finitely many linearly independent such f. $\square$

3.3. Jacquet Functor. If G is any group, let $\mathbb{C}_G$ be the trivial representation. When G is a finite group, it is often usefull to consider the *space of invariants* $= V^G = \mathrm{Hom}_G(\mathbb{C}_G, V)$. It turns out that for l-groups this notion is almost totally useless. However, we often use the space of *coinvariants*, $V_G = V/V(G)$ where $V(G)$ is the subspace spanned by $\pi(g)v - v$. It is easy to see that this is equivalent to $J_G(V)$ where $J_G\colon \mathcal{M}(G) \to$vector spaces is the *Jacquet Functor* defined by $J_G(V) = \mathbb{C}_G \otimes_G V$.

PROPOSITION 10. *The Jacquet functor J has the following properties.*

(1) *J is right exact.*
(2) *If G is compact then J is exact.*
(3) *If G is the union of an increasing family of compact groups, then J_G is exact.*

PROOF. (1) is obvious. (2) We have an exact sequence

$$0 \to V^G \to V \to V_G \to 0.$$

G compact implies that $e_G V = V^G = V_G$. But V^G is clearly left exact. (3) If $G = \bigcup U_i$ then $J_G(V) = \lim J_{U_i}(V)$. But the direct image of exact functors is exact so done by (2). $\square$

4. Irreducible Representations

A representation is *irreducible* if it is algebraically irreducible, that is if contains no invariant subspaces.

EXAMPLE 1. If G is a compact group then every smooth G-module M is completely reducible, that is $M \cong \bigoplus W_\alpha$ where the W_α are irreducible. Thus, the representation theory is entirely controlled by the irreducibles and in a simple way.

EXAMPLE 2. $G = F^*$ (This is "almost" compact.) Let π be a generator for the maximal ideal in the ring of integers $\mathcal{O} \subset F$. Then

$$F^* \cong \mathbb{Z}\pi \oplus \mathcal{O}^*.$$

Here $\mathcal{O}^*$ is compact and $\mathcal{M}(\mathbb{Z}) = \mathcal{M}(\mathbb{C}[t, t^{-1}])$, which is equivalent to the category of sheaves on $\mathbb{C}^*$. Thus,

$$\mathcal{M}(G) \cong \prod_{\substack{\text{irred reps} \\ \text{of } \mathcal{O}^*}} \mathcal{M}(\mathbb{Z}) = \prod_{\substack{\text{irred reps} \\ \text{of } \mathcal{O}^*}} \mathcal{M}(\mathbb{C}[t, t^{-1}]).$$

We see that the structure of the representations is half discrete and half continuous. Specifically, it is a discrete sum of the category of sheaves on some space. This example will be a model for our study. In this section we establish some of the machinery that we will need.

4.1. Decomposing Categories. In this section we discuss categories which satisfy enough conditions so that our definitions make sense. We will only be interested in categories of modules as we have been discussing.

DEFINITION 10. Let $\mathcal{M}$ be a category, $M \in \operatorname{Ob}\mathcal{M}$ an object. Then M is *irreducible* if it has no non-trivial subobjects.

(1) Denote by $\operatorname{Irr}\mathcal{M}$ the set of *equivalence classes* of irreducible objects.
(2) If $M \in \mathcal{M}$, then the *Jordan-Holder content* of M, $\mathrm{JH}(M)$, is the subset of $\operatorname{Irr}\mathcal{M}$ consisting of all irreducible subquotients of M.

REMARK. If G is an l-group, we write $\operatorname{Irr} G$ instead of $\operatorname{Irr}\mathcal{M}(G)$ for the set of equivalence classes of irreducible representations of G.

LEMMA 6. *Let $N, M \in \operatorname{Ob}\mathcal{M}$.*

(1) *If N is a subquotient of M, then $\mathrm{JH}(N) \subset \mathrm{JH}(M)$.*
(2) *$\mathrm{JH}(M) = \emptyset$ if and only if $M = 0$.*
(3) *If $M = \sum_\alpha M_\alpha$ then $\mathrm{JH}(M) = \bigcup_\alpha \mathrm{JH}(M_\alpha)$.*

PROOF. This is straightforward. Note that (2) uses Zorn's lemma. $\square$

If $\mathcal{M}$ is a category, then $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ means that for any object $M \in \mathcal{M}$, there exist subobjects $M_i \in \mathcal{M}_i$ so that $M = M_1 \oplus M_2$. Of course, if $V \in \operatorname{Irr}\mathcal{M}$, then this implies that either $V \in \mathcal{M}_1$ or $V \in \mathcal{M}_2$. This leads to a decomposition

$$\operatorname{Irr}\mathcal{M} = \operatorname{Irr}\mathcal{M}_1 \coprod \operatorname{Irr}\mathcal{M}_2.$$

(Here $\coprod$ means 'disjoint union'.) Conversly, we will see that such a decomposition on the level of sets completely determines the decomposition on the level of categories.

Let $S \subset \operatorname{Irr}\mathcal{M}$. Denote by $\mathcal{M}(S)$ the full subcategory of $\mathcal{M}$ consisting of objects M with $\mathrm{JH}(M) \subset S$.

CLAIM. *If $S, S' \subset \operatorname{Irr}\mathcal{M}$ do not intersect, then the categories $\mathcal{M}(S)$ and $\mathcal{M}(S')$ are orthogonal, i.e. $M \in \mathcal{M}(S)$ and $M' \in \mathcal{M}(S')$ imply $\operatorname{Hom}(M, M') = 0$.*

PROOF. Suppose $\alpha \in \operatorname{Hom}(M, M')$. Set $N = \alpha(M)$. So, $\mathrm{JH}(N) \subset \mathrm{JH}(M) \subset S$ and also $\mathrm{JH}(N) \subset \mathrm{JH}(M') \subset S'$. But $S \cap S' = \emptyset$ so by the last lemma, $N = 0$. $\square$

If $S \subset \operatorname{Irr}\mathcal{M}$, $M \in \mathcal{M}$, we will denote by $M(S)$ the union of all subobjects of M which lie in $\mathcal{M}(S)$. By the lemma, this is the maximal submodule with Jordan-Holder content lying in S.

Definition 11. Let $S \subset \operatorname{Irr}\mathcal{M}$ and $S' = \operatorname{Irr}\mathcal{M} \setminus S$. S is called *splitting* if $\mathcal{M} = \mathcal{M}(S) \times \mathcal{M}(S')$. That is, if $M = M(S) \oplus M(S')$ for each $M \in \mathcal{M}$. In this case we say that S *splits* M.

Claim. *A decomposition of categories $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ is equivalent to a decomposition of sets* $\operatorname{Irr}\mathcal{M} = S \coprod S'$ *where* S is a splitting subset.

Proof. Obvious. □

4.2. Lemmas on Irreducible Representations.

Lemma 7. *Let (ρ, W) be a representation of G. Then (ρ, W) is irreducible if and only if for every open compact subgroup $K \subset G$, $(\rho|_{\mathcal{H}_K}, W^K)$ is either* 0 *or an irreducible representation of $\mathcal{H}_K$. Furthermore, every irreducible representation of $\mathcal{H}_K$ appears in this way for some irreducible ρ. Also, this ρ is unique. In other words, two irreducible representations of G, (ρ_1, W_1) and (ρ_2, W_2), are equivalent if and only if there is some compact open K such that $(\rho_1|_{\mathcal{H}_K}, W_1^K)$ and $(\rho_2|_{\mathcal{H}_K}, W_2^K)$ are non-zero and equivalent.*

Proof. For the first statement, suppose ρ is irreducible. Let w_1, $w_2 \in W$ and denote by $\tilde{w}_1$, $\tilde{w}_2$ their images in W^K. There is an $h \in \mathcal{H}$ such that $hw_1 = w_2$. Then, $e_K h e_K \tilde{w}_1 = \tilde{w}_2$. Thus, W^K is irreducible. The converse is clear: if $Y \subset W$ is an invariant subspace, then $\rho|_{\mathcal{H}_K}$ is reducible whenever $Y^K \neq 0$.

For the next statement, let $V \in \mathcal{M}(\mathcal{H}_K)$ be irreducible. Set $U = \mathcal{H} \otimes_{\mathcal{H}_K} V \in \mathcal{M}(\mathcal{H})$. It is obvious that $V \in \mathrm{JH}(U^K)$. Moreover, taking K-fixed vectors gives an onto map $\mathrm{JH}(U) \to \mathrm{JH}(U^K)$.

For the final claim, assume that $\alpha\colon W_1^K \xrightarrow{\sim} W_2^K$ is an $\mathcal{H}_K$-isomorphism. Then α lifts to an $\mathcal{H}$-isomorphism $W_1' \to W_2'$ where $W_i' = \mathcal{H}W_i^K \subset W_i$, $i = 1, 2$. But W_i irreducible implies $W_i' = W_i$. The converse is trivial. □

As would be expected, there is a version of Schur's lemma for smooth representations of l-groups.

Schur's Lemma. *Suppose G is countable at infinity. Let (ρ, V) be an irreducible representation of G. Then* $\operatorname{End}_G V = \mathbb{C}$.

Proof. Since V is irreducible, $\mathcal{A} = \operatorname{End} V$ is a skew-field. Moreover, $\mathcal{A}$ has countable dimension over $\mathbb{C}$. Indeed, by irreducibility, it is enough to show that the dimension of V is countable. If $\xi \in V$, then V is spanned by the $\rho(g)\xi$ for $g \in G$. But since G is countable at infinity and the function $g \mapsto \rho(g)\xi$ is locally constant (smoothness), we can find a countable spanning set.

Thus we are reduced to proving

Lemma 8. *If $\mathcal{A}$ is a skew-field of countable dimension over $\mathbb{C}$, then $\mathcal{A} = \mathbb{C}$.*

PROOF. Let $a \in \mathcal{A}$. Suppose $a - \lambda \neq 0$ for any $\lambda \in \mathbb{C}$. Since $\mathcal{A}$ has only countable dimension, the elements $(a-\lambda)^{-1}$ cannot be linearly independent. Thus, there are $c_i \in \mathbb{C}$ so that

$$\sum_{i=1}^{k} c_i (a - \lambda_i)^{-1} = 0.$$

Multiplying through by $\prod(a - \lambda_i)$, we get a non-zero polynomial over $\mathbb{C}$ with a as a root. Factoring this polynomial, we see that there are $\mu_j \in \mathbb{C}$ so that

$$\prod_j (a - \mu_j) = 0.$$

Now one of these factors must be zero because otherwise $\mathcal{A}$ would have zero divisors. Hence $a \in \mathbb{C}$. $\square$

$\square$

REMARKS. 1. We will eventually show that the irreducible represetations of any reductive p-adic group are admissible. Then we will be able to stop worrying about technical conditions like "countable dimension". 2. To see how Schur's lemma can fail (really the only way that it can fail), let $\mathcal{K}$ be a field properly containing another field k. Consider the discrete group $G = \mathcal{K}^*$ and its representation in the k-vector space $\mathcal{K}$. This representation is obviously irreducible, but Schur's lemma fails: the intertwining operators are $\mathcal{K}$, strictly bigger than k.

We do, however, have the following extension of Schur's lemma. The key point is that we need some sort of finite-type control to get Schur's lemma. (For a proof see Dixmier *Algebras Envelopantes.*)

QUILLEN'S LEMMA. *Let $\mathcal{K}$ be an algebraically closed field of characteristic 0, $\mathfrak{g}$ a finite dimensional lie algebra over $\mathcal{K}$ with $U = U(\mathfrak{g})$ its universal enveloping algebra. Then for any irreducible U-module M, $End_U(M) \cong \mathcal{K}$.*

Also, if G is a reductive p-adic group, $\mathcal{H}(G, \mathcal{K})$ the Hecke algebra with coefficients in an algebraically closed field of characteristic 0, then Schur's lemma holds. Again, in this case we have some finite-type control.

The next lemma is the statement that our Hecke algebra resembles a semisimple algebra in a crucial sense.

SEPARATION LEMMA. *Suppose that G is countable at infinity. Let $h \in \mathcal{H}(G)$, $h \neq 0$. Then there exists an irreducible representation ρ such that $\rho(h) \neq 0$.*

PROOF. Consider the map $\mathrm{inv}\colon G \to G$ given by $\mathrm{inv}\colon g \mapsto g^{-1}$. This induces a map $\mathrm{inv}\colon \mathcal{H}(G) \to \mathcal{H}(G)$. Set $h^+ = \mathrm{inv}(h)$, and $u = hh^+$.

Let us suppose for the moment that G is unimodular, so that $h = \varphi\mu_G$ for some (two sided) Haar measure μ_G and some $\varphi \in S(G)$. In this case, $h^+ = \varphi^+\mu_G$ where $\varphi^+(g) = \overline{\varphi(g^{-1})}$. Thus, $u = hh^+ = \psi\mu_G$ where

$$\psi(g) = \int_{r\in G} \varphi(r)\overline{\varphi(gr)}dr.$$

Setting $g = 1$, it is obvious that this is not the zero function. What we have shown is that $h \neq 0$ implies $u \neq 0$. (With a bit more care, one can prove this without the assumption that G is unimodular. (Exercise.))

It is enough to find a representation ρ so that $\rho(u) = \rho(h)\rho(h^+) \neq 0$. Note that $u^+ = u$. Thus, from the last paragraph it follows that $u^2 = uu^+ = (hh^+)(hh^+)^+ \neq 0$ and more generally that $u^n \neq 0$. So we are reduced to proving that if $\mathcal{A}$ is a countable-dimensional algebra, $a \in \mathcal{A}$ a non-nilpotent element, then there is a representation of $\mathcal{A}$ that does not kill a. This is lemma 9 below.

Lemma 9. *Let $\mathcal{A}$ be an algebra of countable dimension over $\mathbb{C}$ with unit. Let $a \in \mathcal{A}$ be not nilpotent. Then there exists a simple $\mathcal{A}$-module M such that $a|_M \neq 0$.*

Proof. The proof is similar to that of Schur's lemma. First we establish:

Claim. *There exists $\lambda \in \mathbb{C} \setminus 0$ such that $a - \lambda$ is not invertible in $\mathcal{A}$.*

Proof. If $a \in \mathbb{C}$, this is trivial. Otherwise, by countable-dimensionality, the elements the $(a - \mu)^{-1}$ are linearly dependent. Thus there exists $c_i \in \mathbb{C}$ so that

$$\sum_{i=1}^{k} c_i(a - \mu_i)^{-1} = 0.$$

Multiplying through by $\prod(a - \mu_i)$, we get a non-zero polynomial over $\mathbb{C}$ with a as a root. Thus, there are $\lambda_j \in \mathbb{C} \setminus 0$ and integers $n_j \geq 0$ so that

$$a^{n_0} \prod_j (a - \lambda_j)^{n_j} = 0.$$

As a is not nilpotent, the $(a - \lambda_j)$ are zero divisors, and hence not invertible. □

By the claim, we may suppose that $a - \lambda$ is not invertible. Let M be an irreducible quotient of $\mathcal{A}/(a - \lambda)\mathcal{A}$ which we may take to be non-trivial. Then $(a - \lambda)1 = 0$ in M and so $a1 = \lambda 1 \neq 0$. Hence, a acts non-trivially on M. □

This completes the proof of the Separation Lemma. □

5. Compact Representations

We have two general goals: First, to show that any irreducible representation is admissible. Second, guided by the examples at the beginning of the last section, we wish to show that the category of all representations can be decomposed into managable pieces. In this section, we realize these goals for a special type of representation.

5.1. Definition and Properties.

DEFINITION 12. Let (π, V) be a smooth representation of G. We say that π is *compact* if for every $\xi \in V$ and every open compact subgroup $K \subset G$, the function $\mathcal{D}_{\xi,K} \colon g \mapsto \pi(e_K)\pi(g^{-1})\xi$ has compact support.

REMARKS. 1. The idea here is that compact representations are those which behave like representations of compact groups. In particular, we will see that compact representations are completely reducible 2. Compact representations are sometimes called *finite* representations. 3. It is obvious that if π is compact then so is any subquotient of π.

N.B. Although the definition makes sense for any G, we will use compact representations only in the case G unimodular. (Indeed, it would be reasonable to hypothesise that if G has a compact representation then it must be unimodular.) For this reason we will assume G unimodular when usefull.

If $\xi \in V$, $\tilde{\xi} \in \tilde{V}$ then the function $m_{\tilde{\xi},\xi}(g) =< \tilde{\xi}, \pi(g^{-1})\xi >$ is called a *matrix coefficient*. We may now state an equivalent definition of compact representations:

THEOREM 6. *Every compact representation has compactly supported matrix coefficients. Conversly, if all matrix coefficients of π are compactly supported, then π is compact.*

PROOF. Suppose that $K \subset G$ is a compact open subgroup which stabilizes $\tilde{\xi} \in \tilde{V}$. Then the support of $m_{\tilde{\xi},\xi}$ is contained in the support of $\mathcal{D}_{\xi,K}$. Thus, compact representations have compactly supported matrix coefficients.

For the converse, our strategy is to find a finite number of $\tilde{\xi}_i \in \tilde{V}^K$, $i = 1, \dots, k$, so that $\operatorname{Supp} \mathcal{D}_{\xi,K} \subset \operatorname{Supp} \bigcup_{i=1}^k m_{\tilde{\xi}_i,\xi}$. It is clear that if $\nu \neq 0$ is in the image of $\mathcal{D}_{\xi,K}$, then there is a $\tilde{\xi} \in \tilde{V}^K$ so that $\langle \tilde{\xi}, \nu \rangle \neq 0$. Hence, it is enough to show that the image of $\mathcal{D}_{\xi,K}$ is finite dimensional. If this were false, there would be a sequence of group elements, $g_1, g_2, \dots$, so that the $\nu_i = \mathcal{D}_{\xi,K}(g_i)$ are linearly independent. Observe that the $\{g_i\}$ is not contained in any compact set. Define the functional $\tilde{\nu} \in \tilde{V}^K$ by $\langle \tilde{\nu}, \nu_i \rangle = 1$ and extend by zero. Then $\{g_i\} \subset \operatorname{Supp} m_{\tilde{\nu},\xi}$ which contradicts our assumption that all matrix coefficients are compactly supported. $\square$

PROPOSITION 11. *Any finitely generated compact representation is admissible.*

PROOF. Let (π, V) be such a representation. If V is generated by $\xi_1, \ldots, \xi_k$, then V is spanned by $\pi(g)\xi_i$. Thus, if K is a compact open, $V^K = \pi(e_K)V$ is spanned by $\pi(e_K)\pi(g)\xi_i$. But by the definition of compact representation, there are a finite number of linearly independent such vectors. Hence, V^K is finite dimensional. $\square$

COROLLARY. *Irreducible compact representations are admissible.*

The next theorem may reasonably be called the main theorem on compact representations.

THEOREM 7. *Let W be an irreducible compact representation. Then,*

(1) *$\{W\}$ splits the category $\mathcal{M}(G)$. In other words, every $M \in \mathcal{M}(G)$ can be decomposed as $M = M_W \oplus M_W^{\perp}$ where $\mathrm{JH}(M_W) \subset \{X\}$ and $\mathrm{JH}(M_W^{\perp}) \not\ni W$.*
(2) *M_W is completely reducible. That is, $M_W \cong \bigoplus W_i$ where $W_i \cong W$.*

REMARKS. 1. This theorem is obviously a generalization of the situation for compact groups. 2. This theorem tells us nothing about the irreducible representations themselves.

The proof will follow some preliminaries.

5.2. The Formal Dimension. We assume that G is unimodular so we may find a left- and right-invariant measure μ_G. Using μ_G we may identify $S(G)$ and $\mathcal{H}(G)$ as two sided modules.

PROPOSITION 12. *Let (ρ, W) be an irreducible compact representation of G. Then there exists a natural morphism of $G \times G$-modules*

$$\varphi\colon S(G) \cong \mathcal{H}(G) \to W \otimes \tilde{W}.$$

Moreover, φ is unique up to scaler and can be normalized so that $\operatorname{tr}\rho(h) = <,> \circ(\varphi(h))$. Here $<,>\colon W \otimes \tilde{W} \to \mathbb{C}$ is the natural pairing.

PROOF. Consider $\operatorname{End} W$ as a $G \times G$-module under the action $(g_1, g_2)(a) = \rho(g_1)a\rho(g_2)^{-1}$. Let $\operatorname{End} W_{\mathrm{sm}}$ be the smooth part of this module. Clearly, ρ maps $\mathcal{H}(G)$ to $\operatorname{End} W_{\mathrm{sm}}$. We will show that there is an isomorphism

$$\alpha\colon W \otimes \tilde{W} \xrightarrow{\sim} \operatorname{End} W_{\mathrm{sm}};$$

then $\varphi = \alpha^{-1} \circ \rho$.

LEMMA 10. *The map $\alpha\colon W \otimes \tilde{W} \to \operatorname{End} W_{\mathrm{sm}}$ given by $\alpha(\xi \otimes \tilde{\xi})(w) = < w, \tilde{\xi} > \xi$ is an isomorphism of $G \times G$ modules.*

PROOF. Injectivity is obvious. Recall that irreducible compact representations are admissible. Thus, we may prove surjectivity at each finite stage by counting dimensions. Let $K \subset G$ be an open compact subgroup. Then $(\operatorname{End} W)^{K\times K}$ is naturally a subset of $\operatorname{End} W^K$ which has dimension at most $(\dim W^K)^2 \leq \dim(W \otimes \tilde{W})^{K\times K}$. Hence, surjectivity follows from injectivity. $\square$

To see that φ is unique up to a scaler, we pass to contragredients. Specifically, using proposition 6,

$$\begin{aligned}\operatorname{Hom}_{G\times G}(\mathcal{H}(G), W\otimes\tilde{W}) &= \operatorname{Hom}_{G\times G}(\tilde{W}\otimes W, \widetilde{\mathcal{H}(G)})\\ &\subset \operatorname{Hom}_{G\times G}(\tilde{W}\otimes W, C^\infty(G\times G/\text{diagonal}))\\ &= \operatorname{Hom}_{G\times G}(\tilde{W}\otimes W, C^\infty(G))\\ &= \operatorname{Hom}_{G\times G}(\tilde{W}\otimes W, \operatorname{Ind}_G^{G\times G}\mathbb{C}_G)\\ &= \operatorname{Hom}_G(\tilde{W}\otimes W, \mathbb{C}_G)\end{aligned}$$

by Frobenious reciprocity

$$= \mathbb{C}$$

by Schur's lemma.

Finally, the normalization is obvious from the definition of α. $\square$

If (ρ, W) is an irreducible compact representation then the last proposition gives a map

$$\varphi\colon \mathcal{H}(G) \to W\otimes\tilde{W}.$$

On the other hand, there is a map in the other direction which assigns to two vectors the associated matrix coeficient:

$$\begin{aligned}m\colon W\otimes\tilde{W} &\to S(G) \cong \mathcal{H}(G)\\ (\xi,\tilde{\xi}) &\mapsto m_{\xi,\tilde{\xi}}(g) = <\rho(g^{-1})\xi, \tilde{\xi}> .\end{aligned}$$

It is natural to consider the composition $\varphi\circ m\colon W\otimes\tilde{W} \to W\otimes\tilde{W}$.

PROPOSITION 13. *Given G, ρ and W as above, there exists a nonzero number $d(\rho)$, called the* formal dimension *of (ρ, W), such that $\varphi\circ m = d(\rho)\cdot$ Identity.*

PROOF. As $W\otimes\tilde{W}$ is an irreducible representataion of $G\times G$, the existence of the formal dimension follows from Schur's lemma. We must show that it is non-zero. Let $w \in W\otimes\tilde{W}$ be so that $h = m(w)$ is non-zero. We will be finished if we show that $\varphi(h) \neq 0$. By the definition of φ, it is enough to prove that $\rho(h) \neq 0$. We will prove this be showing that for any irreducible representation, (τ, V), *not* equivalent to ρ, $\tau(h) = 0$. Then, by the separation lemma, $\rho(h) \neq 0$.

LEMMA 11. *Let (τ, V) be any irreducible representataion of G not equivalent to ρ, then $\tau(h) = 0$.*

PROOF. Let $v \in V$ and consider the morphism of G-modules $W \otimes \tilde{W} \to V$ given by $\xi \otimes \tilde{\xi} \mapsto \tau(m(\xi \otimes \tilde{\xi}))v$. Here we have thought of m as a map into $\mathcal{H}(G)$ and (τ, V) as an $\mathcal{H}(G)$-module. As a G-module (acting on the first component), $W \otimes \tilde{W}$ is completely reducible and is a sum of copies of W. Thus, the same is true of $\operatorname{Im} W \otimes \tilde{W}$. In particular, when V is irreducible and not equivalent to W, $\operatorname{Im} W \otimes \tilde{W} = 0$. In particular, $\tau(h)v = 0$ for all $v \in V$. Thus $\tau(h) = 0$. □

□

REMARKS. 1. $d(\rho)$ depends on the choice of Haar measure. 2. If G is compact and we normailize so the measure of G is 1, then the formal dimension is the reciprocal of the actual dimension of the representation. 3. The formal dimension can be defined more generally for representations whose matrix coefficients are L^2 but not necessarily compactly supported. 4. If W is unitrizable, $\xi \in W$ and $\tilde{\xi} \in \tilde{W}$ then it can be shown that

$$\int_G |m_{\xi,\tilde{\xi}}(g)|^2 d\mu_G = d(\rho).$$

This gives another proof that the formal dimension is non-zero.

5.3. Proof of the Main Theorem. In this section, we will prove a theorem which clearly implies the main theorem on compact representations (theorem 7). Recall that we identify $S(G)$ and $\mathcal{H}(G)$ via the Haar measure μ_G and that we have an isomorphism $\alpha\colon W \otimes \tilde{W} \to \operatorname{End} W_{\mathrm{sm}}$.

THEOREM 8. *Let (ρ, W) be an irreducible compact representation with m and φ as in the last section, and let $(\eta, V) \in \mathcal{M}(G)$ be any smooth representation of G. Set*

$$\begin{aligned} \mathcal{E}_{W,K} &= d(\rho)^{-1} m(\varphi(e_K)) \in \mathcal{H}(G) \\ V_0 &= \sum_{K \subset G} \operatorname{Im} \eta(\mathcal{E}_{W,K}) \quad \textit{and} \\ V_1 &= \bigcap_{K \subset G} \operatorname{Ker} \eta(\mathcal{E}_{W,K}) \end{aligned}$$

where K runs through all compact open subgroups. Then,

(1) *V_0 and V_1 are G-submodules of V.*
(2) *$V = V_0 \oplus V_1$.*
(3) *V_1 does not have subquotients isomorphic to W.*
(4) *V_0 is a direct sum of copies of W.*

Proof. (1) and (2) are clear. By proposition 13, $\varphi(\mathcal{E}_{W,K}) = \varphi(e_K)$. Thus, for any subquotient of V isomorphic to (ρ, W) there must be elements not killed by some $\mathcal{E}_{W,K}$. But by construction V_1 does not have any such vectors, proving (3).

For (4), it is enough to show that V_0 is generated by irreducible submodules isomorphic to W. But by definition, V_0 is generated by the images of various maps $W \otimes \tilde{W} \to V_0$. As $W \otimes \tilde{W}$ is completely reducible and a direct sum of copies of W, so is V_0. $\square$

CHAPTER II

Cuspidal Representations

In this chapter we prove the important result that every irreducible representation of an l-group G is admissible. In fact, we will obtain a stronger statement, the so-called Uniform Admissibility theorem, which, if V is an irreducible represention, gives an explicit bound for $\dim V^K$ depending only on G and the open compact subgroup $K \subset G$ (and not on V).

This general result is deduced from the special case of cuspidal representations, that is, those killed by the Jacquet functors J_U where U is a non-trivial standard unipotent subgroup of G. The first step in the study of cuspidal representations is to relate them to the compact representations introduced in the last chapter. This is based on some important geometric results.

We begin by specializing to the case $G = \mathrm{GL}(n)$. Here the geometry is fairly straightforward. For the general case, we will quote at least one hard geometric result without the proof as that would take us too far afield. On the other hand, most of the consequences of the geometry are proved in substantially the same way for the general case as for $\mathrm{GL}(n)$.

In the last section, we use uniform admissibility to prove that the set of irreducible cuspidal representations splits the category $\mathcal{M}(G)$. Consequently, we may divide the problem of understanding representations of G into two parts: first, the study of cuspidal representations, and second, the understanding of the so-called induced representations. Finally, we complete our analysis of the first problem by showing that the subcategory of cuspidal representatins, may be decomposed as the sum of categories of modules over algebras which are explicitly described.

1. The Geometry of $GL(n)$

In light of the results of the previous chapter, the statement that all irreducible smooth representations of an l-group G are admissible is equivalent to the statement that, if $K \subset G$ is a compact open subgroup, then all irreducible representations of the Hecke algebra, $\mathcal{H}_K(G) = e_K\mathcal{H}(G)e_K$, are finite dimensional. The first step is to prove that $\mathcal{H}_K(G)$ is "almost" an algebra of finite type.

DEFINITION 13. Let A be a $\mathbb{C}$-algebra with unit. A is of *finite type* if there is a commutative, finitely generated $\mathbb{C}$-algebra C and a homomorphism $\alpha\colon C \to A$ such that (i) $\alpha(C)$ lies in the center of A, and (ii) A is a finitely generated C-module.

REMARKS. 1. Clearly, any irreducible representation of an algebra of finite type is finite dimensional. 2. Such objects are in the world of algebraic geometry, and so may be studied using algebra-geometric techniques.

Except where otherwise indicated, $G = \mathrm{GL}(n)$.

1.1. Basic Results. Set $K_0 = \mathrm{GL}(n, \mathcal{O})$; this is a maximal compact subgroup. If $M_n(\mathcal{O})$ designates the $n \times n$ matrices with entries in $\mathcal{O}$ and π generates the maximal ideal in $\mathcal{O}$, then for each $i > 0$ we define $K_i = \{1 + \pi^i M_n(\mathcal{O})\}$. The K_i are open compact subgroups and K_0 normalizes each K_i. The K_i, $i > 0$ are called *congruence subgroups*. They clearly form a basis for the topology of G.

Let M_0 be the diagonal subgroup of G. We will denote the maximal compact subgroup of M_0 by M_0°. Clearly, $M_0^\circ = M_0 \cap K_0$, that is all diagonal matrices with entries in $\mathcal{O}^*$. Let $\Lambda = M_0/M_0^\circ \cong \mathbb{Z}^n$. Fix an embedding $\Lambda \hookrightarrow M_0$ by

$$\lambda = (l_1, \ldots, l_n) \mapsto \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix}.$$

Our goal is to describe $\mathcal{H}_K(G)$ for $G = \mathrm{GL}(n)$. First we have some simple lemmas that are true for general G. If $K \subset G$ is an open compact subgroup, we will use the notation

$$a(g) = e_K * \mathcal{E}_g * e_K \in \mathcal{H}_K(G)$$

for any $g \in G$. Here $\mathcal{E}_g$ is the delta distribution at g.

LEMMA 12. *$a(g)$ depends only on the double coset $KgK = \operatorname{Supp} a(g)$.*

LEMMA 13. *Given a double coset $\sigma = KgK \subset G$, $a(g) = e_K * \mathcal{E}_g * e_K$ is the unique K-left-and-right-invariant distribution supported on σ and with integral 1.*

LEMMA 14. *As g runs through a system of representatives for the double cosets $K\backslash G/K$, the $a(g)$ form a basis for $\mathcal{H}_K(G)$.*

In light of these lemmas, to understand the multiplication in $\mathcal{H}_K(G)$ we must study double cosets. For example, $(KgK)(Kg'K) \supset Kgg'K$. If in fact

$$(KgK)(Kg'K) = Kgg'K,$$

then $a(gg') = a(g)a(g')$.

In general, a description of $K\backslash G/K$ is given by the *Cartan decomposition*. For $\mathrm{GL}(n)$ this is as follows:

We know $\Lambda \cong \mathbb{Z}^n = \{(e_1, \dots, e_n)\}$. The *Weyl chamber*, $\Lambda^+ \subset \Lambda$, is given by $\Lambda^+ = \{(l_1, \dots, l_n) \mid l_1 \leq l_2 \leq \cdots \leq l_n\}$. Equivalently

$$\Lambda^+ = \left\{ \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix} \middle| \; l_1 \geq l_2 \geq \cdots \geq l_n \right\}.$$

The Cartan decomposition is

$$G = K_0 \Lambda^+ K_0.$$

The proof follows from simply manipulating the matrices involved.

Fix a congruence subgroup $K = K_i$, $i > 0$. Choose a set of representatives $x_1, \dots, x_r$ for $K\backslash K_0$. Let $\mathcal{H}_0 = \mathcal{H}_K(K_0) \subset \mathcal{H}_K(G)$ be the finite-dimensional subalgebra spanned by the $a(x_j) = e_K * \mathcal{E}_{x_j} * e_K$. Observe that, as K_0 normalizes K, $Kx_j = x_jK$ for all j. Therefore, for any $g \in G$, we have $(Kx_jK)(KgK) = Kx_jKgK = Kx_jgK$. Equivalently, $a(x_jg) = a(x_j)a(g)$. In the same way, $a(gx_j) = a(g)a(x_j)$.

Let C be the span of $\{a(\lambda) | \lambda \in \Lambda^+\}$. The next theorem is fundamental.

THEOREM 9. (1) $\mathcal{H}_K(G) = \mathcal{H}_0 C \mathcal{H}_0$.
(2) *C is a commutative, finitely generated algebra.*

REMARK. This is saying that that $\mathcal{H}_K(G)$ is of finite type in some sense, but it is not generated over C on the left nor on the right, but rather "in the middle".

PROOF OF (1). By the Cartan decomposition, $G = \bigcup_{\lambda \in \Lambda^+} K_0 \lambda K_0$. Moreover, $K_0 = \bigcup_i Kx_i = \bigcup_i x_iK$. Therefore,

$$G = \bigcup_{\substack{\lambda \in \Lambda^+ \\ i,j}} Kx_i \lambda x_j K.$$

This implies that the $a(x_i \lambda x_j)$ form a basis for $\mathcal{H}_K(G)$. But we showed above that $a(x_i \lambda x_j) = a(x_i)a(\lambda)a(x_j)$ which implies (1). $\square$

The proof of part (2) of the theorem will occupy the remainder of this section. It is clearly enough to prove that $a(\lambda\mu) = a(\lambda)a(\mu)$ for $\lambda, \mu \in \Lambda^+$. Essentially, we must show that $(K\lambda K)(K\mu K) = K\lambda\mu K$. This is not trivial because the elements of Λ^+ do not normalize K. The idea is to decompose K into parts that can be moved right or left.

Let U be the standard maximal unipotent subgroup of G (i.e. upper triangular matrices with 1's on diagonal) and $\overline{U}$ the lower triangular unipotent subgroup.

Set $K_+ = K \cap U$ and $K_- = K \cap \overline{U}$. Also, $K_{M_0} = K \cap M_0$, where $M_0 =$ diagonal matrices as before. Here is the decomposition.

PROPOSITION 14. $K = K_+ K_{M_0} K_-$

PROOF. Just do elementary row and column reductions on the elements of K. These correspond to multiplying by K_+ and K_- on the left and right, respectively. $\square$

COROLLARY. $K = K_- K_{M_0} K_+$

PROOF. $K = K^{-1} = K_-^{-1} K_{M_0}^{-1} K_+^{-1} = K_- K_{M_0} K_+$ $\square$

PROPOSITION 15. *If* $\lambda \in \Lambda^+$ *then* $\lambda K_+ \lambda^{-1} \subset K_+$ *and* $\lambda^{-1} K_- \lambda \subset K_-$.

PROOF. Observe that $\mathrm{Ad}(\lambda)|_U$ is (not strictly contracting. In fact, if $\lambda = \mathrm{diag}(\lambda_i)$, then $\lambda(u_{i,j})\lambda^{-1} = (\lambda_i \lambda_j^{-1} u_{i,j})$. But for $j > i$, $\lambda_i \lambda_j^{-1} \in \mathcal{O}$. This proves the first statement. The second is similar. $\square$

We can now prove part (2) of theorem 9. We have reduced the problem to showing $K\lambda K\mu K = K\lambda\mu K$. Now,

$$\begin{aligned} K\lambda K\mu K &= K\lambda K_+ K_{M_0} K_- \mu K \\ &= K(\lambda K_+ \lambda^{-1})\lambda\mu(\mu^{-1} K_- \mu)K \\ &\subset K\lambda\mu K \end{aligned}$$

by the last proposition. The reverse inclusion always holds so theorem 9 (2) is proved.

1.2. Modules. We have shown that $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$ with C commutative and so, in particular, $a(\lambda^n) = a(\lambda)^n$. (Recall that $a(\lambda) = e_K \mathcal{E}_\lambda e_K$.) In this section we use this to study modules. Let (π, V) be a representation of G, π_K the associated representation of $\mathcal{H}_K$ on V^K. We will consider the problem of computing the kernel of $a(\lambda^n)$ on V^K.

First we have a simple lemma. Recall that for any compact subgroup $\Gamma \subset G$, e_Γ is the *unique* distribution which is supported on Γ, bi-Γ invariant and with integral 1; i.e. just Haar measure. Using this uniquness, it is easy to prove

LEMMA 15. *If* $\Gamma = \Gamma_1 \Gamma_2$, *then* $e_\Gamma = e_{\Gamma_1} * e_{\Gamma_2}$. *Moreover,* $e_{g\Gamma g^{-1}} = \mathcal{E}_g * e_\Gamma * \mathcal{E}_{g^{-1}}$.

By the lemma and proposition 14, we have

$$e_K = e_{K_+} * e_{K_{M_0}} * e_{K_-}$$

and

$$\mathcal{E}_\lambda * e_{K_+} * \mathcal{E}_{\lambda^{-1}} = e_{\lambda K_+ \lambda^{-1}}$$

for $\lambda \in \Lambda^+$ (we will often suppress the $*$'s). Moreover, proposition 15 immediatedly implies

LEMMA 16. *If* $\nu \in \Lambda^+$, *then*

(1) $e_K * e_{\nu K_+ \nu^{-1}} = e_K$
(2) $e_{\nu^{-1} K_- \nu} * e_K = e_K$
(3) $e_K * e_{K_{M_0}} = e_{K_{M_0}} * e_K = e_K$.
(4) $e_{\nu K_{M_0} \nu^{-1}} = e_{\nu^{-1} K_{M_0} \nu} = e_{M_0}$

PROPOSITION 16. *If* $\nu \in \Lambda^+$, *then* $\operatorname{Ker} a(\nu)|_{V^K} = \operatorname{Ker} e_{\nu^{-1}K_+\nu}|_{V^K}$.

PROOF. Using the preceding lemmas, we have

$$\begin{aligned} a(\nu) &= e_K * \mathcal{E}_\nu * e_K \\ &= e_{K_+} * \mathcal{E}_\nu * e_{\nu^{-1} K_{M_0} \nu} * e_{\nu^{-1} K_- \nu} * e_K \\ &= \mathcal{E}_\nu * e_{\nu^{-1} K_+ \nu} * e_K. \end{aligned}$$

But on V^K, e_K acts as the identity. Moreover, $\mathcal{E}_\nu$ is invertible. It follows that the kernel is as claimed. $\square$

Recall that we are trying to compute $\operatorname{Ker}(a(\nu))$ for $\nu = \lambda^n$. Proposition 16 suggests that we should look at $\nu^{-1}K_+\nu$. If $(k_{i,j}) \in K_+$, we have

$$\nu^{-1}(k_{i,j})\nu = \lambda^{-n}(k_{i,j})\lambda^n = (\lambda_i^{-n}\lambda_j^n k_{i,j}).$$

Since $K_+ = K \cap U$, $k_{i,j} = 0$ for $i > j$ and $k_{i,j} = 1$ for $i = j$. It remains to consider $i < j$. We begin with a special case.

SPECIAL CASE. Suppose $\lambda = \operatorname{diag}(\pi^{m_1}, \ldots, \pi^{m_n})$ where $m_1 > m_2 > \cdots > m_n$; the m_i are *strictly increasing*. Then $\lambda_i^{-n}\lambda_j^n$ gets big (in valuation) with n for $i < j$. Thus,

$$\bigcup_n \lambda^{-n} K_+ \lambda^n = U.$$

This gives a filtration by compact subgroups $U_1 \subset U_2 \subset \cdots \subset U$ with $\bigcup U_i = U$.

REMARK. Recall from chapter I.3.3 the Jacquet module $J_U(V) = V/V(U)$. Since U has a filtration by compact subgroups, proposition 10 shows that J_U is an exact functor.

We are interested in the kernel of e_{U_n}. Of course, $\operatorname{Ker} e_{U_1} \subset \operatorname{Ker} e_{U_2} \subset \cdots$.

PROPOSITION 17. $\bigcup_i \operatorname{Ker} e_{U_i} = V(U)$, *the subspace of* V *spanned by elements of the form* $\pi(u)v - v$.

PROOF. It is obvious that $\operatorname{Ker} e_{U_i} = V(U_i)$. But $V(U)$ is the union of the $V(U_i)$. $\square$

COROLLARY. $\bigcup_n \operatorname{Ker} a(\lambda^n) \cap V^K = V(U) \cap V^K$.

PROOF. Immdiate from propositions 16 and 17. □

GENERAL CASE. Now we suppose that $\lambda = \operatorname{diag}(\pi^{m_1}, \cdots, \pi^{m_n})$ where the m_i are *not* strictly increasing. Say $m_1 = m_2 = \cdots = m_{n_1} > m_{n_1+1} = \cdots = m_{n_1+n_2} > \cdots$. The n_i's give a partition of n. We will carry out the same analysis as before except that now we must work with an arbitrary standard parabolic subgroup instead of just the minimal parabolic. With this in mind we make an aside on parabolic subgroups.

Parabolic Subgroups. Suppose that G is reductive group. Pick $g \in G$. Set

$$P_g = \{x \in G \mid \{\operatorname{Ad}(g^n)x; n \geq 0\} \text{ is relatively compact in } G\}$$
$$U_g = \{x \in G \mid \operatorname{Ad}(g^n)x \to 1 \text{ as } n \to \infty\}$$

CLAIM. *Let G be any reductive group. For each $g \in G$, P_g and $P_{g^{-1}}$ form a pair of opposite parabolic subgroups. Moreover, if $M = P_g \cap P_{g^{-1}}$, then $P_g = MU_g$ and $P_{g^{-1}} = MU_{g^{-1}}$.*

PROOF. Exercise. □

REMARKS. 1. M is called a *Levi subgroup* and $P = MU$ is the *Levi decomposition.* 2. If g is unipotent or central, $P_g = G$ and $U_g = 1$.

It is not hard to see that in the case $G = \mathrm{GL}(n)$, $g = \lambda \in \Lambda^+$, P_λ is the group of block-upper-triangular matrices corresponding to the partition of n that we associated to λ above. Similarly, $P_{\lambda^{-1}}$ and M_λ are the corresponding block-lower-triangular and block-diagonal groups, respectively. We call P_λ a *standard parabolic subgroup* and U_λ a *standard unipotent subgroup.*

Set $K_+^P = K \cap U_\lambda$, $K_-^P = K \cap U_{\lambda^{-1}}$ and $K_M^P = K \cap M_\lambda$ (we will often suppress the P and λ). Exactly as before, we can prove

PROPOSITION 18. *For $G = \mathrm{GL}(n)$*

(1) $K = K_+^P K_M^P K_-^P$, $\lambda K_+^P \lambda^{-1} \subset K_+^P$ *and,* $\lambda^{-1} K_-^P \lambda \subset K_-^P$
(2) $(\operatorname{Ad}\lambda^n)|_{K_+} \to 1$ *as* $n \to \infty$, *and* $(\operatorname{Ad}\lambda^{-n})|_{K_-} \to 1$ *as* $n \to \infty$
(3) $\bigcup_n \operatorname{Ad}(\lambda^{-n}) K_+^P = U_\lambda$
(4) $\bigcup_n \operatorname{Ker} a(\lambda^n) \cap V^K = V(U_\lambda) \cap V^K$.

In light of the preceding results, it is not surprising that the Jacquet functors J_U where U is a unipotent subgroup are particularly important.

Jacquet Functors. Let G be an *arbitrary* (reductive) l-group. Let $P = MU$ be a standard parabolic subgroup of G. Let (π, V) be a representation of G. So far we have viewed $J_U(V)$ only as a vector space. However, M normalizes U and thus preserves $V(U)$. Consequently, $J_U(V)$ is naturally an M-module. Define the functor

$$r_{M,G}\colon \mathcal{M}(G) \to \mathcal{M}(M)$$

by

$$r_{M,G}(V) = J_U(V).$$

The choice of $P = MU$ is understood here.

The key fact about $r_{M,G}$ is that it is related to induction. First we state an important geometric fact which we do not prove.

Iwasawa Decomposition. *If P is a parabolic subgroup and K_0 is a maximal compact subgroup of G, then $G = K_0P$. In particular, G/P is compact.*

Corollary. *If P is any parabolic subgroup of G, then P is conjugate to a standard parabolic by an element of K_0. Moreover, G/P is compact.*

Theorem 10. *$r_{M,G}$ has a right adjoint functor,*

$$i_{G,M}\colon \mathcal{M}(M) \to \mathcal{M}(G)$$

defined as follows: if W is a representation of M, extend it to P by letting U act trivially. Then

$$i_{G,M}(W) = \operatorname{ind}_P^G(W).$$

Remarks. 1. The Iwasawa decomposition implies that $\operatorname{ind}_P^G = \operatorname{Ind}_P^G$. 2. We will later modify the definitions of r and i slightly. See chapter 3.

Proof. Let $W \in \mathcal{M}(M)$, $V \in \mathcal{M}(G)$. We must show that there is a functorial isomorphism $\operatorname{Hom}_G(V, i_{G,M}(W)) \tilde{\to} \operatorname{Hom}_M(r_{M,G}V, W)$. Well,

$$\operatorname{Hom}_G(V, i_{G,M}(W)) = \operatorname{Hom}_G(V, \operatorname{Ind}_P^G(W))$$

by Frobenious reciprocity

$$= \operatorname{Hom}_P(V, W)$$

as U acts trivially

$$\begin{aligned} &= \operatorname{Hom}_P(V/V(U), W) \\ &= \operatorname{Hom}_P(r_{M,G}V, W) \\ &= \operatorname{Hom}_M(r_{M,G}V, W) \end{aligned}$$

as required. $\square$

Proposition 19. *Let M be a Levi subgroup of G. Then, the functors $r_{M,G}$ and $i_{G,M}$ satisfy the following properties.*

(1) *$r_{M,G}$ is right adjoint to $i_{G,M}$.*

(2) *If N is a Levi subgroup of M, then* $r_{N,M} \circ r_{M,G} = r_{N,G}$ *and* $i_{G,M} \circ i_{M,N} = i_{G,N}$.
(3) $i_{G,M}$ *maps admissible to admissible.*
(4) $r_{M,G}$ *maps finitely generated to finitely generated.*
(5) $i_{G,M}$ *and* $r_{M,G}$ *are exact.*

Proof. We have already proved (1), and (2) is a simple verification. (3) follows from the fact that, as G/P is compact, ind_P^G takes admissible modules to admissible modules (see section I.3.2). Recall from chapter 1 that ind is exact. Hence the first part of (5) is clear. For the second part of (5), we must show that U is the union of compact subgroups. If $G = \mathrm{GL}(n)$ this is discussed before proposition 17. It is also true in general (see section 2.1).

For (4), let (π, V) be a finitely generated G-module. It is enough to show that V is finitely generated as a P-module because then so is $V/V(U)$; but U acts trivially here so $V/V(U) = r_{G,M}(V)$ is finitely generated as an M-module.

Suppose V is generated by $\xi_1, \ldots, \xi_l$. Then $G = PK_0$ implies that $V|_P$ is generated by all $\pi(k)\xi_i$, $k \in K_0$. But because we are considering only smooth representations, $k \mapsto \pi(k)\xi$ is locally constant. As K_0 is compact, this proves that $V|_P$ is finitely generated. □

The essential usefulness of the functors $i_{G,M}$ and $r_{M,G}$ is, first of all, to build representations of G out of representations of M – a smaller and presumably simpler group – and second, to understand which representations may be obtained in this way. One feature of this situation is that even when studying a particular group, such as $\mathrm{GL}(n)$, we are led to consider other groups, such as $M = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$.

1.3. Quasi-Cuspidal Representations.

Definition 14. A representation (π, V) is called *quasi-cuspidal* if for any standard Levi subgroup M except $M = G$, $r_{M,G}(V) = 0$.

Remark. Another formulation is that (π, V) is quasi-cuspidal if for any standard unipotent subgroup U except $U = 1$, $J_U(V) = 0$.

Definition 15. Let $K \subset G$ be a fixed compact subgroup. A representation (π, V) is called *compact modulo center* if whenever $\xi \in V$, $\mathcal{D}_{\xi,K}(g) = \pi(e_K)\pi(g^{-1})\xi$ has compact support modulo center.

It is easy to see that this definition is independent of the choice of K.

Theorem 11. (π, V) *is quasi-cuspidal if and only if it is compact modulo center.*

PROOF. Let (π, V) be a quasi-cuspidal representation, $\xi \in V$. We may assume that $K = K_i$ is a congruence subgroup and, by choosing i large enough, that ξ is K-invariant. Then, $\mathcal{D}_{\xi,K}(g) = \pi(e_K)\pi(g^{-1})\pi(e_K)\xi$.

As in section 1.1, let $\{x_1, \ldots, x_r\}$ be a system of representatives for $K\backslash K_0$. We proved that $g \in Kx_i\lambda x_jK = x_iK\lambda Kx_j$ for some $\lambda \in \Lambda^+$ and some i, j. Thus, it is enough to show that $\lambda \mapsto \pi(x_i)\pi(a(\lambda))\pi(x_j)\xi$ has compact-modulo-center support on Λ^+. This is true precisely if $\lambda \mapsto \pi(a(\lambda))\xi$ has compact-modulo-center support on Λ^+. (Recall that $a(\lambda) = e_K\mathcal{E}_\lambda e_K$.)

Let $\nu_1, \ldots, \nu_n$ be a basis for Λ^+. Let $\lambda = \sum m_i\nu_i$. Since we are assuming V is quasi-cuspidal, for each $\mu \in \Lambda^+$ so that $U_\mu \neq 1$,

$$V^K \cap \bigcup_n \operatorname{Ker} a(\mu^n) = V(U_\mu) \cap V^K = V^K.$$

Hence, for each $\eta \in V$ we can find $k_{\mu,\eta}$ so large that $\pi(a(\mu^k))\eta = 0$ whenever $k \geq k_{\mu,\eta}$. In particular, we can find L so large that $\pi(a(\lambda))\xi = 0$ if any $m_i \geq L$.

We now have an upper bound on the m_i's. If we could similarly put a lower bound on them, we would prove that $\lambda \mapsto \pi(a(\lambda))\xi$ has compact support. We can do this only "modulo center". Observe that by replacing λ with λR where $R = \operatorname{diag}(\pi^r, \ldots, \pi^r)$ is in the center of Λ^+, we can get $m_i \geq 0$.

Conversly, suppose that (π, V) is compact modulo center. By reversing the reasoning given above, we see that $\lambda \mapsto \pi(a(\lambda))\xi$ has compact-modulo-center support in Λ^+. But, for λ non-central, the sequence λ^n eventually leaves all compact-modulo center subsets of Λ^+. That is, $a(\lambda^n)$ eventually acts trivially, and so for large n $\operatorname{Ker} a(\lambda^n) = V^K$. However, we know that

$$V^K \cap \bigcup_n \operatorname{Ker} a(\lambda^n) = V(U_\lambda) \cap V^K.$$

Therefore, $V(U_\lambda) \cap V^K = V^K$ for all congruence subgroups K. As our representations are smooth, this implies that $J_U = V/V(U) = 0$ whenever $U = U_\lambda$ and λ is a non-central element of Λ^+. That is, $J_U(V) = 0$ whenever U is a standard unipotent subgroup except $U = 1$. Therefore, (π, V) is quasi-cuspidal. $\square$

In the last chapter we had many nice results concerning compact representations. As yet we know very little about compact modulo center ones. We now introduce machinery to deal with the "modulo center", and use our information about compact representations to draw conclusions about quasi-cuspidal ones. For now we are most interested in the case $G = \mathrm{GL}(n)$. However, even for this case we will need to consider $G = \mathrm{GL}(n_1) \times \ldots \times \mathrm{GL}(n_k)$.

Suppose $G = \mathrm{GL}(n)$. Set $G^\circ = \{g \in G | \det g \in \mathcal{O}^*\}$. G° is an open, dense, normal subgroup of G with

$$\Lambda(G) \stackrel{\text{def}}{=} G/G^\circ \cong \mathbb{Z} = F^*/\mathcal{O}^*.$$

Moreover, it is easy to see that G° contains all compact subgroups of G.

REMARK. More generally, for G arbitrary, there is a G° which contains all compact subgroups of G and so that $G/G^\circ = \Lambda(G)$ is a lattice. (See section 2.1.) For example, when $G = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$, then $\Lambda(G) \cong \mathbb{Z}^k$.

Let $Z(G)$ be the center of G. Then $Z(G)G^\circ$ is an open subgroup of finite index in G (see section 2.1 for the general case). For example, for $G = \mathrm{GL}(n)$, the index is n.

The idea is that we can study representations of G° and $Z(G)$ separately, and then, up to finite information, make conclusions about representations of G. Moreover, by restricting to G° we no longer have to worry about "modulo center". We first see the simplification in the Cartan decomposition. Take $G = \mathrm{GL}(n)$.

The Cartan decomposition states that $G = K\Lambda^+K$. Moreover, Λ^+ contains $\mathrm{diag}(\pi^r, \ldots, \pi^r)$ for all $r \in \mathbb{Z}$ and thus is not a strict cone. This is precisely why we needed the "modulo center" assumption in the last theorem, as is clear from the proof. Set $\Lambda^{+\circ} = \Lambda^+ \cap G^\circ$. Then,

$$G^\circ = K\Lambda^{+\circ}K.$$

Moreover, $\Lambda^{+\circ} \cong (\mathbb{Z}^+)^l$, $l = n-1$ (and thus is a strict cone). In other words, there is a basis $\nu_1, \ldots, \nu_l$ such that $\Lambda^{+\circ} = \{\sum m_i\nu_i | m_i \geq 0\}$.

HARISH-CHANDRA'S THEOREM. *Representations of G° are compact if and only if they are quasi-cuspidal.*

REMARKS. 1. As all standard unipotent subgroups lie in G°, it makes sense to talk of quasi-cuspidal representations. 2. The key point is compact, not just compact modulo center.

PROOF. The proof is identical to the last theorem, only easier. As before, if (π, V) is quasi-cuspidal, we fix a K compact and $\xi \in V^K$ and easily reduce to proving that $\lambda \mapsto \pi(a(\lambda))\xi$ has compact support in $\Lambda^{+\circ}$.

Let $\nu_1, \ldots, \nu_l$ be a basis for $\Lambda^{+\circ}$. Write $\lambda = \sum m_i\nu_i$ where (unlike last time) $m_i \geq 0$. Next we prove that there is a constant L so that for $\pi(a(\lambda))\xi$ to be nonzero, we must have $m_i \leq L$. Thus, we get finite (and hence compact) support.

The converse similarly parallels the proof of theorem 11. $\square$

DEFINITION 16. A representation is called *cuspidal* if it is both quasi-cuspidal and finitely generated.

COROLLARY. *Any irreducible cuspidal representation of G is admissible.*

PROOF. Let (ρ, W) be the representation. First we show that $W|_{G^\circ}$ is finitely generated. Let $Z = Z(G)$. Because $[G : ZG^\circ]$ is finite, $W|_{ZG^\circ}$ is finitely generated. But by irreducibility, Z acts by scalers. Hence W is finitely generated as a G°-module, as claimed.

By Harish-Chandra's theorem, $W|_{G^\circ}$ is compact. Proposition 11 says that finitely generated compact representations are admissible. Thus, $W|_{G^\circ}$ is admissible. As G° contains all compact subgroups, the corollary follows. $\square$

In order to extend this result to *all* irreducible representations, we use the functors $r_{M,G}$ and $i_{G,M}$, and their properties. We have the following easy but important lemma.

LEMMA 17. *Let (τ, W) be an irreducible representation of G, then there is a parabolic $P = MU$ and an irreducible cuspidal representation (ρ, E) of M, so that there exists an embedding $W \hookrightarrow i_{G,M}(E)$.*

PROOF. Let M be a Levi subgroup, minimal subject to the condition $E' = r_{M,G}(W) \neq 0$. We claim that E' is cuspidal. This follows from proposition 19. Indeed, by part (2), if $N \subsetneqq M$ then $r_{N,M}(E') = r_{N,M} \circ r_{M,G}(W) = r_{N,G}(V) = 0$ by choice of M. This proves E' quasi-cuspidal. Also, W is irreducible and so certainly finitely generated. Thus, by proposition 19 (4), E' is finitly generated.

Let E be an irreducible quotient of E'. E is an irreducible cuspidal representation of M, as required. Moreover, there is a nonzero map $r_{M,G}(W) = E' \to E$. By the adjunction property (proposition 19), we get a non-zero map $W \to i_{G,M}(E)$. As W is irreducible, this must be an embedding. $\square$

REMARK. This use of the adjunction property is typical. Namely, we show that something exists and is non-zero. It never gives more detailed information than that.

THEOREM 12. *Any irreducible representation is admissible.*

PROOF. Let W and E be as in the lemma. E is irreducible cuspidal and therefore admissible by the corollary to Harish Chandra's theorem. By part (3) of proposition 19, $i_{G,M}(E)$ is also admissible. But $W \hookrightarrow i_{G,M}(E)$ so W is admissible. $\square$

1.4. Uniform Admissiblity. The theorem that we just proved says the following: given an open compact subgroup K and an irreducible representation V of G, then V^K is finite dimensional. However, as far as we know, $\dim V^K$ may be arbitrarily large for a given V.

UNIFORM ADMISSIBILITY THEOREM. *Given an open compact subgroup $K \subset G$, then there is an effectively computable constant, $c = c(G, K)$, so that whenever V is an irreducible representation of G, $\dim V^K \leq c$.*

Reformulation. *All irreducible representations of the algebra $\mathcal{H}_K(G)$ have dimension bounded by $c(G,K)$.*

Our main tool will be the decomposition $\mathcal{H}_K(G) = \mathcal{H}_0 C \mathcal{H}_0$. However, we first need some linear algebra. Consider the following question: Given $N \cong \mathbb{C}^m$ and C a commutative subalgebra of $\operatorname{End} N$, what is a reasonable bound for $\dim C$?

Conjecture. *If C is generated by l elements, then $\dim C \leq m + l$.*

Bernstein does not know how to prove this. However, we do have

Proposition 20. *If $N \cong \mathbb{C}^m$, $C \subset \operatorname{End} N$ is commutative and generated by l elements, then*

$$\dim C \leq m^{2-1/2^{l-1}}.$$

Proof. Omitted. See [BZ0] $\square$

We now prove the Uniform Admissibility theorem.

Proof. Let (ρ, V) be an irreducible representation of $\mathcal{H}_K(G)$. Let $k = \dim V$. We want to find $c = c(G,K)$ so that $k \leq c$. We know that $k \leq \infty$. By a general algebraic result (Burnside's Theorem), it follows that $\rho\colon \mathcal{H}_K(G) \to \operatorname{End} V$ is surjective.

We may write $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$ with C commutative and finitly generated (say l generators). Let $d = \dim \mathcal{H}_0$. Clearly, $k^2 = \dim \operatorname{End} V = \dim \rho(\mathcal{H}_K) \leq d^2 \dim \rho(C)$. But by the proposition, $\dim \rho(C) \leq k^{2-1/2^{l-1}}$. Thus,

$$k^2 \leq d^2 k^{2-1/2^{l-1}}.$$

Therefore, if we set $c(G,K) = d^{2^l}$, we have $k \leq c$. $\square$

Consider $G^\circ \subset G$ as before, and consider $K \subset G^\circ$ compact. We have seen that cuspidal representations of G° are compact. Thus, given any irreducible cuspidal representation (ρ, W) of G°, and $\xi \in W$, then $\mathcal{D}_{\xi,K}(g) = \rho(e_K)\rho(g^{-1})\xi$ has compact support in G°. We will now show how the uniform admissibility theorem can strengthen this result.

Proposition 21. *Given $K \subset G^\circ \subset G$ as above, there exists an open compact subset $\Omega \subset \Omega(G,K) \subset G^\circ$ such that* $\operatorname{Supp} \mathcal{D}_{\xi,K}(g) \subset \Omega$ *for all (ρ, W) irreducible cuspidal and $\xi \in W$.*

Proof. It follows from the proof of Harish-Chanda's theorem that compact representations of G° are exactly those for which $\lambda \mapsto \rho(a(\lambda))\xi$ has finite (and hence compact) support in $\Lambda^{+\circ}$. This in turn is equivalent to the statement that for $\xi \in W^K$ there is a constant k_ξ so that any $\nu \in \Lambda^{+\circ}\backslash\{1\}$ satisfies $\rho(\nu^k)\xi = 0$ whenever $k \geq k_\xi$. It is easy to see that our proposition amounts to the statement that these constants can be chosen independent of ξ and W. But this is obvious because we know that there is a constant $c = c(G, K)$ so that $\dim W^K \leq c$. $\square$

Corollary. *Given $K \subset G^\circ$, there are only finitly many equivalence classes of irreducible cuspidal representations of $\mathcal{H}_K(G^\circ)$.*

Reformulation. *There are a finite number of equivalence classes of irreducible cuspidal representations of G° with a K-fixed vector.*
Remark. By working through the proofs in this section, one can obtain a bound on this number.

Proof. It is easy to see that the support of the matrix coefficients of all irreducible cuspidal representations must lie in $\Omega(G, K)$ (see the proof of theorem 6). Hence, the corollary follows from the following general lemma.

Lemma 18. *The matrix coefficients of any set of pairwise non-isomorphic matrix coefficiets are linearly independent functions.*

The proof is standard.
$\square$

2. General Groups

In this section we will discuss many of the results that hold for general reductive groups but so far we have only discussed for $\mathrm{GL}(n)$. We will omit the proofs of many of the geometric results. Most of these are standard although one (Bruhat's theorem) is quite hard.

2.1. Geometric Results. Let G be the $\mathbf{F}$-points of a connected reductive algebraic group where F is a totally disconnected field. Most of the geometric results that were almost obvious for $\mathrm{GL}(n)$ also hold for G but the proofs are more difficult.

Proposition 22. (1) *G is an l-group.*
(2) *Let G° be the subgroup of G generated by all compact subgroups. Then G° is an open normal subgroup with compact center.*
(3) *$G/G^\circ = \Lambda(G)$ is a lattice.*
(4) *If Z is the center of G then $G^\circ Z$ is of finite index in G.*

DEFINITION 17. A *parabolic subgroup* is one of the form

$$P = P_g = \{x \in G \mid \{\operatorname{Ad} g^n(x)\} \text{ is relatively compact}\}.$$

Fix a minimal parabolic P_0. A *standard parabolic* is a parabolic containing P_0.

PROPOSITION 23. (1) *Any subgroup which contains P_0 is a parabolic.*
(2) *Any parabolic is conjugate to exactly one minimal parabolic.*
(3) *Parabolic subgroups are equal to there own normalizers.*

IWASAWA DECOMPOSITION. *There is a maximal compact subgroup K_0 so that $G = P_0K_0$.*

COROLLARY. *If P is any parabolic subgroup of G, then P is conjugate to a standard parabolic by an element of K_0. Moreover, G/P is compact.*

LEVI DECOMPOSITION. *There is a unipotent group, U_0, and a reductive group M_0, so that $P_0 = M_0U_0$.*

CLAIM. *M_0 is compact modulo center if and only if M_0° is compact.*

We prove this because it introduces a technical point that will be important later.

PROOF. Let $\Lambda = \Lambda(M_0)$ be the lattice M_0/M_0°. In case $G = \mathrm{GL}(n)$, there is a natural lifting of Λ to the center $Z(M_0)$ (see section 1.1). In general, however, the situation is more subtle.

Let Λ_Z be the image of $Z(M_0)$ in Λ. Λ_Z *can* be lifted to the abelian group $Z(M_0) \subset M_0$. Moreover, it follows from proposition 22 that Λ_Z is a lattice of finite index in Λ. Thus, modulo the center, M_0 is compact if and only if M_0° is. $\square$

It is important to note that we may write

$$\Lambda(M_0) = \bigcup \Lambda_Z \mu_i$$

and then choose a *setwise* lifting of Λ to M_0 so that restricted to Λ_Z it is a group lifting.

Choose a minimal parabolic with fixed Levi decomposition, $P_0 = M_0U_0$. (Note that this also determines a Levi decomposition for $P \supset P_0$; namely, choose M containing M_0.) K_0 is a maximal compact subgroup so that $G = P_0K_0$.

Fix a lift of Λ to M_0 as above. Let $\Lambda^+ = \{\lambda \in \Lambda \mid \operatorname{Ad}(\lambda)|_{U_0} \text{ is (not strictly) contracting}\}$. As with $\mathrm{GL}(n)$, there is a subtlty when G does not have compact center. In this case, $Z(G) \cap \Lambda^+$ is non-trivial. However, most of the results below are trivially true for central elements. Of course, we can always work with G° which has compact center. In this case, Λ^+ is a cone. Note that for $\lambda \in \Lambda^+$, $P_0 \subseteq P_\lambda$.

In general, the proofs of the results in this section are nearly identical to the case $G = \mathrm{GL}(n)$ discussed in previous lectures. The next theorem, however, is hard.

BRUHAT'S THEOREM. *There are arbitrarily small congruence subgroups K satisfying*

(1) *K is normalized by K_0.*
(2) *For any standard parabolic subgroup $P = MU$, we have a decomposition, $K = K_{\overline{U}} K_M K_U$, where $K_{\overline{U}} = K \cap \overline{U}$, $K_M = K \cap M$ and $K_U = K \cap U$.*
(3) (a) *K_M is $\mathrm{Ad}\,\Lambda$-invariant.*
    (b) *K_U is $\mathrm{Ad}\,\Lambda^+$-invariant.*
    (c) *$K_{\overline{U}}$ is $(\mathrm{Ad}\,\Lambda^+)^{-1}$-invariant.*

REMARKS. 1. Sometimes condition (2) is expressed by saying that P and K are in *good position*. For $\lambda \in \Lambda^+$, condition (3) says that λ is *dominant* with respect to the pair (P, K). 2. We will sometimes need to consider the lattice $\Lambda(M)$ for M a non-minimal (standard) Levi subgroup, say $P = MU$. (Our convention is that $\Lambda = \Lambda(M_0)$.) Although $\Lambda(M)$ is not necessarily contained in Λ, we do have $\Lambda(Z(M)) \subset \Lambda(Z(M_0))$. (This situation is somewhat clarified by the introduction of the root system, section IV.2.1.)

3. Let $\Lambda(M)^{++}$ be the set of λ so that $P = P_\lambda$; these are called *strictly dominant*. Equivalently, we could require that $\mathrm{Ad}(\lambda)|_U$ and $\mathrm{Ad}(\lambda^{-1})|_{\overline{U}}$ are *strictly* contracting, and that the family of operators $\mathrm{Ad}(\lambda^n)|_M$, $n \in \mathbb{Z}$, is uniformly bounded. Suppose that P and K are in good position. If $\lambda \in \Lambda(M)^{++}$ and furthermore λ is dominant with respect to (P, K), then we say that λ is *strictly dominant with respect to the pair* (P, K). Note that this set includes $\Lambda(M)^+ \cap \Lambda^+$ but will in general be larger. We will denote the set of λ which are strictly dominant with respect to (P, K) by $\Lambda(M, K)^{++}$

LEMMA 19. *For any standard parabolic $P = MU$, we may find arbitrarily small K so that P and K are in good position and there exists λ strictly dominant with respect to the pair (P, K).*

PROOF. Follows from Bruhat's theorem and the preceding remarks. □

COROLLARY. *Any standard unipotent subgroup U has a filtration by compact subgroups.*

PROOF. Suppose $P = MU$. Then by the lemma there is an open compact subgroup K and a λ strictly dominant with respect to the pair (P, K). Let $U_n = \lambda^{-n} K_U \lambda^n$ with $K_U = K \cap U$ as in Bruhat's theorem. Then $U_1 \subset U_2 \subset \cdots$ and $\bigcup_n U_n = U$. □

As for was the case for $\mathrm{GL}(n)$, an important role is played by distributions of the form $a(\lambda) = e_K \mathcal{E}_\lambda e_K$, $\lambda \in \Lambda^+$.

LEMMA 20. *Suppose $\lambda, \mu \in \lambda^+$. Then $a(\lambda\mu) = a(\lambda)a(\mu)$.*

REMARK. the proof is the same as for $\mathrm{GL}(n)$ except that, as discussed above, the lifting $\Lambda^+ \hookrightarrow M_0$ is only as sets, not as groups. However, this does not effect the proof.

We need one more geometric result.

CARTAN DECOMPOSITION.

$$G = K_0 \Lambda^+ K_0$$

If we write $\Lambda = \bigcup \Lambda_Z \mu_i$ as before, μ_i will also denote the corresponding (setwise) lift to M_0. Let $N(K)$ be the normalizer of K in G. Let $x_1, \ldots, x_r \in K_0$ be a (setwise) lift of the image of $K_0 \to N(K)/K$.

The last two results easily imply

THEOREM 13. *Let $C = \mathrm{Span}\{a(\lambda) \mid \lambda \in \Lambda_Z^+\}$, $D = \mathrm{Span}\{a(\mu_i)\}$ and $\mathcal{H}_0 = \mathrm{Span}\{a(x_i)\}$. Then*

(1) $\mathcal{H}_K(G) = \mathcal{H}_0 D C \mathcal{H}_0$.
(2) *C is a commutative algebra.*

REMARK. Unlike in the case of $\mathrm{GL}(n)$, the decomposition in (1) is not a product of subalgebras; D and $\mathcal{H}_0$ are finite dimensional vector spaces but not in general subalgebras.

2.2. Representation Theory. Theorem 13 is weaker than the result that we obtained for $\mathrm{GL}(n)$ (theorem 9). Nevertheless, it is suficient to imply in general the results that followed theorem 9. In particular, as for $\mathrm{GL}(n)$, we have

PROPOSITION 24.

$$\bigcup_n \operatorname{Ker} a(\lambda^n) \cap V^K = V(U_\lambda) \cap V^K$$

As before, the next step is to apply this to quasi-cuspidal representations. For each standard parabolic $P = MU$, we can define the functors $i_{G,M} \colon \mathcal{M}(M) \to \mathcal{M}(G)$ and $r_{M,G} \colon \mathcal{M}(G) \to \mathcal{M}(M)$ as before. A representation is *quasi-cuspidal* if $r_{M,G}(\pi) = 0$ for all Levi subgroups $M \subsetneqq G$. Cuspidal representations are quasi-cuspidal and finitely generated, as before.

THEOREM 14. *A representation (π, V) is quasi-cuspidal if and only if it is compact modulo center.*

HARISH-CHANDRA'S THEOREM. *(π, V) is a quasi-cuspidal representation of G if and only if $\pi|_{G^\circ}$ is compact.*

The proofs of these theorems for GL(n) can be easily modified to handle the gerneral case. The proofs of all the remaining results are exactly the same as for GL(n).

Corollary. *Any cuspidal irreducible representation is admissible.*

Theorem 15. *Any irreducible representation is admissible.*

Uniform Admissibility Theorem. *Given $K \subset G$ a congruence subgroup, there is a constant $c = c(G, K)$ such that for any irreducible (ρ, W), $\dim W^K < c$.*

Theorem 16. *Given $K \subset G$, there is a compact modulo center subgroup $\Omega = \Omega(G, K) \subset G$ which contains the supports of all matrix coeffiecients of the form $m_{\tilde{\xi},\xi}(g) =< \tilde{\xi}, \rho(g^{-1})\xi >$ where $\tilde{\xi}, \xi$ are K-invariant and lie in some quasi-cuspidal representation (ρ, W).*

Corollary. *There are a finite number of equivalence classes of irreducible cuspidal representations of G° with a K-fixed vector.*

3. Cuspidal Components

3.1. Relations between G and G°. Here we investigate to what extent the representation theory of G° controls the representation theory of G. Note that $\Lambda(G) = G/G^\circ$ is a lattice.

Definition 18. An *unramified character* of G is a character $\psi\colon G \to \mathbb{C}^*$ which is trivial on G°. The set of unramified characters is denoted $\Psi(G)$.

Remark. $\Psi(G) = \mathrm{Hom}(\Lambda(G), \mathbb{C}^*) \cong (\mathbb{C}^*)^l$. In this way, we introduce (complex) algebraic geometry into the study of G.

Proposition 25. *Let (ρ, V) and (ρ', V') be irreducible representations of G. Then*

(1) *$\rho|_{G^\circ}$ is semisimple of finite length.*
(2) *The following are equivalent*
 (a) $\rho|_{G^\circ} \cong \rho'|_{G^\circ}$
 (b) $\mathrm{JH}(\rho|_{G^\circ}) \cap \mathrm{JH}(\rho'|_{G^\circ}) \neq \emptyset$
 (c) *$\rho' = \psi\rho$ for some unramified character $\psi \in \Psi$*

Proof. (1) Let Z be the center of G. $\rho|_{G^\circ Z}$ is semisimple of finite length because ρ on G is and $G^\circ Z$ has finite index in G. But irreducibility implies that Z acts as a scaler so $\rho|_{G^\circ}$ is also semisimple.

(2) (a)⇒(b) and (c)⇒(a) are obvious. Thus, it is enough to show that (b)⇒(c). Let $H = \mathrm{Hom}_{G^\circ}(V, V')$. There is an action, τ, of G on H given by $\tau(g)f = \rho'(g)f\rho(g)^{-1}$ where $g \in G$ and $f \in H$. By definition of H, $\tau|_{G^\circ}$ is the identity.

Thus, we may think of τ as a representation of $\Lambda(G)$. Let $h \in H$ be an eigenfunction for τ with eigencharacter ψ; i.e. $\tau(g)h = \psi(g)h$ for all $g \in G$. From the definition of τ, it is now obvious that as a map of $\mathbb{C}$ vector spaces, $h\colon V \to V'$, h intertwines the G-action of $\psi\rho$ on V with the G-action of ρ' on V'. As both actions are irreducible, it follows that $\rho' = \psi\rho$. $\square$

We have an action of the algebraic group $\Psi(G)$ on the set $\operatorname{Irr} G$, namely $\psi\colon \pi \to \psi\pi$. We now investigate the orbits of this action.

Lemma 21. *Every orbit of* $\Psi(G)$ *on* $\operatorname{Irr} G$ *has a finite stationary subgroup.*

Proof. Let $Z = Z(G)$, $\Lambda = \Lambda(G)$ and Λ_Z the image of

$$Z \to G \to \Lambda$$

as in the last section. Let $Z' \subset Z$ be a lifting of Λ_Z. As $Z' \cong \Lambda_Z$, there will be no confusion if we write $\Psi(Z')$ for both the characters of Z' and $\operatorname{Hom}(\Lambda_Z, \mathbb{C}^*)$.

The group $\Psi(G)$ acts on $\Psi(Z')$ via the restriction $\Psi(G) \to \Psi(Z')$. Furthermore, there is a $\Psi(G)$ equivariant map

$$\operatorname{Irr} G \to \Psi(Z')$$

which takes an irreducible representation to its central character restricted to Z'. Thus, to prove the lemma, it is enough to show that the action of $\Psi(G)$ on $\Psi(Z')$ has finite stabilizers. But this is clear from the fact that Λ_Z has finite index in Λ. $\square$

Definition 19. A *cuspidal component* is an orbit, D, of $\Psi(G)$ in the set $\operatorname{Irr}_c G$ of cuspidal representations of G.

Remark. This makes sense becase, by Harish-Chandra's theorem, $\psi\rho$ is cuspidal whenever ρ is.

We have shown that each cuspidal component D has the form $\Psi(G)/\mathcal{G}$ where $\mathcal{G}$ is a finite subgroup. Therefore, D has the structure of a connected complex algebraic variety. Moreover, the action of $\Psi(G)$ is compatible with this structure.

3.2. Splitting $\mathcal{M}(G)$.

Proposition 26. *Let* $D \subset \operatorname{Irr} G$ *be a cuspidal component. Then* D *splits the category* $\mathcal{M}(G)$.

Recall that this means that every $V \in \mathcal{M}(G)$ can be written $V = V_D \oplus V_D^\perp$ where $\operatorname{JH}(V_D) \subset D$ and $\operatorname{JH}(V_D^\perp) \cap D = \emptyset$.

PROOF. We begin by restricting the situation to G°. By proposition 25, all elements of D are equivalent upon restriction to G°, say $D|_{G^\circ} = \{\rho\}$. Of course, ρ may no longer be irreducible. But by the same proposition, it is semisimple and of finite length, say $\rho_1, \ldots, \rho_r$. These are irreducible cuspidal representations of G° and therefore compact (Harish-Chandra's theorem). Moreover, compact representations are splitting (the main theorem on cuspidal representations, section I.5.1). Thus, on the level of G°-modules, there is a decomposition $V = V_D \oplus V_D^\perp$ where $\mathrm{JH}(V_D|_{G^\circ}) \subset \{\rho_1, \ldots, \rho_r\}$ and $\mathrm{JH}(V_D^\perp|_{G^\circ}) \cap \{\rho_1, \ldots, \rho_r\} = \emptyset$.

It only remains to observe that this decomposition is preserved by the action of G. But this follows from the fact that G permutes the ρ_i. $\square$

The next result is a "uniform" version of the proposition.

THEOREM 17. *The subset of irreducible cuspidal representations,* $\mathrm{Irr}_c\, G \subset \mathrm{Irr}\, G$*, splits* $\mathcal{M}(G)$*. In other words, if* $V \in \mathcal{M}(G)$*, we may write* $V = V_c \oplus V_i$ *where* $\mathrm{JH}(V_c)$ *consists only of cuspidal representations and* V_i *has no cuspidal Jordan-Holder components.*

PROOF. Let K be a congruence subgroup. It is a corollary of the uniform admissibility theorem that there are only a finite number of irreducible representations of G° with K-invariant vectors. By proposition 25, this implies that there are only finitly many cuspidal components, say $D_1, \ldots, D_{j(K)}$, which include representations with K-fixed vectors. Furthermore, if one representation in D has a K-fixed vector, then all representations in D contatain such a vector.

Proposition 26 says that the D's split $\mathcal{M}(G)$. Therefore, for $V \in \mathcal{M}(G)$,

$$V = V_{D_1} \oplus V_{D_{j(K)}} \oplus V_{c,K}^\perp = V_{c,K} \oplus V_{c,K}^\perp.$$

Consider a decreasing sequence of congruence subgroups, $K_1 \supset K_2 \supset \ldots$. Set

$$V_c = \bigcup_{K_i} V_{c,K_i}$$
$$V_i = \bigcap_{K_i} V_{c,K_i}^\perp$$

Obviously, $\mathrm{JH}(V_c) \subset \mathrm{Irr}_c\, G$ and $\mathrm{JH}(V_i) \cap \mathrm{Irr}_c\, G = \emptyset$.

It only remains to show that $V = V_c \oplus V_i$. Let $\xi \in V$. Then for one of our congruence subgroups, say K, $\xi \in V^K$. We have $\xi = \xi_{c,K} \oplus \xi_{c,K}^\perp$. We will write ξ' for $\xi_{c,K}^\perp$. It is obvious that $\xi_{c,K} \in V_c$. We will be done if we show that $\xi' \in V_i$.

Let $V' \subset V$ be the module generated by ξ'. We must show that $\mathrm{JH}(V')$ does not intersect any cuspidal components D. Since ξ is K-invariant, any projection of ξ (and so, a fortiori, ξ') onto a representation without K-fixed vectors must be zero. Thus, $\mathrm{JH}(V')$ does not intersect any D without K-fixed vectors. On the other hand, ξ' is, by definition, the part of ξ which has zero projection onto the cuspidal components with K-fixed vectors. Thus, $\mathrm{JH}(V')$ does not intersect any D *with* K-fixed vectors. This proves $\xi' \in V_i$ as needed. $\square$

We will use the notation $\mathcal{M}_{\text{cusp}}$ for the category of cuspidal representations and $\mathcal{M}_{\text{ind}}$ for the remaining ones. These are called *induced* representations. We have shown

Corollary. *$\mathcal{M}(G) = \mathcal{M}_{\text{cusp}} \times \mathcal{M}_{\text{ind}}$. Moreover, $\mathcal{M}_{\text{cusp}} = \prod_D \mathcal{M}(D)$ where D runs through the cuspidal components of G.*

This result separates the problem of understanding representations of G, that is the category $\mathcal{M}(G)$, into two parts, namely $\mathcal{M}_{\text{cusp}}$ and $\mathcal{M}_{\text{ind}}$. Furthermore, we have decomposed $\mathcal{M}_{\text{cusp}}$ into a sum of much smaller categories. In the next section we will give a description of the $\mathcal{M}(D)$ as a category of modules over an explicitly described ring. In this sense, we can say a lot about $\mathcal{M}_{\text{cusp}}$. The category $\mathcal{M}_{\text{ind}}$, on the other hand, is more subtle. Nevertheless, a similar though somewhat less complete analysis may be given for $\mathcal{M}_{\text{ind}}$. This is the topic of chapter 3.

3.3. The Category $\mathcal{M}(D)$. In this secion we investigate $\mathcal{M}(D)$, the category of representations whose Jordan-Holder components are contained in a cuspidal component D, say

$$D = \{\psi\rho \mid \rho = \text{fixed cuspidal representation}, \psi \in \Psi(G)\}.$$

Recall that $\Psi(G)$ is the set of unramified characters on G. Let F be the algebra of regular functions on $\Psi(G)$. F is a G-module in the obvious way. Clearly,

$$F \cong \mathbb{C}[\Lambda = G/G^\circ] = \operatorname{ind}_{G^\circ}^G \mathbb{C}.$$

Definition 20. Let Π be a projective object in a category $\mathcal{M}$. We say Π is a *generator* if the functor

$$\mathcal{F}_\Pi \colon X \to \operatorname{Hom}(\Pi, X)$$

is faithful.

Remark. Since $\mathcal{F}_\Pi$ is exact, it is enough to check that $X \neq 0$ implies $\mathcal{F}_\Pi(X) \neq 0$.

Proposition 27. *Let $\Pi(D) = F \otimes \rho$. Then*

(1) *$\Pi(D) \in \mathcal{M}(D)$.*
(2) *$\Pi(D)$ is a projective object in $\mathcal{M}(G)$, and hence in $\mathcal{M}(D)$.*
(3) *$\Pi(D)$ is finitely generated.*
(4) *$\Pi(D)$ is a generator of the category $\mathcal{M}(D)$*

Proof. For (1), just observe that $\mathrm{JH}(\Pi(D)|_{G^\circ}) \subset \mathrm{JH}(\rho|_{G^\circ})$. For (2), we must show that $X \mapsto \mathrm{Hom}_G(\Pi(D), X)$ is exact. It is easy to see that $\Pi(D) = \mathrm{ind}_{G^\circ}^G(\mathbb{C}) \otimes \rho = \mathrm{ind}_{G^\circ}^G(\rho|_{G^\circ})$. The idea now is to use Frobenieus reciprocity to get

$$\mathrm{Hom}_G(\Pi(D), X) = \mathrm{Hom}_{G^\circ}(\rho|_{G^\circ}, X|_{G^\circ}).$$

We know that $\rho|_{G^\circ}$ is semisimple (proposition 25) and compact (Harish Chandra's theorem). Furthermore, irreducible compact representations are splitting. Exactness follows, proving (2).

(3) is obvious. For (4), suppose that $X \in \mathrm{Ob}\,\mathcal{M}(D)$, $X \neq 0$. Then $\mathrm{JH}(X) \neq \emptyset$ and so X has an irreducible subquotient $\tau = \psi\rho$. Then, $\mathcal{F}_\Pi(\tau)$ is a subquotient of $\mathcal{F}_\Pi(X)$. But it is obvious that $\mathcal{F}_\Pi(\tau) \neq 0$. Therefore $\mathcal{F}_\Pi(X) \neq 0$. $\square$

This proposition is a powerful tool for illucidating the structure of $\mathcal{M}(D)$ when combined with the following general lemma.

Lemma 22. *Let $\mathcal{M}$ be an abelian category with arbitrary direct sums. Let $\Pi \in \mathcal{M}$ be a finitly generated, projective generator. Set $\Lambda = \mathrm{End}_{\mathcal{M}}\,\Pi$. Then $\mathcal{M} \cong {}^r\mathcal{M}(\Lambda) =$ category of right Λ-modules.*

Proof. Define

$$\begin{aligned} \mathcal{F}\colon \mathcal{M} &\to {}^r\mathcal{M}(\Lambda) \text{ by} \\ X &\mapsto \mathrm{Hom}(\Pi, X) \end{aligned}$$

which is naturally a right Λ module. As Π is projective and finitely generated, $\mathcal{F}$ is exact and commutes with arbitrary direct sums. Therefore, theorem 5 implies that $\mathcal{F}$ has a left adjoint, say $\mathcal{G}\colon {}^r\mathcal{M}(\Lambda) \to \mathcal{M}$ with

$$\mathrm{Hom}_\Lambda(L, \mathcal{F}(X)) \cong \mathrm{Hom}(\mathcal{G}(L), X)$$

for $X \in \mathcal{M}$ and $L \in {}^r\mathcal{M}(\Lambda)$. In particular, it follows that $\mathcal{G}$ is right exact and commutes with direct sums (proposition 8).

By definition, $\mathcal{F}(\Pi) = \Lambda$. We claim that $\mathcal{G}(\Lambda) = \Pi$. There is an adjunction morphism

$$\alpha\colon \mathcal{G}(\Lambda) = \mathcal{G}\mathcal{F}(\Pi) \to \Pi.$$

This map has the property that the composition $\mathcal{F}\Pi \to \mathcal{F}\mathcal{G}\mathcal{F}\Pi \xrightarrow{\mathcal{F}\alpha} \mathcal{F}\Pi$ is the identity (see section I.3.1). As $\mathcal{F}$ is faithful, this implies that α is onto. Furthermore, Π is projective so $\mathcal{G}\mathcal{F}\Pi = \Pi \oplus K$ where $K = \mathrm{Ker}\,\alpha$.

On the other hand, $\mathcal{G}\mathcal{F}\Pi = \mathcal{G}(\Lambda)$ and $\mathcal{G}$ maps projective objects to projective objects (proposition 8). Obviously, Λ is projective in ${}^r\mathcal{M}(\Lambda)$. This shows that $\mathcal{G}\mathcal{F}\Pi$ is projective, and hence so is K. Using this and the fact that Π is a generator, it is easy to see that $K \neq 0$ implies that $\mathrm{Hom}(K, \Pi) \neq 0$. Thus, $\mathrm{Hom}(\mathcal{G}\mathcal{F}\Pi, \Pi) \neq \Lambda$ which is false. This proves that $\mathcal{G}\mathcal{F}\Pi \cong \Pi$.

We will be done if we show that the adjunction map is an isomorphism, $\alpha\colon \mathcal{G}\mathcal{F}X \cong X$, for any object $X \in \operatorname{Ob}\mathcal{M}$. But since Π is a generator there is an exact sequence

$$\bigoplus_{S_1} \Pi \to \bigoplus_{S_2} \Pi \to X \to 0.$$

Since $\mathcal{G}$ and $\mathcal{F}$ commute with direct sums and $\mathcal{G}\mathcal{F}\Pi \cong \Pi$, we get a commuting diagram

$$\begin{tikzcd}
\bigoplus_{S_1} \mathcal{G}\mathcal{F}\Pi \arrow[r] \arrow[d, "\cong"'] & \bigoplus_{S_2} \mathcal{G}\mathcal{F}\Pi \arrow[r] \arrow[d, "\cong"'] & \mathcal{G}\mathcal{F}X \arrow[r] \arrow[d, "\alpha"'] & 0 \\
\bigoplus_{S_1} \Pi \arrow[r] & \bigoplus_{S_2} \Pi \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

By the five lemma, α is an isomorphism. $\square$

In our case, this lemma implies that

$$\mathcal{M}(D) \cong {}^{r}\mathcal{M}(\Lambda(D))$$

where $\Lambda(D) = \operatorname{End}(\Pi(D))$.[1] Our next goal is to describe this ring explicitly. First we have a digression on how to think about $\Pi(D)$.

Digression. Think of $\Pi = F \otimes \rho$ as a family of 1-dimensional representations of G parametrized by the points of Ψ.

More generally, let B be the algebra of regular functions on some algebraic variety M. Define a (G, B)-module to be a B-module together with a smooth G-action which commutes with B. If V is a (G, B)-module, then we can think of it as a family of G-modules parametrized by the points of M. Namely, if $x \in M$ and $m_x \subset B$ is the associated maximal ideal, we define the *specializ ation* of V at x to be $V_x = V \otimes B/m_x = V/m_xV$.

We now describe the ring $\Lambda(D) = \operatorname{End}(\Pi(D))$.

Simple Case. Here we assume that $\psi\rho \not\cong \rho$ for $\psi \neq 1$. In this case we will prove that $\Lambda(D) = F$, the ring regular functions on $\Psi(G)$.

Consider the G-module $\Pi = F \otimes \rho$ as a (G, F)-module in the obvious way. Then it is easy to see that for each $\psi \in \Psi$, the G-module $\Pi_\psi \cong \psi\rho$. Note that this is irreducible. Now suppose that $\alpha\colon \Pi \to \Pi$ is an element of $\Lambda(D)$. According to our philosophy, we should think of α as a collection of maps

$$\alpha\colon \Pi_\psi \to \Pi_\phi.$$

Now we use our assumption to conclude that $\psi = \phi$, and so by Schur's lemma, this map is multiplication by a constant. In other words, giving an element $\alpha \in \Lambda$ is equivalent to giving a function on Ψ. This correspondence yields the isomorphism, $\Lambda(D) \cong F$.

[1] We have also used Λ to denote lattices such as M_0/M_0°. It will be clear from context which is meant.

General Case. Let $\mathcal{G} \subset \Psi(G)$ be the subgroup of ψ so that $\psi\rho \cong \rho$. We proved in section 3.1 that $\mathcal{G}$ is finite. For each $\psi \in \mathcal{G}$, pick an intertwining operator $\nu_\psi \colon \rho \to \psi\rho$. This determines maps $\Pi_\phi \to \Pi_{\phi\psi}$ and so an element of $\Lambda(D)$ which we will also denote by ν_ψ Of course, these choices are not cannonical so there are constants $c_{\psi\phi}$ so that $\nu_\psi\nu_\phi = c_{\psi\phi}\nu_{\psi\phi}$.

In this case, $\alpha \in \Lambda(D)$ leads to maps

$$\alpha \colon \Pi_\phi \to \Pi_{\phi\psi}$$

for any $\psi \in \mathcal{G}$. Of course, then $\alpha\nu_{\psi^{-1}} \in F$. This proves part (1) of

Proposition 28. *Let D be a cuspidal component. Let $\mathcal{G}$, ν_ψ be as above.*

(1) *$\Lambda(D)$ is isomorphic to*

$$\bigoplus_{\psi \in \mathcal{G}} F\nu_\psi.$$

(2) *There are the following relations:*
   (a) *If $f \in F$ and $\psi \in \mathcal{G}$, then*

$$f\nu_\psi = \nu_\psi \tilde{f}$$

   where $\tilde{f}$ is f translated by ψ.
   (b) $\nu_\psi\nu_\phi = c_{\psi\phi}\nu_{\psi\phi}$.

Proof. The only thing that remains to be checked is (2)a. But this is obvious. □

Remark. We have seen that if A is ring with unit which is a projective generator of a category $\mathcal{M}$, then $\mathcal{M} \cong {}^r\mathcal{M}(\operatorname{End} A)$. It is important to keep in mind that there may be more than one projective generator and consequently different realizations of the category. For example, we took $\Pi = \operatorname{ind}_{G^\circ}^G(\rho|_{G^\circ})$ as our projective generator for $\mathcal{M}(D)$. We could also have taken $\Pi' = \operatorname{ind}_{G^\circ}^G \tau$ for some $\tau \subset \rho|_{G^\circ}$.

CHAPTER III

General Representations

Let V be an irreducible representation of a reductive p-adic group G. We have shown that there is a cannonical decompostion $V = V_c \oplus V_i$ associated to the decomposition $\mathrm{Irr}(G) = \mathrm{Irr}_c(G) \bigcup \mathrm{Irr}_i(G)$. Also, Irr_c is the disjoint union of the cuspidal components D and we have shown that $V_c = \bigoplus V_D$. In this section, we work towards a similar decomposition for Irr_i.

1. Induction and Restriction

In order to pass from cuspidal representations to general representations, we must understand the functors $i_{G,M}$ and $r_{M,G}$. First, we normalize these functors. This leads to many nice formulas throughout the subject. A very important instance of this is the basic geometric lemma which expresses the composition $r_{N,G} i_{G,M}$ in geometric terms.

1.1. Normalization. Let $P = MU$ be a parabolic subgroup. Then we have defined functors

$$\begin{aligned} i_{G,M} &\colon \mathcal{M}(M) \to \mathcal{M}(G) \\ r_{M,G} &\colon \mathcal{M}(G) \to \mathcal{M}(M). \end{aligned}$$

Let $X = P \backslash G$. Let $e = P \in X$. Recall from section I.3.2 that there is a bijection between smooth representations (τ, V) of M and G-equivariant sheaves on X which have fiber at the point e the representation τ considered as a P-module (U acts trivially). Call this sheaf $\mathcal{F}_\tau$. One realization of $i_{G,M}(\tau)$ is as the space of sections of $\mathcal{F}_\tau$. To see that this definition should be altered, observe that $i_{G,M}$ does not commute with taking contragredients; i.e. $i_{G,M}(\tilde{\tau})$ is not necessarily isomorphic to $\widetilde{i_{G,M}(\tau)}$. For example, take $G = \mathrm{GL}(2)$, $M =$ diagonal matrices, and $\tau = \mathbb{C}_M$. Then $\tau \cong \tilde{\tau}$ but $i_{G,M}(\mathbb{C}_M) \not\cong \widetilde{i_{G,M}(\mathbb{C}_M)}$ as is easily checked.

In this section we will normalize $i_{G,M}$ by twisting the sheaf $\mathcal{F}$ by an appropriate character so that we get $i_{G,M}(\tilde{\tau}) \cong \widetilde{i_{G,M}(\tau)}$, among other results.

To construct our isomorphism, it is enough to find a pairing $i_{G,M}(\tilde{\tau}) \times i_{G,M}(\tau) \to \mathbb{C}$. We know that over each point there is a pairing of the fibers $(\mathcal{F}_{\tilde{\tau}})_x \times (\mathcal{F}_\tau)_x \to \mathbb{C}$, and hence a pairing $\mathcal{F}_{\tilde{\tau}} \times \mathcal{F}_\tau \to$ sheaf of functions on X. We could try to obtain the required pairing by composing this with integration over X. The problem is that there is no G-equivariant measure on X so we cannot get a G-equivariant pairing this way.

Let Δ be the sheaf of locally constant measures (distributions) on X. The key point is that Δ is is a 1-dimensional G-equivariant sheaf. Thus, unlike for the sheaf of functions, there is a natural G-equivariant morphism $S(\Delta) \to \mathbb{C}$ given by evaluation at the identity function. (Here $S(\Delta)$ is the space of compactly supported sections.) The next lemma is easy to prove

Lemma 23. *There is a G-equivariant sheaf δ and an isomorphism $\delta^2 \cong \Delta$.*

We will also write δ for the associated equivalence class of representations of P. It is clear that Δ is trivial on U, so we may think of δ as a representation of M extended trivially to P.

Remark. In general, δ may not be unique. Over $\mathbb{C}$, it is possible to give Δ a positivity structure which makes the choice of square root cannonical. Over other algebraically closed fields, it is not clear how to do this. (The situation is similar with half-spin structures on curves and the half sum of positive roots in Lie theory.)

Definition 21. The functor $i_{G,M}\colon \mathcal{M}(M) \to \mathcal{M}(G)$ is given by

$$i_{G,M}(V) = \operatorname{ind}_P^G(\delta \otimes V)$$

where (τ, V) is extended trivially to P. Also, $r_{M,G}\colon \mathcal{M}(M) \to \mathcal{M}(G)$ is given by

$$r_{M,G}(V) = \delta^{-1} \otimes V/V(U).$$

Of course, an essential feature of these definitions is that they preserve the properties of $i_{G,M}$ and $r_{M,G}$ that we have already established. Furthermore, now $i_{G,M}(\tilde{\tau}) \cong \widetilde{i_{G,M}(\tau)}$. We see this as follows:

If $\mathcal{F}'_\tau = \delta \otimes \mathcal{F}_\tau$ then $i_{G,M}(\tau)$ may be considered as the space of compactly supported sections of $\mathcal{F}'_\tau$. There is a pairing $i_{G,M}(\tilde{\tau}) \times i_{G,M}(\tau) \to \mathbb{C}$ because we have $\mathcal{F}_{\tilde{\tau}} \times \mathcal{F}_\tau \to \delta^2 = \Delta$, fiberwise. Thus,

$$S(\mathcal{F}_{\tilde{\tau}}) \times S(\mathcal{F}_\tau) \to S(\Delta) \to \mathbb{C}$$

which is equivariant.

We summarize the properties of these functors:

Proposition 29. *Induction and restriction (Jacquet functor) have the following properties:*

(1) $i_{G,M}$ *is right adjoint to* $r_{M,G}$.
(2) $r_{M,G}$ *and* $i_{G,M}$ *are exact.*
(3) *If* $G \subset M \subset N$ *are Levi subgroups, then there are cannonical isomorphisms* $i_{G,N} \cong i_{G,M} \circ i_{M,N}$ *and* $r_{N,G} \cong r_{N,M} \circ r_{M,G}$.
(4) *There is a cannonical isomorphism* $i_{G,M}(\tilde{\tau}) \cong \widetilde{i_{G,M}(\tau)}$.

Remark. If we fix an identification of δ with the field of the representation ($\mathbb{C}$ in our case), then an equivalent definition for $i_{G,M}$ is the space of functions $f: G \to V$ satisfying $f(mug) = \tau(m)\Delta(m)^{1/2}f(g)$ for $m \in M$ and $u \in U$. This is a less satisfactory formulation because it is less canonical. Note that under such an identification, Δ becomes the usual modulus character. In particular, if $U_1 \subset U_2 \subset \cdots \subset U$ is a filtration of U by compact subgroups (see section II.1.2), then $\Delta(m)$ equals the index $[U_i : m^{-1}U_i m]$ for any i.

1.2. Basic Geometric Lemma.

Lemma 24. *Let* $Q = NV$ *and* $P = MU$ *be parabolics in* G. *Then* Q *has finitely many orbits on* $X = P\backslash G$.

Proof. This follows from our general geometric results (section II.2.1). $\square$

Remark. Let $w_1, \ldots, w_k$ be representatives for the orbits (=double cosets). For each $w = w_i$, let $N' = N \cap w^{-1}Mw$ and $M' = M \cap wNw^{-1}$. Then N', M' are Levi subgroups for N, M respectively, and $\operatorname{Ad} w : N' \to M'$ is an isomorphism.

Basic Geometric Lemma. *Let* $Q = NV$, $P = MU$ *and* $w_1, \ldots, w_k$ *be as above. Let*

$$\Gamma = r_{N,G} \circ i_{G,M} : \mathcal{M}(M) \to \mathcal{M}(N).$$

Then there is a finite filtration of Γ *by subfunctors with quotients*

$$\Gamma_r = i_{N,N'} \circ \tilde{w}_r \circ r_{M',M}$$

where N', M' *are Levi subgroups of* N, M *as above,* $\operatorname{Ad} w_i : N' \to M'$ *is an isomorphism as above, and* $\tilde{w}_i : \mathcal{M}(M') \to \mathcal{M}(N')$ *is the associated equivalence of categories.*

Proof. Set $X = P\backslash G$. Let E be an M-module and $\mathcal{F}_E$ the associated G-equivariant sheaf on X so that $\mathcal{F}_E(e) = \delta \otimes E$. Recall that

$$i_{G,M}(E) = S(X, \mathcal{F}_E)$$

the space of compactly supported sections. Our strategy is to define a filtration on $i_{G,M}(E)$ and then show that it is preserved by $r_{M,G}$.

We will denote the orbit w_iQ on X by $\mathcal{O}_i$. As discussed in section I.1.1, we can order the orbits so that $\mathcal{O}_1$, $\mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$ etc. are open. Set

$$\mathcal{S}_r = S(\bigcup_{i \leq r} \mathcal{O}_i, \mathcal{F}_E).$$

Then we have a filtration

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_r = S(X, \mathcal{F}_E) = i_{G,M}(E).$$

Moreover,

$$\mathcal{S}_r / \mathcal{S}_{r-1} \cong S(\mathcal{O}_r, \mathcal{F}_E).$$

Let us for now ignore our normalization of the Jacquet functor. Thus we will write $\Gamma(E) = J_V(\mathcal{S}_r)$. As J_V is exact, $\Gamma(E)$ is filtered by $J_V(\mathcal{S}_r)$ with quotients $J_V(S(\mathcal{O}_r, \mathcal{F}_E))$. Examining the action of V gives

$$J_V(S(\mathcal{O}_r, \mathcal{F}_E)) = S(\mathcal{O}_r/V, \mathcal{F}')$$

where $\mathcal{F}'$ is the sheaf associated to $r_{M',M}(E)$. Another way to write this is as $S(N/Q', w_r\mathcal{F}')$ where Q' is a parabolic inside N with Levi component N'. But this is precisely what it means to take $i_{N,N'}$ of $\tilde{w}_r r_{M',M}(E)$.

We have now proven that the functor Γ is filtered by subfunctors with quoteints that differ from Γ_r by at most a character (because we have ignored the normalizations). But it is easy to see that the normalizations cancel. $\square$

Remark. To each orbit of Q in $P\backslash G$ we have associated a functor which turns out to be a subquotient of Γ. It follows from the proof of the basic geometric lemma that in case the orbit is closed, it is actually a quotient if Γ; in case the orbit is open, we get a subfunctor of Γ.

For example, when $P = Q$, there is a unique closed orbit, namely $e = P \in X$. The associated functor is trivial so we see that Γ has a trivial quotient. For example, when ρ is a cuspidal representation of M, there is a map

$$r_{G,M} i_{M,G}(\rho) \to \rho.$$

This is, in fact, the adjunction morphism coming from Frobenius reciprocity.

On the other hand, when $Q = \overline{P}$, the parabolic opposite P, there is a unique open orbit, namely the image of $\overline{P}$ in X. Again the associated functor is trivial. Thus, in this case Γ has a trivial subfunctor. We will return to this situation in section 3.1.

2. Classification of Non-cuspidal in Terms of Cuspidal

2.1. Cuspidal Data.

DEFINITION 22. A *cuspidal data* is a pair (M, ρ) where $M \subset G$ is a Levi subgroup and $\rho \in \mathrm{Irr}_c(M)$. Two cuspidal data, (M, ρ), (M', ρ') are *associate* if there is a $g \in G$ so that

$$\mathrm{Ad}\, g\colon M \tilde{\to} M' \quad \text{and}$$
$$\mathrm{Ad}\, g\colon \rho \tilde{\to} \rho'$$

REMARK. We do *not* assmue that g maps the standard parabolic $P \supset M$ to itself. For example, $M = \begin{pmatrix} * & \\ & * \end{pmatrix}$, $P = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, and $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

CLAIM. *Let κ be an irreducible representation of G. Then there is a standard Levi subgroup M, an irreducible cuspidal representation ρ and a surjection $r_{M,G}(\kappa) \twoheadrightarrow \rho$.*

PROOF. As κ is irreducible, quasi-cuspidal is the same as cuspidal. But by the definition of quasi-cuspidal, either κ itself is quasi-cuspidal (so we take $M = G$) or some $r_{M,G}(\kappa)$ is non-zero and so has an irreducible quotient, ρ. To insure that ρ is cuspidal, the transitivity of $r_{M,G}$ implies that we can take M minimal so that $r_{M,G}(\kappa) \neq 0$. $\square$

COROLLARY. *If κ is an irreducible representation of G, then there is a cuspidal data (M, ρ) so that*

$$\kappa \hookrightarrow i_{G,M}(\rho).$$

PROOF. Follows from the claim and Frobenius reciprocity. $\square$

THEOREM 18. *Let κ be an irreducible representation of G. Then all cuspidal data (M, ρ) such that $\rho \in \mathrm{JH}(r_{M,G}(\kappa))$ are associate.*

The theorem and the last corollary immediately give:

COROLLARY. *Up to associate, there exists a unique cuspidal data with $\kappa \hookrightarrow i_{G,M}(\rho)$.*

For the proof of theorem 18 we will need the basic geometric lemma from the last section and the following lemma.

LEMMA 25. *Suppose τ is a representation of $M \subsetneq G$. Set $\pi = i_{G,M}(\tau)$. Then $\pi_{\mathrm{cusp}} = 0$.*

REMARK. Here we have written π_{cusp} instead of π_c for emphasis. See section II.3.2

PROOF. Write $\pi = \pi_{\text{cusp}} \oplus \pi_i$. If $\pi_{\text{cusp}} \neq 0$, then $\operatorname{Hom}(\pi_{\text{cusp}}, \pi) \neq 0$. But by adjunction (Frobenius reciprocity),

$$\operatorname{Hom}(\pi_{\text{cusp}}, \pi) = \operatorname{Hom}(r_{M,G}(\pi_{\text{cusp}}), \tau) = 0$$

by definition of cuspidal. $\square$

Now we prove the theorem.

PROOF. We know that $\kappa \hookrightarrow i_{G,M}(\rho)$ for some (M, ρ). Let (N, σ) be any cuspidal data so that

$$\sigma \in \mathrm{JH}(r_{N,G}(\kappa)_{\text{cusp}}) \subset \mathrm{JH}(F(\rho)_{\text{cusp}})$$

where $\Gamma = r_{N,G} \circ i_{G,M}$. We will be finished if we show that (N, σ) and (M, ρ) are associate.

By our basic geometric lemma, $\Gamma(\rho)_{\text{cusp}}$ is filtered by $(i_{N,N'} \circ \tilde{w} \circ r_{M',M})(\rho)_{\text{cusp}}$ where N', M' are Levi subgroups of N, M respectively and $w\colon M' \to N'$ is an isomorphism. By the last lemma, this is nonzero only if $N' = N$. Also, by definition of cuspidal, we must have $M = M'$. This proves that

$$\sigma \in JH(F(\rho)_{\text{cusp}}) \subset \bigcup_w \{w\rho\}$$

where the $w\colon N \to M$ are isomorphisms. Thus, (N, σ) is associate to (M, ρ) via w, as needed. $\square$

Let $\Omega(G)$ be the set of cuspidal data, (M, ρ), up to associate. Theorem 18 suggests that $\Omega(G)$ is an important object to study. In fact, we will see that $\Omega(G)$ has the structure of an algebraic variety whose geometry has great significance for the representation theory of G.

Define

$$\begin{aligned} \mathrm{pr}\colon \operatorname{Irr} G \to \Omega(G) \quad & \textit{by} \\ \kappa \mapsto \{(M, \rho) \mid \rho \in \mathrm{JH}(r_{M,G}(\kappa))\}. \end{aligned}$$

pr is well defined by Theorem 18.

PROPOSITION 30. pr *is a finite-to-one epimorphism.*

PROOF. Let (M, ρ) be a cuspidal data. We want to show that there are only finitely many irreducible κ so that $\rho \in \mathrm{JH}(r_{M,G}(\kappa))$. It is enough to show that this implies a surjection $r_{M,G}(\kappa) \twoheadrightarrow \rho$, because then by adjunction, $\kappa \hookrightarrow i_{G,M}(\rho)$ which has finite length.

It is convenient to prove this in the form of a general lemma.

LEMMA 26. *If π and τ are representations of G with π of finite length, τ cuspidal, and $\tau \in \mathrm{JH}(\pi)$, then there is a surjection $\pi \twoheadrightarrow \tau$.*

To prove the proposition, apply the lemma to the M-modules $\pi = r_{M,G}\kappa$ and $\tau = \rho$.

PROOF. We wish to show that $\mathrm{Hom}_G(\pi, \tau) \neq 0$. If we restrict our representations to G°, then τ is compact and so splits off as a direct summand of π. It follows that the space $S = \mathrm{Hom}_{G^\circ}(\pi, \tau)$ has finite dimension; that is $0 < \dim S < \infty$. Let $\Lambda = \Lambda(G) = G/G^\circ$. Then Λ acts on S is such a way that the invariants $S^\Lambda = \mathrm{Hom}_G(\pi, \tau)$. In this language, the assumption that $\tau \in \mathrm{JH}(\pi)$ implies that S has a quotient module T with $T^\Lambda \neq 0$. It is now an exercise in commutative algebra to show prove that $S^\Lambda \neq 0$. □

□

PROPOSITION 31. *$\Omega(G)$ is an algebraic variety.*

PROOF. The first step is

LEMMA 27. *If M is any group, $\mathrm{Irr}_c M$ is an algebraic variety (with an infinite number of components).*

PROOF. Irr_c is the union of the cuspidal components, D, which we know are connected algebraic varieties (section II.3.1). □

Let $M_1, \ldots, M_k$ be representatives of the conjugacy classes of the standard parabolic subgroups. Let $W_i = W(M_i)$ be the finite group $\mathcal{N}(M_i)/M_i$ where $\mathcal{N}(M_i)$ is the normalizer of M_i in G. By the lemma, $\mathrm{Irr}_c M_i$ is an algebraic variety, and, as the quotient of an algebraic variety by the action of a finite group is again an algebraic variety, so is $(\mathrm{Irr}_c M_i)/W_i$. But it is obvious that

$$\Omega(G) = \bigcup_{i=1}^{k} (\mathrm{Irr}_c M_i)/W_i$$

proving the proposition. □

REMARK. Each connected component of $\Omega(G)$ can be expressed as $D/W(M, D)$ where $W(M, D)$ is the subgroup of $W(M)$ which preserves D. In particular, each component is a cuspidal component modulo a finite group. However, the choice of D – even M – is not unique. Indeed, (M, D) may be replaced by any (M', D') which is associate in the obvious sense.

If Ω is a component of the variety $\Omega(G)$, then we will denote by Irr_Ω its pre-image under the surjection $\mathrm{Irr}\, G \to \Omega(G)$. We will also say that Irr_Ω is a *component* of $\mathrm{Irr}\, G$. We get

$$\mathrm{Irr}\, G = \bigcup \mathrm{Irr}_\Omega\,.$$

2.2. The Decomposition Theorem. In this section we show that the decompostion on the level of irreducible representations, $\operatorname{Irr} G = \bigcup \operatorname{Irr}_\Omega$ (which comes from the decomposition of $\Omega(G)$ into its connected components, Ω), leads to a decompoistion on the level of categories.

Let V be a G-module. Set $V(\Omega) =$ maximal submodule such that $\mathrm{JH}(V(\Omega)) \subset \operatorname{Irr}_\Omega$. We say that V is *split* if $V = \bigoplus V(\Omega)$.

DECOMPOSITION THEOREM. *Each component* $\operatorname{Irr}_\Omega$ *splits the category* $\mathcal{M}(G)$*. In other words, every* $V \in \mathcal{M}(G)$ *is split. We will write*

$$\mathcal{M}(G) = \prod_\Omega \mathcal{M}(\Omega).$$

As a first step in the proof, there is the following lemma.

LEMMA 28. *If* $V' \subset V$ *and* V *is split, then* V' *is split.*

PROOF. $V'(\Omega) = V(\Omega) \cap V'$. Set

$$W = V' / \bigoplus V'(\Omega).$$

Then our claim will be proved if we show that $\mathrm{JH}(W) = 0$ (because then $W = 0$).

Fix a component Ω_0 and consider

$$p_{\Omega_0} \colon V \to \bigoplus_{\Omega \neq \Omega_0} V(\Omega).$$

We will write $\overline{p}_{\Omega_0}$ for the corresponding map

$$\overline{p}_{\Omega_0} \colon W \to (\bigoplus_{\Omega \neq \Omega_0} V(\Omega)) / \operatorname{Im}(\bigoplus V'(\Omega)).$$

Since V is split, $\operatorname{Ker} p_{\Omega_0} = V(\Omega_0)$. Consequently,

$$\operatorname{Ker}(p_{\Omega_0}|_{V'} \to \bigoplus V(\Omega)) = V'(\Omega_0).$$

Thus, $\overline{p}_{\Omega_0}$ is an *injection* of W into a subquotient of $\bigoplus_{\Omega \neq \Omega_0} V(\Omega)$. Therefore,

$$\mathrm{JH}(W) \subset \bigcup_{\Omega \neq \Omega_0} \operatorname{Irr}_\Omega.$$

But this holds for all Ω_0 and Irr is the union of the $\operatorname{Irr}_\Omega$. So, $\mathrm{JH}(W) = \emptyset$. $\square$

The idea for the proof of the decomposition theorem is that we can show that a module splits by embedding it into one that we know splits, and then using the lemma.

Define

$$\mathcal{M}(\text{cusp}) = \bigoplus_{M \subset G} \mathcal{M}(M)_{\text{cusp}}$$

where M runs through the standard Levi subgroups of G. Define functors

$$I\colon \mathcal{M}(\text{cusp}) \to \mathcal{M}(G)$$
$$R\colon \mathcal{M}(G) \to \mathcal{M}(\text{cusp})$$

by $I((M, \rho_M)) = \bigoplus_M i_{G,M}(\rho_M)$ and $R(\pi) =$ the set of $r_{M,G}(\pi)_{\text{cusp}}$ for $M \subset G$.

Lemma 29. (1) *R is left adjoint to I.*

(2) *R is exact, faithful and maps finitely generated objects to finitely generated ones.*

(3) *For each $V \in \operatorname{Ob}\mathcal{M}(G)$, the adjunction morphism $\alpha\colon V \to IR(V)$ is an embedding.*

Proof. Parts (1) (2) follow from the corresponding properties for i and r (see proposition 19). For part (3), suppose α is not an embedding, say $\tau \subset \operatorname{Ker}\alpha$. Then, as I kills nothing, $R\tau = 0$. We want to show that this is impossible. We may as well assume τ is irreducible. Let M be a minimal parabolic so that $r_{M,G}(\tau) \neq 0$. Then $r_{M,G}(\tau)$ is cuspidal and so $R\tau \neq 0$. Contradiction. $\square$

We now come to the proof of the decomposition theorem.

Proof. Let $V \in \operatorname{Ob}\mathcal{M}(G)$. We must show that V is split. By lemma 28 it is enough to embed into a split module. Lemma 29 shows that V embeds in a representation of the form $\bigoplus_{M\subset G} i_{G,M}(\tau_M)$ where τ_M is a cuspidal representation of M. Thus, to prove the theorem, it is enough to prove that the $i_{G,M}(\tau_M)$ are split. But as τ_M is cuspidal, we may write $\tau_M = \bigoplus_D \tau(D)$ where the D run through the cuspidal components of M. This reduces our problem further; we must prove that $i_{G,M}(\tau(D))$ is split. We will show, in fact, that there is a component Ω so that $\mathrm{JH}(i_{G,M}(\tau(D))) \subset \mathrm{Irr}_\Omega$.

Let Ω be the connected component of $\Omega(G)$ which is a quotient of D (see remark following the proof of proposition 31). If $\pi \in \mathrm{JH}(i_{G,M}(\tau(D)))$ then $\mathrm{pr}(\pi)$ is some cuspidal data (M, ρ) with $\rho \in D$. Therefore, as an element of $\Omega(G)$, $\mathrm{pr}(\pi) \in \Omega$. $\square$

To some extent, the decomposition theorem does for general representations what we did for cuspidal representations, namely break the category into smaller pieces parametrized be the connected components of some algebraic variety. In the cuspidal case, however, we went further; we obtained a description of the pieces as categories of modules over explicitly determined rings, the endomorphism rings of projective generators. So far, we have nothing like this in general.

Naively one could try to find a projective generator for $\mathcal{M}(\Omega)$ by inducing the projective generator of $\mathcal{M}(D_M)$, for an appropriate Levi subgroup M and cuspidal component D_M. In fact, this approach works, but there are many things to check. One of the most difficult turns out to be the statement that what you get is projective; it is not clear that $i_{G,M}$ takes projective objects to projective objects. However, this is an immediate corollary of the deep fact, proved in the next section, that $i_{G,M}$ has a right adjoint. We will use this result in section 4 to obtain a description of $\mathcal{M}(\Omega)$ as a category of modules over an endomorphism ring.

To finish this section, here are some further consequences of lemma 29 for the category $\mathcal{M}(G)$.

Definition 23. A Noetherean category is one in which every finitely generated object is Noetherean.

Remark. Of courese Noetherean objects are always finitely generated.

Proposition 32. *$\mathcal{M}(G)$ is a Noetherean category.*

Proof. We have defined $\mathcal{M}(\text{cusp}) = \bigoplus \mathcal{M}(M)_{\text{cusp}}$. Of course, each $\mathcal{M}(M)_{\text{cusp}}$ may be further decomposed according to its cuspidal components. Thus, we may view R as a functor

$$R\colon \mathcal{M}(G) \to \prod \mathcal{M}(D_M)$$

where the product on the right runs through all cuspidal components of all standard Levi subgroups fo G. In chapter II we showed that $\mathcal{M}(D_M)$ is equivalent to the category of right modules over a Noetherean ring. As it is easy to see that a product of Noetherean categories is Noetherean, $\mathcal{M}(\text{cusp})$ is Noetherean.

Suppose that V is a finitely generated object of $\mathcal{M}(G)$ with a chain

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots .$$

Lemma 29 states that R is exact and faithful. Hence, if this chain did not stabilize then neither would

$$R(V_1) \subset R(V_2) \subset \dots$$

in $R(V)$. But this chain must stabilize because it is in a Noetherean category and $R(V)$ is finitely generated (lemma 29, again). $\square$

Proposition 33. *The functors $r_{M,G}$ and $i_{G,M}$ map fintely generated modules into finitely generated modules.*

Proof. We already know this for $r_{M,G}$ (proposition **??**). Suppose that $V \in \mathcal{M}(M)$ is finitely generated. By the last result, we may assume that V is Noetherean. We wish to show that $i_{G,M}V$ is also Noetherean.

Suppose for the moment that V is cuspidal. Then the basic geometric lemma implies that $r_{M,G}i_{G,M}V$ has a filtration with quotients of the form wV. These are clearly Noetherean. Thus, $r_{M,G}i_{G,M}V$ is Noetherean. But $r_{G,M}$ is exact and in this case acts faithfully. Hence, $i_{G,M}V$ is Noetherean, as needed.

In the general case, lemma 29 provides an embedding

$$V \hookrightarrow IR(V) = \bigoplus_{N \subset M} i_{M,N}(\rho_N)$$

for appropriate (cuspidal) ρ_N. Applying $i_{G,M}$ to both sides, the last paragraph shows that we get an embedding of $i_{G,M}V$ in a Noetherean module. Thus V is Noetherean. $\square$

3. A Right Adjoint for $i_{G,M}$

3.1. The Statement. Let $P = MU$ be a parabolic subgroup of G. We know that $r_{M,G}$ is left adjoint to $i_{G,M}$. Thus, if π is a representation of M,

$$\mathrm{Hom}_M(r_{M,G}i_{G,M}(\pi), \pi) = \mathrm{Hom}_G(i_{G,M}(\pi), i_{G,M}(\pi)) \neq \emptyset.$$

In other words, the functor $\Gamma = r_{M,G} \circ i_{G,M}$ has a trivial quotient. In fact, this is obvious from the filtration of Γ in the basic geometric lemma as we now explain.

The basic geometric lemma gives a filtration of the functor Γ whose subquotients correspond to the orbits of P acting on $X = P\backslash G$. There is a distinguished orbit of this action, namely the point $\{P\} \in X$, which is the only closed orbit. It is clear that this orbit corresponds to the trivial (i.e. identity) functor, and, as the orbit is closed, the trivial functor is a quotient (rather than just a subquotient) of Γ. (See remark at end of section 1.2.) The point is that the adjointness property is related to the existence of the distinguished orbit.

Let $\overline{P} = M\overline{U}$ be the parabolic opposite to P. Consider the action of $\overline{P}$ on X. In this case also there is a distinguished orbit, namely, the unique open orbit. It is generated by $\overline{P} \hookrightarrow G \twoheadrightarrow X$. Call this orbit $\mathcal{O}$. Again, it is clear that the functor associated with this orbit is trivial. Moreover, as $\mathcal{O}$ is open, it is a subfunctor. Next we show how the existence of this functor leads to an adjunction property.

Set, $\overline{r}_{M,G} = r^{\overline{P}}_{M,G}$. That is $\overline{r}_{M,G}(V) = V/V(\overline{U})$. In this case, $\Gamma = \overline{r}_{M,G} \circ i_{G,M}$. We have shown that Γ has a trivial subfunctor. In other words, for τ any representation of M, there is an embedding $\tau \hookrightarrow \Gamma(\tau)$. Now suppose $\varphi \in \mathrm{Hom}_G(i_{G,M}(\tau), \pi)$. Then we get a morphism of M-modules by considering the composition

$$\beta(\varphi)\colon \tau \to \Gamma(\tau) = \overline{r}_{M,G} \circ i_{G,M}(\tau) \xrightarrow{\varphi} \overline{r}_{M,G}(\pi).$$

In other words, there is a map

$$\beta\colon \mathrm{Hom}_G(i_{G,M}(\tau), \pi) \to \mathrm{Hom}_M(\tau, \overline{r}_{M,G}(\pi)).$$

Theorem 19. *β is an isomorphism. In other words, $\overline{r}_{M,G}$ is right adjoint to $i_{G,M}$.*

The proof will be given in the next section. This is a deep and somewhat mysterious theorem. Here we give some applications.

Corollary. *$r_{M,G}$ commutes with infinte direct products. In particular, any product of quasi-cuspidal representations is quasi-cuspidal.*

Proof. As $r_{M,G}$ has a left adjoint we may apply theorem 5. $\square$

To give some idea of the power of this result, we use it to derive the uniform admissibility theorem:

Fix $K \subset G$ compact. Recall that ρ is a quasi-cuspidal representation if and only if the matrix coefficients of ρ are supported on some compact modulo center set $S(\rho)$ If uniform admissibility failed then there would be a sequence of cuspidal representations $\rho_1, \rho_2, \ldots$ so that the sets $S(\rho_1), S(\rho_2), \ldots$ grow without bound. However, if $\pi = \prod \rho_i$, then π is quasi-cuspidal by the corollary. Moreover, $S(\rho_i) \subset S(\pi)$ which is compact modulo center. (c.f. theorem 16.)

Here is another important corollary of the adjointness.

Corollary. *$i_{G,M}$ maps projective objects to projective objects.*

Proof. Since $i_{G,M}$ has a right adjoint which is exact, this follows from proposition 8. $\square$

We will use this result in our study of the category $\mathcal{M}(G)$. Let Ω be a connected component of $\Omega(G)$. There is a Levi subgroup M and a cuspidal component of M, say D, so that Ω is the quotient of D by a finite subgroup. We have defined a representation $\Pi(D)$ which is a finitely generated projective generator of $\mathcal{M}(D)$. Set

$$\Pi(\Omega) = i_{G,M}(\Pi(D)).$$

By the last corollary, $\Pi(\Omega)$ is projective. It will turn out that $\Pi(\Omega)$ is a finitely generated projective generator for $\mathcal{M}(\Omega)$ and so by lemma 22, $\mathcal{M}(\Omega) \cong {}^r\mathcal{M}(\Lambda(\Omega))$ where $\Lambda(\Omega) = \operatorname{End} \Pi(\Omega)$. We are not quite ready to prove this yet.

3.2. Proof of Adjointness.

Theorem 20. *$i_{G,M}$ is left adjoint to $\overline{r}_{M,G}$. In other words, for any smooth representations of M, τ and π, there is a functorial isomorphism*

$$\operatorname{Hom}_G(i_{G,M}(\tau), \pi) \xrightarrow{\sim} \operatorname{Hom}_M(\tau, \overline{r}_{M,G}(\pi)).$$

Remark. In the last section we constructed a map which we claimed was an isomorphism. Here we will define another map, show that it is an isomorphism, and then show that the maps coincide.

We will not prove theorem 20 directly. Instead we will prove

THEOREM 21. *For σ a smooth representation of G*

$$\overline{r}_{M,G}(\tilde{\sigma}) = \widetilde{r_{M,G}(\sigma)}.$$

CLAIM. *Theorem 20 and theorem 21 are equivalent.*

PROOF. Let τ be any representation of G. Recall proposition 6 (2) states that if γ and κ are representations of G, then

$$\mathrm{Hom}_G(\kappa, \tilde{\gamma}) = \mathrm{Hom}_G(\gamma, \tilde{\kappa})$$

This fact, together with our normalization of induction (section 1.1) and Frobenius reciprocity imply

$$\begin{aligned} \mathrm{Hom}_G(i_{G,M}(\tau), \tilde{\sigma}) \quad &= \mathrm{Hom}_G(\sigma, \widetilde{i_{G,M}(\tau)}) \\ &= \mathrm{Hom}_G(\sigma, i_{G,M}(\tilde{\tau})) \\ &= \mathrm{Hom}_M(r_{M,G}(\sigma), \tilde{\tau}) \\ &= \mathrm{Hom}_M(\tau, \widetilde{r_{M,G}(\sigma)}). \end{aligned}$$

On the other hand, theorem 20 implies that

$$\mathrm{Hom}_M(\tau, \overline{r}_{M,G}(\tilde{\sigma})) = \mathrm{Hom}_G(i_{G,M}(\tau), \tilde{\sigma}).$$

Consequently,

$$\mathrm{Hom}_M(\tau, \overline{r}_{M,G}(\tilde{\sigma})) = \mathrm{Hom}_M(\tau, \widetilde{r_{M,G}(\sigma)}).$$

Finally, it is easy to see that this statement for every τ is equivalent to theorem 21.

For the converse statement, first observe that by reversing each step in the preceding argument, we get that theorem 21 implies the special case of theorem 20 in which π has the form $\tilde{\sigma}$ for some representation σ (in particular, for π admissible). However, this case is sufficient because of the following trick:

For general π, set $\pi_1 = \tilde{\tilde{\pi}}$ and $\pi_2 = \tilde{\tilde{\pi}}_1$. Observe that π_1 and π_2 are each the contragredient of something so theorem 20 is valid for π_1 and π_2. Also, we have a sequence

$$0 \hookrightarrow \pi \hookrightarrow \pi_1 \hookrightarrow \pi_2.$$

Using general functorial nonsense, the theorem for π follows from the theorem for π_1 and π_2. $\square$

REMARK. This last part of the proof is clearly somewhat unsatisfying.

The proof of theorem 21 rests on a classical result known as Jacquet's lemma. However, the usual formulation of Jacquet's lemma assumes that our representations are admissible. The main step in our proof of theorem 21 will be to show that Jacquet's lemma holds without assuming admissibility. This in turn relies on the Stabilization theorem which we will prove in the next section. This is a deep and difficult theorem.

Let $K \subset G$ be a congruence subgroup, $P = MU$ a parabolic, and $\lambda \in \Lambda^{++}(M, K)$ strictly dominant with respect to (P, K). (See section II.2.1.)

Jacquet's Lemma (Preliminary Version). *With the notation as above, let $K_M = K \cap M$. Then, if V is an admissible representation of G, the natural map*

$$V^K \to J_U(V)^{K_M}$$

is onto.

Notation. We will write J for J_U and $\overline{J}$ for $J_{\overline{U}}$. Let p denote either the projection $V \to J(V)$ or the projection $V^K \to J(V)^{K_M}$.

Proof. Let $\xi' \in J(V)^{K_M}$. The first step is to show that for large enough n, $\lambda^n \xi' \in \operatorname{Im} V^K$. Let ξ be a lift of ξ' to V^{K_M}. Let K' be a congruence subgroup so small that ξ is invariant under K'. Then $\eta = \lambda^n \xi$ is invariant with $\lambda^n K'_{\overline{U}} \lambda^{-n}$ as well as K_M. However, $\bigcup_n \lambda^n K'_{\overline{U}} \lambda^{-n} = \overline{U}$. In particular, we may choose n so large that η is invariant with respect to $K_{\overline{U}}$. Set $\phi = e_{K_U} \eta$. Then $\phi = e_{K_U} e_{K_M} e_{K_{\overline{U}}} \eta = e_K \eta$ and so $\phi \in V^K$. Moreover, the action of e_{K_U} is killed by J and so $p(\phi) = p(\eta) = \lambda^n \xi'$. Thus, for large enough n, $\lambda^n \xi' \in \operatorname{Im} V^K$.

Let $A\colon J(V)^{K_M} \to J(V)^{K_M}$ be the action of $\lambda \in M$. We have shown that each element of $J(V)^{K_M}$ is mapped to $\operatorname{Im}(V^K)$ by a sufficiently large power of A. However, as A is invertible and $J(V)^{K_M}$ is finite dimensional (by admissibility), this implies that $\operatorname{Im}(V^K)$ is the whole space. $\square$

Remark. Recall (from section II.2.1) that $a(\lambda) = e_K \mathcal{E}_\lambda e_K$. It is clear that, under $p\colon V^K \to J(V)^{K_M}$, the action of $a(\lambda)$ gets mapped to A. Using this observation, we can give an equivalent version of Jacquet's lemma. First we introduce some terminology:

Definition 24. Let L be a linear space and $a \in \operatorname{End}(L)$. The *localization* of L with respect to a will be denoted by (L_a, A). Let $L' = L/\kappa$ where $\kappa = \bigcup_n \operatorname{Ker}(a^n)$. Let a' be the endomorphism of L' corresponding to a. Then L_a is an extension of L' together with an invertible operator A so that

(1) A coincides with a' on $L' \subset L_a$, and
(2) $L_a = \bigcup_n A^{-n}(L')$.

The following lemma is obvious:

Lemma 30. *If L_a is finite dimensional, then $L_a = L'$.*

If λ is as before and $a = a(\lambda)$, proposition 24 says that $\bigcup_n \operatorname{Ker} a^n|_{V^K} = V(U) \cap V^K$. Thus, the fist step in the proof of Jacquet's lemma is equivalent to the statement

$$(J_U(V)^{K_M}, \lambda) \textit{ is the localization of } V^K \textit{ with respect to } a. \qquad (\bigstar)$$

The second step is lemma 30, together with admissibility. In particular, the only place that we used admissibility was the final argument. If we could find a substitute for this argument, that is, a broader condition to insure $L_a = L'$, we could strengthen Jacquet's lemma. This is possible using the stabilization theorem.

Definition 25. Let L be a linear space, $a \in \operatorname{End}(L)$. We say that (L, a) is *stable* if $L = \operatorname{Ker} a \oplus \operatorname{Im} a$ and on $\operatorname{Im} a$, a is invertible.

Remark. The last condition is equivalent to $\operatorname{Ker} a^2 = \operatorname{Ker} a$ and $\operatorname{Im} a^2 = \operatorname{Im} a$.

Stabilization Theorem. *Let $K \subset G$ be a congruence subgroup, $P = MU$ a parabolic, $\lambda \in \Lambda(M, K)^{++}$ (i.e. λ strictly dominant with respect to (P, K)). Suppose V is any smooth representation of G, then for $n \geq c(G, K)$, $a(\lambda^n) \in \operatorname{End}(V^K)$ is stable.*

Remark. 1. The constant c appearing here is the same one as in section II.2.2. We will use the notation $V_0^K = \operatorname{Ker} a(\lambda^n)$ and $V_*^K = \operatorname{Im} a(\lambda^n)$.

Jacquet's Lemma (Final Version). (1) *Jacquet's lemma holds without assuming admissibility. That is, if $K_M = K \cap M$ and V is* any *smooth representation, then the natural map*

$$p \colon V^K \to J_U(V)^{K_M}$$

is onto.

(2) *Furthermore, p has a natural inverse. In other words, $J_U(V)^{K_M}$ may be realized as a direct summand of V^K in a way that is functorial in V.*

Proof. As before, we have an invertible map $A \colon J(V)^{K_M} \to J(V)^{K_M}$ so that each element of $J(V)^{K_M}$ is mapped into $\operatorname{Im}(V^K)$ by a sufficiently large power of A. Moreover, by the stabilization theorem, there is a power of A, say $\mathcal{A} = A^m$, which is stable and in particular, $\operatorname{Im} \mathcal{A}^n = \operatorname{Im} \mathcal{A}$. Thus, $\operatorname{Im} \mathcal{A} \subset \operatorname{Im}(V^K)$. But $\mathcal{A}$ is invertible so $J(V)^{K_M} = \operatorname{Im}(V^K)$.

For the second claim, the stabilization theorem says that $V^K = V_0^K \oplus V_*^K$. But we have just seen that $J(V)^{K_M} = V_*^K$. $\square$

Remarks. 1. Although the proof of Jacquet's lemma involved a specific choice of λ, the result itself is independent of this choice. 2. This argument essentially shows that, in the language of lemma 30, if (L, a) is stable, then $L_a = L'$.

We will now prove theorem 21:

PROOF. We have shown that $V^K = V_0^K \oplus V_*^K$ and $V_*^K \cong J(V)^{K_M}$. Similarly, $\tilde{V}^K = \tilde{V}_0^K \oplus \tilde{V}_*^K$ and $\tilde{V}_*^K \cong \overline{J}(\tilde{V})^{K_M}$. To prove the theorem, we must find a pairing between $J(V)$ and $\overline{J}(\tilde{V})$. We do this as follows: given $\xi \in J(V)$ and $\tilde{\xi} \in \overline{J}(\tilde{V})$, find a K so small that they are invariant under K. View them as elements of V^K and $\tilde{V}^K$ and take the natural pairing. It is clear that this does not depend on the choice of K and gives a non-degenerate pairing. Finally, the definition of the pairing is independent of the choice of λ so we may take λ in the center of M. It is then clear that the pairing is M-equivariant. We summarize:

LEMMA 31. *There is a unique non-degenerate, M-equivariant pairing of $\overline{J}(\tilde{V})$ with $J(V)$ such that for $\tilde{\xi} \in \tilde{V}$, $\xi \in V$ then*

$$\langle \tilde{\xi}, \pi(\lambda^n)\xi \rangle = \langle p(\tilde{\xi}), J(\pi)(\lambda^n)p(\xi) \rangle$$

for all $n > n_0$. Here n_0 depends only on the stabliizer of $\tilde{\xi}$, ξ.

Two things remain to be checked. First, that $J(V)$ is the *complete* contragredient of $\overline{J}(\tilde{V})$. Namely, for any compact subgroup K_M, $\overline{J}(\tilde{V}^{K_M}) = (J(V)^{K_M})^*$, or equivalently, $\tilde{V}_*^K = (V_*^K)^*$. But this follows from the (nearly obvious) fact that $(V^K)^* = (V_0^K)^* \oplus (V_*^K)^*$.

Second, we must check that the normalizations cancel out so that we get a nice pairing between $\overline{r}_{M,G}(\tilde{\sigma})$ and $r_{M,G}(\sigma)$. This amounts to finding a *cannonical* isomorphism $\delta_{\overline{P}} \otimes \delta_P \cong \mathbb{C}$. This can be done (but we will not discuss it). $\square$

REMARK. For admissible representations, theorem 21 was proved by Casselman using (the original version of) Jacquet's lemma. For the more general result, the stabilization theorem is needed.

IMPORTANT COMMENT ON THEOREM 20. The proof of Theorem 20 as stated is now complete. However, at the end of the last section we introduced a specific map

$$\beta\colon \operatorname{Hom}_G(i_{G,M}(\tau), \pi) \to \operatorname{Hom}_M(\tau, \overline{r}_{M,G}(\pi))$$

coming from our basic geometric lemma. We would like to see that this is the same as the isomorphism above. It will be sufficient if we show that, for each M-module τ, the pairing between $\overline{r}_{M,G} i_{G,M}(\tilde{\tau})$ and $r_{M,G} i_{G,M}(\tau)$ determined by the basic geometric lemma coincides with the one coming from theorem 21. We will use the notation $A = \overline{r}_{M,G} i_{G,M}(\tilde{\tau})$, $B = r_{M,G} i_{G,M}(\tau)$, and $\langle a, b\rangle_{\mathrm{BGL}}$ and $\langle a, b\rangle$ for the pairings coming from the basic geometric lemma and theorem 21, respectively.

The basic geometric lemma gives filtrations of A and B. For example, $\tilde{\tau}$ is a submodule of A and τ is a quotient of B; $\langle a, b\rangle_{\mathrm{BGL}}$ induces the canonical pairing on τ and $\tilde{\tau}$. Similarly, a submodule of $A/\tilde{\tau}$ is paired with a quoteint of $\operatorname{Ker}(B \to \tau)$ etc. It is obvious that the pairings on the filtrations determines $\langle a, b\rangle_{\mathrm{BGL}}$. Thus, it is enough to check that $\langle a, b\rangle$ respects the filtrations in the same way. We will do this at the first stage: $\tilde{\tau} \hookrightarrow A$. The others are similar.

Let $X = G/P$. Recall that $\tilde{\tau}$ and τ may be realized as G-equivariant sheaves on X. Then $i_{G,M}(\tilde{\tau})$ and $i_{G,M}(\tau)$ are the spaces of compactly supported sections of certain sheaves $\tilde{\mathcal{F}}$ and $\mathcal{F}$. (See section III.1.1.) A and B are then quotients of these spaces. The elements $\tilde{\tau} \hookrightarrow A$ are those which come from sections supported on the image of $\overline{U}$ in X.

The stalks of $\tilde{\mathcal{F}}$ and $\mathcal{F}$ are in duality. We will write $\langle\cdot,\cdot\rangle_{\mathcal{F}}$. The pairing between $i_{G,M}(\tilde{\tau})$ and $i_{G,M}(\tau)$ is

$$\int_G \langle\cdot,\cdot\rangle_{\mathcal{F}}.$$

For $a \in A$ and $b \in B$, $\langle a, b\rangle$ may be described roughly as follows: pick special liftings of a and b (coming from Jacquet's lemma), s_a and s_b to $i_{G,M}(\tilde{\tau})$ and $i_{G,M}(\tau)$ and then

$$\langle a, b\rangle = \int_G \langle s_a, s_b\rangle_{\mathcal{F}}.$$

(This is not quite accurate because really we must work with K-invariant elements where K is in good position with respect to $(P, \overline{P})$. However, since we may take arbitrarily small K, this is essentially correct.)

Suppose that $a \in \tilde{\tau} \subset A$ and $b \in B$. Let s_a and s_b be any sections of $\tilde{\mathcal{F}}$ and $\mathcal{F}$ which project to a and b. Then it is not hard to see that

$$\langle a, b\rangle_{\mathrm{BGL}} = \int_{\overline{U}} \langle s_a, s_b\rangle_{\mathcal{F}}.$$

This is independent of the choice of s_a and s_b. Moreover, since s_a is supported on $\overline{U}$, it would be the same to write this as an integral over G. But then it is precisely the same as $\langle a, b\rangle$.

3.3. Proof of Stabilization. We begin with some terminology. Recall that a map $t\colon V \to V$ is *stable* if $V = V_0 \oplus V_*$ with $tV_0 = 0$ and t invertible on V_*. We say that t is *eventually stable* if some power of t is stable. It turns out that the direct sum of two stable modules is stable. Moreover, if φ intertwines two stable maps, (t, V) and (t', V'), then $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are also stable. In other words, *stable modules form an abelian category.*

Recall that if B is a commutative Noetherean algebra, a (G, B)-module is a G-module which is also a B-module so that the actions commute. It follows that if V is a (G, B)-module and $K \subset G$ is a compact subgroup, then V^K is again a B-module. We say that V is *B-admissible* (or just *admisible* where there is no confusion) if for each compact $K \subset G$, V^K is a finitely generated B-module. Exactly as for the usual concept of admissible, we can prove

Lemma 32. *If $M \subset G$ is a Levi subgroup, then the functor*

$$i_{G,M}\colon \mathcal{M}(M,B) \to \mathcal{M}(G,B)$$

maps admissible modules to admissible modules.

Here $\mathcal{M}(G,B)$ is the category of smooth (G,B)-modules. We will need the following algebraic lemma.

Lemma 33. *Let B be a commutative Noetherean algebra, L a finitely generated B-module, and α an element of $\mathrm{End}_B(L)$. If L localized at α is a finitely generated B-module, then L is eventually stable with respect to α.*

Proof. Recall the notation (L_α, A) for L localized at α. By the definition of localization, we may consider $A^{-1} \in \mathrm{End}_B(L_\alpha)$. If L_α is a finitely generated B-module, then so is $\mathrm{End}_B(L_\alpha)$. Consequently, there is an equation

$$A^{-i} + b_1 A^{-i+1} + \cdots + b_i = 0,$$

which implies $A^{-1} = b_1 + b_2 A + \cdots + b_i A^{i-1}$. Thus, $A^{-1} \in B[A]$.

We claim that the map $L \to L_\alpha$ is onto. Indeed, set $L' = \mathrm{Im}(L) \subset L_\alpha$. Then L' is A-invariant. But $A^{-1} \in B[A]$ and so L' is also A^{-1}-invariant. Thus, $L_\alpha = \bigcup_n A^{-n}(L') = L'$.

Let κ be the kernel of the map $L \to L_\alpha$. By definition each element of κ is killed by some power of α. Moreover, since both L and L_α are finitely generated B-modules, so is κ. This implies that there is a constant m so that $\alpha^m \kappa = 0$. We claim that L is α^m-stable. Set $I = \mathrm{Im}\,\alpha^m$. We must show that α^m is invertible on I. This follows from the fact that the map $L \to L_\alpha$ is onto because then $(I, \alpha^m) \cong (L_\alpha, A^m)$ and A^m is invertible. $\square$

Now we turn to the stabilization theorem. Recall that $\Lambda(M,K)^{++}$ is the set of λ that are strictly dominant with respect to the pair (P,K), which are in good position (section II.2.1). The statement is

Theorem 22. *Let $\lambda \in \Lambda(M,K)^{++}$. Let $a = a(\lambda)$. Then there exists a constant $b = b(G,K)$ depending only on G and K, such that for any $V \in \mathcal{M}(G)$, a^b is stable on V^K. Moreover, we may choose $b \leq c(G,K)$.*

Proof. First, we prove the theorem for the special case of the representation $\Pi = i_{G,M}(\Pi(D))$ introduced in section 3.1. The key lemma is nothing more than our algebraic lemma (number 33) together with ($\bigstar$) from the last section (3.2).

Lemma 34. *Let V be an admissible (G,B)-module and suppose that $r_{M,G}(V)$ is an admissible (M,B)-module. Then V^K is eventually stable with respect to a.*

PROOF. Set $K_M = K \cap M$. By ($\bigstar$) from the last section (which does not require admissibility), the localization $(V^K)_a \cong J_U(V)^{K_M}$, which our assumptions insure is a finitly generated B-module. The result now follows from lemma 33. $\square$

Let D be a cuspidal component associated with some irreducible cuspidal (and hence admissible) representation ρ of a standard Levi subgroup N. Recall from section II.3.3 the module $\Pi(D) = F \otimes \rho$ where F is the algebra of functions on the variety of unramified characters on N. Clearly, $\Pi(D)$ is an admissible (N, F)-module. So, by lemma 32, $\Pi = i_{G,N}(\Pi(D))$ is an admissible (G, F)-module. Moreover, as $\Pi(D)$ is cuspidal, the basic geometric lemma implies that $r_{M,G}(\Pi)$ has a filtration with quotients $i_{M,N_w} \circ w(\Pi(D))$. Therefore, $r_{M,G}(\Pi)$ is an admissible (M, F)-module. We may now apply lemma 34 and conclude that Π *is eventually stable with respect to* a.

We have proved that there exists a constant b so that Π is a^b-stable. In other words, $\Pi = \Pi_0 \bigoplus \Pi_*$ so that $a^b\Pi_0 = 0$ and a^b is invertible on Π_*. We would like to show that $b \leq c$. It suffices to show that $a^c\Pi_0 = 0$.

As Π is a (G, F)-module, it is convenient to think of it as a collection of representations indexed by the points of $\Psi(N)$; i.e. by the unramified characters of N. For $\psi \in \Psi$, $\Pi_\psi = \Pi \otimes_F F/m_\psi \cong i_{G,N}(\psi\rho)$, where $m_\psi \subset F$ is the maximal ideal of functions vanishing at ψ. *In the next chapter (section IV.1.2), we will use analytic techniques to prove*

LEMMA 35. *For almost all $\psi \in \Psi$, Π_ψ is irreducible.*

As irreducible representations are admissible, $\dim \Pi_\psi^K \leq c$ for almost all $\psi \in \Psi$. In particular, almost all Π_ψ are a^c-stable. Thus, for almost all ψ, the image of Π_0 under the natural projection $\Pi_0 \subset \Pi \twoheadrightarrow \Pi_\psi$ is killed by a^c. It follows that $a^c\Pi_0 = 0$. Thus, $b \leq c$.

Having established the theorem for Π, the next step is all induced representations. Let $M \subset G$ be a Levi subgroup and D a cuspidal component associated to M. Suppose that L is an M-module. Then we claim that $i_{G,M}(L)$ is a^c stable.

Let $\Pi(D)$ be the projective generator for $\mathcal{M}(D)$ discussed in chapter II (and above). Then, for some index sets α and β, there is an exact sequence

$$\prod_\alpha \Pi(D) \to \prod_\beta \Pi(D) \to L \to 0.$$

As induction has a left adjoint, $i_{G,M}$ commutes with products (Theorem 5). Hence there is an exact sequence

$$\prod_\alpha \Pi \to \prod_\beta \Pi \to i_{G,M}(L) \to 0$$

where $\Pi = i_{G,M}(\Pi(D))$. We have thus presented $i_{G,M}(L)$ as the cokernel of two a^c-stable modules; hence, $i_{G,M}(L)$ is also a^c-stable.

Finally, we consider the general case. If V is any G-module, then $V = \bigoplus_\Omega V(\Omega)$ (section 2.2). Thus, we may assume that $V \in \mathcal{M}(\Omega)$ where Ω is a fixed component coming from a cuspidal component, say D, associated with a Levi subgroup, say M. It is a feature of this situation (lemma 29) that we may embed V in a sum of induced modules, say

$$V' = \bigoplus_{N \sim M} i_{G,N}(L_N).$$

Similarly, we may embed the cokernel of the map $V \to V'$ in a sum of induced modules, say V''. Notice that the theorem holds for V' and V''. Therefore, the exact sequence

$$0 \to V \to V' \to V''$$

realizes V as the kernel of a map between two a^c-stable modules. Hence, the theorem holds for V. □

Remark. Consider the sequence of left ideals $I_n = \mathcal{H}_K a(\lambda^n)$. The stabilization theorem implies that the sequence $I_0 \supset I_1 \supset I_2 \supset \cdots$ is stable after $n = c(G, K)$. In other words,

$$I_n = I_c \text{ for } n \geq c \qquad (*)$$

Conversely, this statement implies the stabilization theorem: Let V be any smooth representation of G. Then (*) immediately implies that in V^K $\operatorname{Ker} a(\lambda^n) = \operatorname{Ker} a(\lambda^c)$ and $\operatorname{Im} a(\lambda^n) = \operatorname{Im} a(\lambda^c)$ for $n \geq c$, as claimed.

It is natural to ask whether we may use this equivalence to find a reasonable bound for c. (Recall that so far our only bound on c relies on proposition 20 and is very excessive.) As (*) is a purely geometrical statement, one might hope for a purely geometrical proof which would also provide the estimate. Bernstein has found this for GL(2), but not for other groups.

One may proceed more intrinsically (i.e. without choosing λ) as follows: given a compact subgroup $C \subset U$, consider the ideal $I_C \subset \mathcal{H}_K$ given by $I_C = e_K \mathcal{H} e_C e_K$. It is easy to check that the stabilization theorem is equivalent to the statement that I_C is independent of C for C sufficintly large. The problem now becomes to find a relatively small subgroup C which gives the minimal ideal I_C.

4. The Category $\mathcal{M}(\Omega)$

4.1. The Projective Generator. Let Ω be a component of $\Omega(G)$. In section III.2.1 we associated to Ω (not uniquely) a pair (M, D) where M is a Levi subgroup, D is a cuspidal component of M, and Ω is the quotient of D by a finite group.

Recall that $\Pi(D)$ is a finitely generated projective generator of $\mathcal{M}(D)$. We have defined

$$\Pi(\Omega) = i_{G,M}(\Pi(D)).$$

At this point, this seems like bad notation because (M, D) is not uniquely associated to Ω. We will eventually show that this does not matter.

Having proved the stabilization theorem, we have now completed the proof that $\overline{r}_{M,G}$ is right adjoint to $i_{G,M}$ and hence that $i_{G,M}$ maps projective objects to projective objects. We already knew that it maps finitely generated objects to finitely generated objects. Consequently, $\Pi(\Omega) \in \mathcal{M}(\Omega)$ is a finitely generated projective object. We would like to say that it also generates the category. In this direction we have:

Proposition 34. *Consider all pairs (N, D) which yield Ω. Then $\{i_{G,N}(\Pi(D))\}$ is a set of generators for the category $\mathcal{M}(\Omega)$.*

Proof. Since these objects are projective, it is enough to show that any irreducible representation, $\pi \in \mathcal{M}(\Omega)$, is a quotient of one of the $i_{G,N}(\Pi(D))$.

Choose a Levi subgroup N so that $\overline{r}_{N,G}(\pi)$ is cuspidal. By the second adjunction, associated to the identity map $\overline{r}_{N,G}(\pi) \to \overline{r}_{N,G}(\pi)$ there is an adjunction morphism

$$i_{G,N}\overline{r}_{N,G}(\pi) \to \pi.$$

This map is surjective. As π is irreducible, there is a $\rho \in \mathrm{JH}(\overline{r}_{N,G}(\pi))$ so that

$$i_{G,N}(\rho) \twoheadrightarrow \pi.$$

Let D be the cuspidal component of N containing ρ. Then it is clear that (N, D) yields Ω. Moreover, $\rho \in \mathcal{M}(D)$ so there is a surjection $\Pi(D) \twoheadrightarrow \rho$. Using the exactness of i, we get a surjection

$$i_{G,M}\Pi(D) \twoheadrightarrow i_{G,M}\rho.$$

Putting the last two maps together,

$$i_{G,N}(\Pi(D)) \twoheadrightarrow \pi.$$

□

In the next chapter we will prove:

Proposition 35. *If (M, D) and (N, D') are associated with the same component Ω, then $i_{G,M}(\Pi(D))$ and $i_{G,N}(\Pi(D'))$ are (not cannonically) isomorphic. Thus, $\Pi(\Omega)$ is a well defined elemetnt of $\mathcal{M}(\Omega)$.*

Putting these propositions together gives

Theorem 23. $\Pi(\Omega)$ *is a finitely generated projective generator of $\mathcal{M}(\Omega)$ and thus, by our general lemma 22, $\mathcal{M}(\Omega) \cong {}^{r}\mathcal{M}(\Lambda)$ where $\Lambda = \Lambda(\Omega) = \mathrm{End}(\Pi(\Omega))$.*

REMARK. Using Frobenius reciprocity,

$$\begin{aligned}\Lambda = \mathrm{End}(\Pi(\Omega)) &= \mathrm{Hom}(i_{G,M}(\Pi(D)), i_{G,M}(\Pi(D))) \\ &= \mathrm{Hom}(r_{M,G} i_{G,M}(\Pi(D)), \Pi(D))\end{aligned}$$

But using the basic geometric lemma, we see that this has a filtration with quotients $\mathrm{Hom}(w\Pi(D), \Pi(D))$ where $w \in W(M)$. Observe that when $wD \neq D$, $\mathrm{Hom}(w\Pi(D), \Pi(D)) = 0$. Thus, Λ has a canonical filtration with quotients $\mathrm{Hom}(w\Pi(D), \Pi(D))$ where $w \in W(M, D)$. In section 5.1 we will give an interpretation of this in terms of intertwining operators.

4.2. The Center.

DEFINITION 26. Let $\mathcal{A}$ be an abelian category. The *center* of $\mathcal{A}$ is $\mathrm{Center}(\mathcal{A}) = \mathrm{End}(\mathrm{Id}_{\mathcal{A}})$ where Id is the identity operator. More precisely, an element $\varphi \in \mathrm{Center}(\mathcal{A})$ is a set of maps $\varphi_X \colon X \to X$ for $X \in \mathrm{Ob}(\mathcal{A})$ so that for all $\alpha \colon X \to Y$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "\alpha"] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\varphi_Y"] & Y \end{tikzcd}$$

The next result explains the notation:

CLAIM. *If $\mathcal{A} = \mathcal{M}(H)$ for some algebra H with unit, then* $\mathrm{Center}\,\mathcal{A} = \mathrm{Center}\,H$.

PROOF. As H is an algebra with unit, any morphism of left H modules is multiplication on the right by an element of H. In particular, multiplication by $c \in \mathrm{Center}(H)$ defines an endomorphism of $\mathrm{Id}_{\mathcal{A}}$. Thus there is a map $\mathrm{Center}(H) \to \mathrm{Center}(\mathcal{A})$.

Conversly, if $\varphi \in \mathrm{Center}(\mathcal{A})$, then, by what we said above, $\varphi \colon H \to H$ is multiplication by some $c \in H$. Of course, c commutes with multiplication by other elements of H so $c \in \mathrm{Center}(H)$. Moreover, it follows from the commutative diagram that φ_X is also multiplication by c for any H module X. $\square$

It is clear that the same proof will work if we assume only that H is an idempotented algebra, so long as we restrict our attention to non-degenerate modules, as we do in the definition of $\mathcal{M}(H)$. More precisely:

CLAIM. *If H is an idempotented algebra,* $\mathrm{Center}(\mathcal{M}(H)) = \{a \colon H \to H$ *which commute with the left and right action of* $H\}$.

Our goal is to prove:

THEOREM 24. *Let Ω be a connected component of the variety of cuspidal data $\Omega(G)$. Then* $\mathrm{Center}(\mathcal{M}(G))$ *and* $\mathrm{Center}(\mathcal{M}(\Omega))$ *may be described as the set of regular functions on $\Omega(G)$ and on Ω, respectively.*

NOTATION: $\mathcal{O}(\Omega) =$ regular functions on Ω.

The point of this theorem is that it implies that there is a large supply of central elements. To see why this is important, consider the situation in real groups. Here, a main tool is to pass to the universal enveloping algebra which has a large center, including, for example, the Laplacians. These elements are useful for decomposing representations.

In our case, this theorem will provide a large center. Unfortunately, it is not as explicit as in the real case where we have a description in terms of the group itself (the universal enveloping algebra) rather than the representations. It would be interesting to describe, to the extent possible, $\mathrm{Center}\,\mathcal{M}(G)$ directly in terms of G. There are some remarks in this direction at the end of this section.

Preliminaries.

Let (M, D) be a Levi subgroup together with a cuspidal component. Recall that $\Pi(D)$ is a projective generator for $\mathcal{M}(D)$, and that that $\Lambda(D) = \mathrm{End}(\Pi(D))$ may be described as follows: Suppose $D = \{\psi\rho | \psi \in \Psi(M)\}$. Let $\mathcal{G}$ be the finite subgroup of $\Psi(M)$ consisting of ψ so that $\psi\rho \cong \rho$. For each $\psi \in \mathcal{G}$, pick an intertwining operator $\nu_\psi\colon \psi\rho \to \rho$. Then

$$\Lambda(D) \cong \bigoplus_{\psi\in\mathcal{G}} F\nu_\psi$$

where $F = \mathcal{O}(\Psi(M))$. Moreover, if $f \in F$ and $\psi \in S$, then $f\nu_\psi = \nu_\psi\tilde{f}$ where $\tilde{f}$ is f translated by ψ. In particular, the center of $\Lambda(D)$ is $F^{\mathcal{G}}$.

Let Ω be the component of $\Omega(G)$ associated to (M, D). There are surjections

$$\Psi(M) \twoheadrightarrow D \twoheadrightarrow \Omega = D/W(M, D)$$

(see end of section 2.1). Correspondingly, there are natural identifications

$$\mathcal{O}(\Omega) = \mathcal{O}(D)^{W(M,D)} \text{ and,}$$
$$\mathcal{O}(D) = \mathcal{O}(\Psi)^{\mathcal{G}} = F^{\mathcal{G}} = \mathrm{Center}(\Lambda(D)).$$

Proof of the Theorem.

By far the deepest fact used in this proof is that $\Pi = i_{G,M}\Pi(D)$ is a projective generator for ${}^r\mathcal{M}(\Omega)$. Thus, $\mathcal{M}(\Omega) \cong \mathcal{M}(\Lambda)$ where $\Lambda = \mathrm{Hom}(\Pi, \Pi)$, and so it is enough to show $\mathrm{Center}(\Lambda) = \mathcal{O}(\Omega)$.

As we have

$$\Lambda = \mathrm{Hom}(\Pi, \Pi) = \mathrm{Hom}(i_{G,M}(\Pi(D)), i_{G,M}(\Pi(D))),$$

it is obvious that

$$\Lambda(D) \hookrightarrow \Lambda.$$

It is easy to see that $\mathrm{Center}(\Lambda) \subset \mathrm{Center}(\Lambda(D))$. Therefore, when computing the center of Λ, we may work in the category of $\Lambda(D)$ bi-modules and compute instead $\mathcal{C}_{\Lambda(D)}(\Lambda)$. Here our notation is that, if A is a bi-module over R, then $\mathcal{C}_R(A) = \{\beta \in R \mid \beta a = a\beta \, \forall a \in A\}$ is the commutant.

By Frobenius reciprocity,

$$\Lambda = \mathrm{Hom}(r_{M,G} i_{G,M}(\Pi(D)), \Pi(D)).$$

By the basic geometric lemma, there is a filtration of $r_{M,G} i_{G,M}(\Pi(D))$ with quotients $w\Pi(D)$, where $w \in W(M)$. As these modules are projective, this filtration induces one on Hom. In other words, there is a filtration of Λ with quotients

$$\Lambda_w = \mathrm{Hom}(w\Pi(D), \Pi(D)).$$

Of course, here we only need to consider $w \in W(M, D)$ (see end of section 4.1). Observe that Λ and Λ_w are $\Lambda(D)$ bi-modules in an obvious way, and the filtration holds in this category. Moreover, it is obvious that Λ_w is isomorphic to $\Lambda(D)$ except that the right action is twisted by w. It follows that

$$\bigcap_w \mathcal{C}_{\Lambda(D)}(\Lambda_w) = \mathcal{O}(D)^{W(M,D)} = \mathcal{O}(\Omega).$$

To deduce from this information about Λ, we use the following simple algebraic lemma.

Lemma 36. *Let R be a ring and consider the category, $\mathfrak{C}$, of R bi-modules. Suppose that $A \in \mathrm{Ob}\,\mathfrak{C}$ has a filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n \supset \emptyset$ with quotients $C_0, \ldots, C_{n-1}, C_n = A_n$. Suppose that $i \neq j$ implies that $\mathrm{Mor}(C_i, C_j) = 0$. Then $\mathcal{C}_R(A) = \bigcap_{i=0}^n \mathcal{C}_R(C_i)$.*

Proof. By induction on n, we may assume that $\mathcal{C}_R(A_1) = \bigcap_{i=1}^n \mathcal{C}_R(C_i)$. There is a short exact sequence

$$0 \to A_1 \to A \to C_0 \to 0.$$

It is obvious that $\mathcal{C}_R(A) \subset \mathcal{C}_R(A_1) \cap \mathcal{C}_R(C_0)$. Conversely, if $\beta \in \mathcal{C}_R(A_1) \cap \mathcal{C}_R(C_0)$, then there is a map $a \mapsto \beta a - a\beta$, from $C_0 \cong A/A_1 \to A_1$. This must be trivial as our assumption implies that $\mathrm{Mor}(C_0, A_1) = 0$. Hence, $\beta \in \mathcal{C}_R(A)$. $\square$

To apply the lemma and hence prove the theorem, it only remains to observe that $w \neq w'$ implies $\mathrm{Mor}(\Lambda_w, \Lambda_{w'}) = 0$ in the category of $\Lambda(D)$ bi-modules. This is clear.

Remark. We may construct an explicit map from the center of the category $\mathcal{M}(\Omega)$ to $\mathcal{O}(\Omega)$ as follows. If $\varphi \in \mathrm{Center}(\mathcal{M}(\Omega))$, then for each $X \in \mathrm{Ob}\,\mathcal{M}(\Omega)$

there is a map $\varphi_X\colon X \to X$. In particular, let us consider the objects $\Pi_\phi = i_{G,M}(\phi\rho)$ where $\phi \in \Psi(M)$. These representations may all be realized on the same space, say E. One can show that this determines a holomorphic map $z(\varphi)\colon \Psi \to \mathrm{End}(E)$; c.f. section 5.2. However, by lemma 35 almost all of the $i_{G,M}(\phi\rho)$ are irreducible. Hence, there is a dense open set of Ψ whose image in $\mathrm{End}(E)$ is contained in the constants. As this is a closed condition, we have $z(\varphi)\colon \Psi \to \mathbb{C}$. Furthermore, it is not difficult to show (see section 5.1) that this function descends to a regular function $z(\varphi)\colon \Omega \to \mathbb{C}$. Thus, we have a map $z\colon \mathrm{Center}(\mathcal{M}(\Omega)) \to \mathcal{O}(\Omega)$. It is quite easy to show that this map is injective. It is not so clear that its image is large. However, it follows from theorem 24 that z is actually an isomorphism.

The Infinitesimal Character.

Using theorem 24, we can give another interpretation of $\Omega(G)$. Set $\mathcal{Z}(G) = \mathrm{Center}(\mathcal{M}(G)) = \mathrm{End}_{G\times G}(\mathcal{H}(G))$. We have shown that $\mathcal{Z} = \mathcal{O}(\Omega(G))$. Turning this around, we get

$$\Omega(G) = \mathrm{Spec}\,\mathcal{Z}(G).$$

With this interpretation, it makes sense to call the map

$$\mathrm{Irr}\,G \to \Omega(G)$$

the *infinitesimal character*, in analogy with the real case.

Further Remarks.

As we discussed, in the representation theory of real groups, there are two descriptions for the center: one in terms of the representations and, the other directly in terms of the group as the center of the universal enveloping algebra. In our case, however, we have only the description in terms of representations. Here are some ideas for giving a more explicit description.

Let $S^*(G)^{\text{inv.}}_{\text{e.c.}}$ be the set of distributions $\mathcal{E}$ which are (1) invariant with respect to conjugation, and (2) essentially compact. That is, for $h \in \mathcal{H}(G)$, $h * \mathcal{E} = \mathcal{E} * h$ has compact support.

Claim. $S^*(G)^{\text{inv.}}_{\text{e.c.}} \cong \mathrm{End}_{G\times G}(\mathcal{H}(G)) = \mathrm{Center}(\mathcal{M}(G))$.

Proof. In one direction the morphism is $\mathcal{E} \in S^*(G)^{\text{inv.}}_{\text{e.c.}}$ goes to the map $\mathcal{H} \to \mathcal{H}$ given by $h \mapsto \mathcal{E} * h$. Conversely, given $\alpha\colon \mathcal{H} \to \mathcal{H}$, we use $\mathcal{H} \cong S(G)$ and set $\mathcal{E}(f) = \alpha(f)(e)$. $\square$

Problem: Explicitly describe some elements of $S^*(G)^{\text{inv.}}_{\text{e.c.}}$.

Example. $G = \mathrm{SL}(n, \mathbf{F})$. Let $\psi\colon \mathbf{F}^* \to \mathbb{C}^*$ be a non-trivial additive character of $\mathbf{F}$. Consider $\mathcal{E}(g) = \psi(\mathrm{tr}\, g)$. Then $\mathcal{E}(g)$ is clearly invariant. It can by shown by a computation that it is also esssentially compact. [More generally, if φ is a locally constant function on $\mathbf{F}^*$ such that the average over C of φ is zero for some open compact subgroup $C \subset \mathbf{F}^*$, then $\varphi(\mathrm{tr}\, g)$ is essentially compact.]

5. Applications

5.1. Intertwining Operators. [THIS SECTION NEEDS SOME WORK!]

Fix $\omega \in \Omega$. Let us consider the irreducible representations of G with infinitesimal character ω, say $\{\pi_\alpha\}_{\alpha\in S}$. Proposition 30 states that S is finite. Let $(M_1,\rho_1),(M_2,\rho_2),\ldots,(M_r,\rho_r)$ be the cuspidal data which determine ω. By definition, for each $\alpha \in S$, there is an i with $\pi_\alpha \hookrightarrow i_{G,M_i}(\rho_i) \stackrel{\text{def}}{=} \tau_i$. Furthermore, it follows from proposition 35 (which will be proved in the next chapter) that for this we may choose an M and cuspidal component D and consider only those cuspidal data with $M_i = M$ and $\rho_i \in D$. As usual, let $W = W(M,D)$ be the subgroup of the Weyl group of G which preserves M and D. The action of W permutes the ρ_i transitively. It is natural to ask how the action of W is reflected on the π_α, or at least the τ_i. The goal of this section is to construct wherever possible intertwining maps between the τ_i corresponding to the $w \in W$. We will present two approaches to this construction, and then show that they are equivalent.

Remark. Let $\Pi = \Pi(\Omega)$. As follows from the results of sections 3.3 and 4.1, the representations $i_{G,M}(\rho_i)$ have the form Π_{ψ_i} for some ψ_i. In particular, they are generically irreducible (lemma 35). In other words, for almost every ω, the maps $\pi_\alpha \to \tau_i$ are all isomorphisms.

In the first approach, we exploit the realization of $\mathcal{M}(\Omega)$ as the category of (right) modules of $\Lambda = \mathrm{Hom}(\Pi,\Pi)$. Roughly speaking, we will find that associated to each $w \in W$, there is a canonical element A_w in a certain localization of Λ. Multiplication by A_w gives the intertwining operator between various quotients of Π. The fact that these are not always well defined is a reflection of the fact that A_w is *not* in Λ; it has a denominator.

As we saw in section 4.2, Frobenius reciprocity and the basic geometric lemma imply that

$$\Lambda = \mathrm{Hom}(i_{G,M}\Pi(D), i_{G,M}\Pi(D))$$

has a canonical filtration with quotients

$$\Lambda_w = \mathrm{Hom}(w\Pi(D), \Pi(D)).$$

Here $w \in W$. The filtration holds in the category of $\Lambda(D)$ bi-modules. Moreover, as a bi-module Λ_w is isomorphic to $\Lambda(D)$ with the right action twisted by w. It is necessary to be somewhat more precise here. Recall that $\Pi(D) = F \otimes \rho$. The choice of ρ is somewhat arbitrary. According to our philosophy (section II.3.3), we should view of $\Pi(D)$ as a collection of representations of M parametrized by the points of $\Psi(M)$. Thus, we should think of the choice of ρ as a choice of base point. From this point of view, passing from $\Pi(D)$ to $w\Pi(D)$ has two effects. First, w acts on Ψ in the obvious way. Second, the basepoint changes from ρ to $w\rho$.

Let $A_w \in \Lambda_w$ be the canonical map $w\Pi(D) \to \Pi(D)$. Unless $w = 1$, $A_w \notin \Lambda$. On the other hand, let $\mathcal{K}$ be the field of fractions of $F \subset \Lambda(D)$ (see section II.3.3), and set $\Lambda_{\mathcal{K}} = \Lambda \otimes_F \mathcal{K}$ and $\Lambda_{w,\mathcal{K}} = \Lambda_w \otimes_F \mathcal{K}$.

Claim. *As $\Lambda(D)$ bi-modules*

$$\Lambda_{\mathcal{K}} = \bigoplus \Lambda_{w,\mathcal{K}}.$$

Proof. For the moment, let us work in the catogory of F bi-modules. $\Lambda_{\mathcal{K}}$ is a vector space over $\mathcal{K}$ on the right; denote this action by $r(k)(\lambda)$ for $k \in \mathcal{K}$ and $\lambda \in \Lambda_{\mathcal{K}}$. Denote the left action of F by $l(f)(\lambda)$. Let $\mathcal{G}$ be defined as in section II.3.3. Denote the action of $\mathcal{G}$ on F by $g\colon f \mapsto f^g$ for $g \in \mathcal{G}$ and $f \in F$. Similarly, we write $f \mapsto f^w$ for $w \in W$.

For each $w \in W$ and $g \in \mathcal{G}$, let $\iota(g, w)\colon F \to \mathcal{K}$ denote the injection $f \mapsto (f^g)^w$. It is important that these are all distinct. We have an action of F on a $\mathcal{K}$ vector space and it is a general fact that eigenspaces corresponding to distinct characters split off as direct summands. More precisely, set $\Lambda_{\mathcal{K}}^{(g,w)} = \{\lambda \in \Lambda_{\mathcal{K}} | l(f)(\lambda) = r(\iota(g, w^{-1})(f))(\lambda)\, \forall f \in F\}$. Then $\bigoplus_{(g,w)} \Lambda_{\mathcal{K}}^{(g,w)} \subset \Lambda_{\mathcal{K}}$. Moreover, it is clear from our explicit description of $\Lambda(D)$ as an F module and the filtration on Λ that this is an equalty. Finally, we return to the setting of $\Lambda(D)$ bi-modules. It is clear that the action of $\Lambda(D)$ merely permutes the $\Lambda_{\mathcal{K}}^{(g,w)}$ with w fixed and that $\bigoplus_g \Lambda_{\mathcal{K}}^{(g,w)}$ is isomorphic (as a $\Lambda(D)$ bi-module) to $\Lambda_{w,\mathcal{K}}$. Thus $\Lambda_{\mathcal{K}} = \bigoplus \Lambda_{w,\mathcal{K}}$ as needed. $\square$

As follows from the claim, we may regard the A_w as canonical elements of $\Lambda_{\mathcal{K}}$. To see how this leads to an intertwining operator, think of an element $\lambda \in \Lambda$ as a family of morphisms $\lambda_\psi\colon \Pi \to \Pi_\psi$ in the obvious way: first apply λ to Π and then quotient by $m_\psi \Pi$. Here the notation is as in section II.3.3: $m_\psi \subset F$ is the maximal ideal vanishing at $\psi \in \Psi$. Notice that $\Pi/m_\psi\Pi \cong i_{G,M}(\psi\rho)$ where ρ is the "base point", as discussed above.

Obvoisly, there is a polynomial P in F so that $PA_w \in \Lambda$. We have,

Proposition 36. *The kernel of $(PA_w)_\psi$ contains $m_{w^{-1}(\psi)}\Pi$. Furthermore, PA_w changes the base point from ρ to $w\rho$,*

Proof. Clearly, an element $\alpha \in \Lambda$ is in the kernel if $(PA_{w^{-1}})_\alpha \in m_\psi\Pi$. Thus, the statement follows from the fact that $A_w \in \Lambda_{w,\mathcal{K}} \cong \bigoplus_g \Lambda_{\mathcal{K}}^{(g,w)}$. But by definition, elements $\lambda \in \Lambda_{\mathcal{K}}^{(g,w)}$ satisfy $\lambda f = (f^{g^{-1}})^w \lambda$ for $f \in F$. Using the definition of $\mathcal{G}$, this implies that they map $m_{w^{-1}(\psi)}$ to m_ψ. The first statement of the proposition follows at once. The second statement is clear. $\square$

It is now clear that whenever P does not vanish at $w(\psi)$, A_w determines a non-zero intertwining operator from $i_{G,M}(\psi)$ to $i_{G,M}(w\psi)$. It is useful to change the point of view slightly. It is easy to see that we may realize each of the $i_{G,M}(\psi\rho)$

on the same space E (see section 5.2). Then A_w defines a rational map from Ψ to $\operatorname{End}(E)$; denote it by $A_w(\psi\rho)$. As long as we stay away from the poles of A_w, then $A_w(\psi\rho)$ is a canonical intertwining map from $i_{G,M}(\psi\rho)$ to $i_{G,M}(w(\psi\rho))$.
Remark. By Frobenius reciprocity, a map from $i_{G,M}(\psi\rho)$ to $i_{G,M}(w(\psi\rho))$ is the same as an element of $\operatorname{Hom}(r_{M,G}i_{G,M}(\psi\rho), w(\psi\rho))$. Of course, by the basic geometric lemma, $r_{M,G}i_{G,M}(\psi\rho)$ has a filtration with quotients $y(\psi\rho)$ for $y \in W(M)$. It is not hard to see that for generic ψ, the central characters of these quotients are all distinct (we will refer to this as the *regular* case). Thus, the filtration reduces to a direct sum. (Very similar arguments are given in section IV.1.2.) In this case, there is a canonical projection operator in $\operatorname{Hom}(r_{M,G}i_{G,M}(\psi\rho), w(\psi\rho))$, and this operator corresponds to $A_w(\psi\rho)$ under Frobenius reciprocity.

It is natural to consider the composition of A_w and $A_{w^{-1}}$. Let $c_w(\psi) = A_{w^{-1}}(w(\psi\rho))\circ A_w(\psi\rho)$. A priori, $c_w(\psi)$ is a rational map from Ψ to $\operatorname{End}(E)$. However, since for almost every ψ $i_{G,M}(\psi\rho)$ is irreducible, then for almost every (and hence for every) ψ, $c_w(\psi)$ is constant. In other words, $c_w(\psi)$ is a rational function on Ψ.

We know that that none of the operators $A_w(\psi\rho)$ is ever identically zero. As a conseqence, if for some w $c_w(\psi) = 0$, then either $i_{G,M}(\psi\rho)$ or $i_{G,M}(w(\psi\rho)$ is reducible. [NOW PROVE THAT THESE ARE THE SAME.] Conversly, suppose that $i_{G,M}(\psi\rho)$ is reducible, say $i_{G,M}(\psi\rho) \twoheadrightarrow \sigma$, with σ irreducible. But there must be some irreducible cuspidal representation of M, ρ', so that $\sigma \hookrightarrow i_{G,M}(\rho')$. Consequently, there is a map

$$\alpha\colon i_{G,M}(\psi\rho) \to i_{G,M}(\rho')$$

which is neither trivial nor an isomorphism. By Frobenius reciprocity, α corresponds to some non-zero map in $\operatorname{Hom}(r_{M,G}i_{G,M}(\psi\rho), \rho')$. By the basic geometric lemma, $\rho' = w(\psi\rho)$ for some $w \in W$. Furthermore, at least in the regular case, this implies that α is a non-zero multiple of $A_w(\psi\rho)$. Thus, $A_w(\psi\rho)$ is not an isomorphism and in particular, $c_w(\psi) = 0$. We have established

Proposition 37. *In the regular case, $i_{G,M}(\psi\rho)$ is irreducible if and only if all $A_w(\psi\rho)$ are isomorphisms, or equivalently, if and only if all $c_w(\psi) \neq 0$.*

The second (and more standard) approach to defining intertwining operators corresponding to the $w \in W$, uses a fixed realization of the $i_{G,M}(\rho, V)$ as the smooth vectors in the space $\{h\colon G \to V | h(mug) = \rho(m)\Delta(m)^{1/2}h(g)\}$ (see end of section 1.1 and section I.3.2). We are trying to define a map from $i_{G,M}(\rho, V)$ to $i_{G,M}(w\rho, V)$. Naively, we may proceed as follows: pick a representative of w in G, which we also denote by w. Consider the map $h \mapsto {}^w\hat{h}$ where ${}^w\hat{h}(g) = h(wg)$. This nearly transforms correctly under M: ${}^w\hat{h}(mg) = w\rho(m)\Delta(wmw^{-1})^{1/2}\,{}^w\hat{h}(g)$. However, unless w happens to preserve U, ${}^w\hat{h}$ does not transform correctly under left translation by $u \in U$. Once again we proceed in the naive way, namely, average ${}^w\hat{h}$ over $U/U \cap w^{-1}Uw$ so that it becomes invariant. More precisely, set

$$^w h(g) = \int_{U/U\cap w^{-1}Uw} h(wug)du.$$

Putting aside questions of convergence, it is simple to check that ${}^w h(mug) = w\rho(m)\Delta(m)^{1/2}\,{}^w h(g)$. Thus, assuming that the integral makes sense, $h \mapsto {}^w h$ defines an intertwining operator $i_{G,M}(\rho) \to i_{G,M}(w\rho)$.

The trouble is that usually the integral does not converge. One can check that it does converge if we twist ρ by a character ϕ sufficiently far into the positive Weyl chamber (see section IV.2.1). Thus, there is some region where we have a well-defined intertwining operator. Then, using a (fairly long) calculation, it can be proved that the operator is a rational function in ϕ, and hence may be extended almost everywhere. This is the usual definition of the intertwining operator; we will write $\tilde{A}_w$.

We would like to check that the A_w and $\tilde{A}_w$ coincide. As they are both rational functions, it is enough to show that they coincide in the region where the integral converges. Here, the map $r_{M,G}i_{G,M}(\psi\rho) \to w(\psi\rho)$ corresponding to $\tilde{A}_w$ under Frobenius reciprocity may be written as

$$h \mapsto \int_{U/U\cap w^{-1}Uw} h(wu)du.$$

But working through the proof of the basic geometric lemma, one checks that this is just the same canonical projection which corresponds to A_w, as in the remark above. As they correspond to the same thing under Frobenius reciprocity, A_w and $\tilde{A}_w$ must coincide.

5.2. Paley-Wiener Theorem. Let $h \in \mathcal{H}(G)$. Then, for any representation (π, V), we have $\pi(h) \in \operatorname{End} V$. It is natural to ask for a characterization of these endomorphisms. More precisely, suppose that ρ is an irreducible, cuspidal representation of a Levi subgroup M. Obviously, all $\psi\rho$ may be realized on the same space, say L. Thus, a fixed element $m \in M$ determines a map $\Psi \to \operatorname{End}(L)$. This map is clearly regular. Of course, all $\pi_\psi = i_{G,M}(\psi\rho)$ may also be realized on the same space, say E. Furthermore, since ψ is trivial on all compact subgroups of M, it is not hard to see that we may arrange for K_0 to act on E in the same way for each π_ψ. It follows from the definition of induction and what we just said that for fixed $g \in G$, the associated map $\Psi \to \operatorname{End}(E)$ is regular. Similarly, a fixed element $h \in \mathcal{H}(G)$ determines a regular map $\Psi \to \operatorname{End}(E)$. We are seeking a characterization of this map.

THEOREM 25. *Suppose that, for almost any pair M, ρ, there is a family of operators*

$$\gamma_{M,\rho} \colon (\pi, E) \to (\pi, E)$$

where $(\pi, E) = i_{G,M}(\rho)$, satisfying the following:

(1) *Fix M and consider representations $\psi\rho$. Then $\gamma_{M,\psi\rho} \colon E \to E$, defined for almost all ψ, extends to a holomorphic function in ψ, $\Psi \to \operatorname{End} E$.*
(2) *There is an open compact subgroup $K \subset G$ such that $e_K \gamma_\psi = \gamma_\psi = \gamma_\psi e_K$.*
(3) *Given M, ρ, M', ρ' and an isomorphism*

$$\alpha \colon i_{G,M}(\rho) \xrightarrow{\approx} i_{G,M'}(\rho'),$$

then $\gamma\alpha = \alpha\gamma$.

Then there is an $h \in \mathcal{H}(G)$ such that $\gamma_{M,\rho} = \pi(h)$.

As usual, the key fact is that $\mathcal{M}(\Omega) = \mathcal{M}(\Lambda)$ where $\Lambda = \operatorname{End}(\Pi(\Omega))^\circ$. We will need the following lemma from category theory.

LEMMA 37. *Let B be an algebra with unit, Π a finitely generated projective generator for $\mathcal{M}(B)$, and $\Lambda = (\operatorname{End}\Pi)^\circ$. Then Π is an B-Λ bimodule with*

$$\operatorname{End}_\Lambda \Pi = B.$$

PROOF. As Π is a projective generator, there are equivalence of categories

$$F \colon \mathcal{M}(B) \to \mathcal{M}(\Lambda)$$
$$G \colon \mathcal{M}(\Lambda) \to \mathcal{M}(B)$$

given by

$$F(M) = \operatorname{Hom}(\Pi, M)$$
$$G(N) = \Pi \otimes_\Lambda N = \operatorname{Hom}_\Lambda(\Pi^*, N),$$

where $\Pi^* = \operatorname{Hom}(\Pi, B) = F(B)$. Thus,

$$B = G(\Pi^*) = \operatorname{Hom}_\Lambda(\Pi^*, \Pi^*) = \operatorname{Hom}(\Pi, \Pi)^\circ.$$

□

Now we prove the theorem.

PROOF. As usual, we will work with Ω rather than $\Omega(G)$. Write $\mathcal{M}(G) = \prod \mathcal{M}(\Omega)$ and $\mathcal{H}(G) = \bigoplus \mathcal{H}(\Omega)$. Fix Ω.

If K is an open compact subgroup, then $\mathcal{H}_K = \bigoplus \mathcal{H}_K(\Omega)$. We may choose K so small that Ω is generated by K-invariant vectors. Then, $B = \mathcal{H}_K(\Omega)$ is an algebra with unit and $\mathcal{M}(\Omega) = \mathcal{M}(B)$. Since $\Pi = \Pi(\Omega)^K \in \mathcal{M}(B)$, the lemma implies that $B = \operatorname{End}_\Lambda(\Pi)$ where $\Lambda = \operatorname{End}\Pi$.

Now, an algebraic family of maps $a_\psi \in \mathrm{End}(\pi_\psi)^K$ determines an element of $\mathrm{End}_{\Lambda(D)} \Pi$. However, we are also assuming that these maps commute with intertwining operators, in particular, with the A_w discussed in the last section. But the A_w generate Λ over $\Lambda(D)$. Therefore, $a \in \mathrm{End}_\Lambda \Pi = B = \mathcal{H}_K(\Omega)$. In other words, $a_\psi = \pi_\psi(a)$.

$\square$

CHAPTER IV

Additional Topics

The goal of this chapter is to introduce some analytic and cohomological topics and at the same time supply proofs for two results that were used in chapter 3. The proofs were defered because they require unitary techniques which is our first topic. Next we introduce tempered representations and state Langlands classification. The first cohomological result is that $\mathcal{M}(\Omega)$ has finite cohomological dimension. Using this, we establish a cohomological duality.

The first result defered from chapter 3 is that if $\rho \in \mathrm{Irr}_c(M)$, then for almost all $\psi \in \Psi(M)$, $i_{G,M}(\psi\rho)$ is irreducible. This result was needed for the proof of the stabilization theorem (section III.3.3). The proof is in section 1.2.

The other result that we need is

THEOREM 26. *Let Ω be a connected component of $\Omega(G)$, the variety of cuspidal data up to associate. Let D be a cuspidal component for some Levi subgroup M so that Ω is a quotient of D by a finite group. Then $\Pi(\Omega) = i_{G,M}(\Pi(D))$ does not depend on D, only Ω.*

This result was used in section III.4.1 to establish that $\Pi(\Omega)$ is a generator for the category $\mathcal{M}(\Omega)$. We will call it the uniqueness theorem. The proof is in section 3.

1. Unitary Structure

1.1. Unitary Representations.

DEFINITION 27. A G-module (π, V) is *unitary* if it is equiped with a positive definite, G-invariant scaler product $\langle\cdot,\cdot\rangle \colon V \otimes \overline{V} \to \mathbb{C}$.

REMARKS. 1. We do not assume that V is complete with respect to this structure. 2. It is clear that if (π, V) is irreducible unitary, then its complex conjugate, $({}^{c}\pi, \overline{V})$, is naturally isomorphic to its contragredient, $(\tilde{\pi}, \tilde{V})$.

Proposition 38. *Let P be a parabolic subgroup of G, M the associated Levi subgroup. Suppose (ρ, W) is a unitary representation of M. Then $\pi = i_{G,M}(\rho)$ is also unitary.*

Remark. For this proposition, we must use normalized induction as in section III.1.1. For this reason, $i_{G,M}$ with this normalization is often called *unitary induction*.

Proof. Recall that vectors in π are sections of a sheaf on G/P whose fibers are isomorphic to $\delta \otimes W$, where δ is a square root of the G-invariant distributions, Δ. We define

$$\langle \xi, \nu \rangle = \int_{G/P} \langle \xi_x, \nu_x \rangle d\mu \in (\delta_x)^2 = \Delta_x.$$

□

This proposition is usefull only if we have some unitary representations to start with. Our source will be the following

Proposition 39. *Suppose G has compact center. Let (π, V) be an irreducible representation with square-integrable matrix coeffiecients. Then V has an (essentailly unique) unitary structure.*

Remark. We will call a representation with square integrable matrix coefficients a *square integrable representation*.

Proof. Recall the definition of matrix coefficients: given $\xi \in V$ and $\tilde{\xi} \in \tilde{V}$, set $m_{\tilde{\xi},\xi}(g) = \langle \tilde{\xi}, \pi(g^{-1})\xi \rangle$. For a fixed $\tilde{\xi}$ we get a G-equivariant map $\tau \colon V \to C^\infty(G)$. In our case, $\tau \colon V \to L^2(G)$ and, as V is irreducible, this is an embedding. Thus we get a unitary structure on V.

That this scaler product does not depend on the choice of $\tilde{\xi}$ follows from the following version of Schur's lemma.

Lemma 38. *If G is reductinve and V is an irreducible G-module, then up to a constant there exists no more than one G-invariant scaler product on V.*

Proof. Let V^+ be the anti-linear dual of V, that is the space of anti-linear functionals. An Hermitian scaler product on V is equivalent to a representation $V \to V^+$. As V is smooth, we may replace V^+ by its smooth part. Moreover, it is easy to see V admissible irreducible implies V^+ admissible irreducible. Thus, Schur's lemma implies that $\mathrm{Hom}(V, V^+) = \mathbb{C}$. □

□

In the proof of the proposition we did not use the assumption that G is has compact center. However, LEMMA 39. *If G has non-compact center then it has no square-integrable representations.*

PROOF. Suppose V is an irreducible representation of G. If V had square integrable matrix coefficients then, by proposition 39, it is unitary. Thus, its central character, $\chi(x)$, is unitary. That is, $|\chi(z)| = 1$ for $z \in Z(G)$. But $m_{\tilde{\xi},\xi}(zg) = \chi^{-1}(z)m_{\tilde{\xi},\xi}(g)$. Thus, $Z(G)$ not compact implies that the matrix coefficients are not square integrable. $\square$

The technical difficulty that this lemma identifies is very much like one we encountered with compact representations, and just as we introduced compact modulo center representations, here we introduce representations that are square integrable modulo center. To be precise,

DEFINITION 28. Suppose that (π, V) is a representation of G with *unitary* central character $\chi(z)$. We say that (π, V) is *square integrable modulo center* if

$$\int_{G/Z} \mid m_{\tilde{\xi},\xi}(g)|^2 d\mu < \infty$$

for all $\xi \in V$ and $\tilde{\xi} \in \tilde{V}$.

PROPOSITION 40. *Unitary representations have unitary central characters. Conversly, a representation with a unitary central character and which is square integrable modulo center is unitary.*

PROOF. The only thing to be done is to define an inner product for square integrable modulo center representations. Fix $\tilde{\xi}$; then

$$\langle \xi, \nu \rangle = \int_{G/Z} m_{\tilde{\xi},\xi} \overline{m_{\tilde{\xi},\nu}} d\mu.$$

$\square$

REMARK. Experience suggests that it is somehow more difficult to classify unitary reresentations than general ones. In particular, there are not many ways of constructing unitary representations. The only general procedure is to find a space X with a G-action. Then G acts on functions on X and we can find $V \subset L^2(X)$. However, it is not clear how to find such X. The two natural choices are $X =$ point, which gives the trivial representation, and $X = G$, which is essentially what we have considered in this section.

1.2. Applications.

PROPOSITION 41. *Let V be an admissible unitary representation of G. Then V is completely reducible. That is, $V = \bigoplus V_i$ where the V_i are irreducible unitary.*

REMARK. This is false for non-admissible representations.

PROOF. Suppose $W \subset V$ is a submodule. Then the orthogonal complement, $W^\perp \subset V$ is also a submodule and $W \cap W^\perp = 0$. It remains to check that $W + W^\perp = V$. For this it is enough to check that $W^K + (W^\perp)^K = V^K$ for all compact open subgroups $K \subset G$. But using admissibility, this follows by counting dimensions. □

This proposition gives a method for establishing that some representations are irreducible. For this reason, unitary representations can be useful for proving statements about general representations. An important example is the next theorem which was used in the proof of the stabilization theorem.

THEOREM 27. *Let ρ be a cuspidal representation of a Levi subgroup $M \subset G$. Then $\pi_\psi = i_{G,M}(\psi\rho)$ is irreducible for generic $\psi \in \Psi(M)$.*

PROOF. STEP 0: *It is enough to check that π_ψ is irreducible for some ψ.*

This is because irreducibility is an open condition. More precisely, we may realize all of the representations π_ψ on the same space E and with the same action of open compact subgroups (see section III.5.2). Then, for $K \subset G$ open compact, we have a biregular map

$$\begin{aligned} \mathcal{H}_K(G) \times \Psi(M) &\to \mathrm{End}(E^K) \\ (h, \psi) &\mapsto \pi_\psi(h). \end{aligned}$$

If for a fixed ψ this is onto, then it is onto for a zariski dense set in $\Psi(M)$. Now let K shrink.

STEP 1: *We may assume that ρ has unitary central character. That is, there is $\psi \in \Psi(M)$ such that $\psi^{-1}\rho$ has unitary central character.*

Let $Z(M)$ be the center of M and and let χ be the central character of ρ.

Now, χ is a map $Z(M) \to \mathbb{C}^*$. Thus, $|\chi|^2 \colon Z(M) \to \mathbb{R}^{+*}$. Moreover, as $\mathbb{R}^{+*}$ has no compact subgroups, $|\chi|^2$ restricted to $Z(M)^\circ$ is identically 1. Thus,

$$|\chi|^2 \colon Z(M)/Z(M)^\circ \to \mathbb{R}^{+*}.$$

We know (section II.2.1) that $\Lambda(Z(M)) = Z(M)/Z(M)^\circ \hookrightarrow \Lambda(M) = M/M^\circ$ has finite index. Thus, the set of morphisms, $\mathrm{Mor}(\Lambda(M) \to \mathbb{R}^{+*}) = \mathrm{Mor}(\Lambda(Z(M)) \to \mathbb{R}^{+*})$. In particular, there exists a map $|\psi|^2 \colon \Lambda(M) \to \mathbb{R}^{+*}$ so that $|\psi|^2 = |\chi|^2$ on $\Lambda(Z(M))$.

Let ψ be the positive square root of $|\psi|^2$. Then $|\psi^{-1}\chi|^2 = 1$ on $Z(M)$. Thus, $\psi^{-1}\chi$ is unitary and ψ is as claimed.

STEP 2. *If ρ has unitary central character, then ρ is unitary.* This follows from proposition 40 because, as ρ is cuspidal, it is compact modulo center (theorem 14) and so a fortiori, ρ is square integrable modulo center.

STEP 3. *For unitary ψ, $\pi_\psi = i_{G,M}(\psi\rho)$ is unitary and hence completely reducible.* This follows from steps 1 and 2 as well as propositions 38 and 41.

Recall that $r_{M,G}(\pi_\psi)$ has a filtration with quotients $w(\psi\rho)$ for $w \in W(M)$.

STEP 4. *For generic ψ, $w(\psi\rho)$ and $\psi\rho$ have different central characters.*

As above, set $\chi =$ central character of ρ. The central character of $\psi\rho = \chi\psi|_{Z(M)}$. Moreover, the central character of $w(\psi\rho) = w(\chi\psi|_{Z(M)})$. Since we are only interested in generic ψ and $\Lambda(Z(M))$ has finite index in $\Lambda(M)$, it is enough to show that for a generic character α of $\Lambda(Z(M))$, $\chi\alpha \neq w(\chi\alpha)$. But this follows from the fact that $W(M)$ is finite and each $w \in W(M)$ acts non-trivially on $\Lambda(Z(M))$.

STEP 5. *For generic ψ,* $\mathrm{Hom}(\pi_\psi, \pi_\psi) = \mathbb{C}$.

We need the following lemma:

LEMMA 40. *Filtrations of G-modules whose quotients have distinct central characters reduce to direct sums.*

PROOF. We obviously get a direct summand as a $Z(G)$-module. But G commutes with $Z(G)$ (of course) and so preserves the summands corresponding to distinct characters of $Z(G)$. □

Applying step 4 and the lemma, we get $r_{M,G}(\pi_\psi) = \bigoplus w(\psi\rho)$. Using Frobenius reciprocity,

$$\begin{aligned} \mathrm{Hom}(\pi_\psi, \pi_\psi) &= \mathrm{Hom}(r_{M,G}(\pi_\psi), \psi\rho) \\ &= \mathrm{Hom}(\bigoplus w(\psi\rho), \psi\rho) \end{aligned}$$

which is clearly one dimensionsal.

STEP 6. *For ψ unitary and generic, π_ψ is irreducible.* As π_ψ is completely reducible, this follows from step 5.

The theorem follows from step 0 and step 6. □

REMARK. We summarize the procedure used to show that a representation is irreducible: describe its endomorphisms using the basic geometric lemma; if this is one dimensional and if the representation is unitary then it is irreducible.

We conclude this section with two exercises which extend the result of this theorem. The first is easy; the second was worked out by Bernstein using the Langlands classification (section 2.3) and even then was difficult.

(1) Let ρ be an irreducible unitary representation of M, then $i_{G,M}(\psi\rho)$ is irreducible for generic ψ.

(2) Let ρ be *any* irreducible representation of M. Then $i_{G,M}(\psi\rho)$ is irreducible for generic ψ.

2. Central Exponents

[SEVERAL PLACES NEED WORK]

2.1. The Root System. Recall that, for any group G, $\Lambda(G) = G/G^\circ$ is a lattice with $\Lambda(Z(G))$ as a sublattice of finite index. Set

$$\mathfrak{a}(G) = \Lambda(G) \otimes_{\mathbb{Z}} \mathbb{R} = \Lambda(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}.$$

A simple but important observation is that the dual space, $\mathfrak{a}(G)^*$, satisfies

$$\mathfrak{a}(G)^* = \mathrm{Hom}(G, \mathbb{R}^{+*}).$$

Thus, for any character ψ, $|\psi| \in \mathfrak{a}(G)^*$.
Remark. In this situation, it is usual to take logarithms. Thus, $\mathfrak{a}(G)^* \cong \mathrm{Hom}(G,\mathbb{R})$ and $\log|\psi| \in \mathfrak{a}(G)^*$).

Definition 29. If π is an ireducible representation of G with central character χ_π, the *central exponent* of π is $e(\pi) = \log|\chi_\pi| \in \mathfrak{a}(G)^*$. More generally, the *set of central exponents* of π is $\{e(\sigma) \mid \sigma \in \mathrm{JH}(\pi)\}$.

In this section we study representations π with certain conditions on their central exponents, and on the central exponents of the $r_{M,G}(\pi)$. The conditions are *negative* (giving tempered representations, section 2.3) and *strictly negative* (leading to square integrable representations, section 2.2). We now explain what these mean. For simplicity, we assume that G has compact center.

Let $P_0 = M_0U_0$ be a minimal parabolic subgroup. Let P be a standard parabolic; $P = MU$, $P \supset P_0$, $M \supset M_0$. Corresponding to the injections $M_0 \hookrightarrow M$ and $Z(M) \hookrightarrow Z(M_0)$, there are obvious maps $\mathfrak{a}(M)^* \to \mathfrak{a}(M_0)^*$ and $\mathfrak{a}(M_0)^* \to \mathfrak{a}(M)^*$. Furthermore, it is easy to see that the composition $\mathfrak{a}(M)^* \to \mathfrak{a}(M_0)^* \to \mathfrak{a}(M)^*$ is the identity. In particular, we have an injection

$$\mathfrak{a}(M)^* \hookrightarrow \mathfrak{a}(M_0)^*.$$

By this observation, we may view the central exponents of a representation of any standard Levi subgroup as living in $\mathfrak{a}(M_0)^*$. Henceforth we will write $\mathfrak{a}$ (respectively $\mathfrak{a}^*$) for $\mathfrak{a}(M_0) = \Lambda(M_0) \otimes \mathbb{R}$ (respectively $\mathfrak{a}(M)^* = \mathrm{Hom}(G,\mathbb{R})$).

Suppose U is a unipotent subgroup normalized by M_0, and α is a character of M_0 which appears in the adjoint action of M_0 on U. Then $\log|\alpha| \in \mathfrak{a}^*$ is called a *root*. The collection of the roots, Σ, is the *root system*. The choice of P_0 corresponds to a choice of positive roots $\Sigma' \subset \Sigma$. Suppose that Σ is generated by the simple roots $\alpha_1, \ldots, \alpha_l \in \Sigma'$. The standard unipotent subgroups correspond to subsets of the α_i. If $P = MU$ is a standard parabolic with $\alpha_1, \ldots, \alpha_r$ the simple roots appearing in the adjoint action on U, then $\mathfrak{a}(M)^* \subset \mathfrak{a}^*$ is the subspace generated by $\alpha_1, \ldots, \alpha_r$.

Define $\mathfrak{a}^+ = \{\lambda \in \mathfrak{a} \mid \alpha(\lambda) \leq 0 \ \forall \alpha \in \Sigma'\}$ or equivalently, $\mathfrak{a}^+ = \{\lambda \in \mathfrak{a} \mid \alpha_i(\lambda) \leq 0 \ \forall i\}$. Clearly, $\mathfrak{a}^+$, is a cone in $\mathfrak{a}$. The dual cone, $\mathfrak{a}^{*+} \subset \mathfrak{a}^*$, consists of elements that are positive on $\mathfrak{a}^+$. We may describe $\mathfrak{a}^{*+}$ as the set of linear combinations $\sum c_i \alpha_i$ with $c_i \leq 0$. (Thus $\mathfrak{a}^{*+}$ is the opposite of the usual Weyl chamber.) Obviously, each face of the cone $\mathfrak{a}^{*+}$ is itself a cone in one of the subspaces $\mathfrak{a}(M)^* \subset \mathfrak{a}^*$.

Definition 30. A central exponent $e \in \mathfrak{a}(M)^*$ is *(strictly) negative* if it lies in (the interior of) $-\mathfrak{a}(M)^{*+}$.

It will be important to clarify the relationship bewteen the various $\Lambda(M)$ for the standard Levi subgroups M. Suppose $M \subset N$ are Levi subgroups, then we have the following inclusions:

$$\begin{array}{ccll}
\mathfrak{a}(N) & \subset & \mathfrak{a}(M) & \text{vector spaces} \\
\cup & & \cup & \\
\Lambda(N) & & \Lambda(M) & \text{lattices} \\
\cup & & \cup & \\
\Lambda(Z(N)) & \subset & \Lambda(Z(M)) & \text{lattices}
\end{array}$$

The only thing that we have not proved is the inclusion on the last line. But this follows from the fact that for an abelian group Z, Z° is the same as the maximal compact subgroup of Z. Thus, an inclusion of abelian groups $Z_1 \hookrightarrow Z_2$ leads to an inclusion $\Lambda(Z_1) \hookrightarrow \Lambda(Z_2)$. As a particular case of this diagram, we see that $\Lambda(Z(M)) \subset \Lambda$ for any standard Levi subgroup M. Thus,

$$\Lambda(Z(M)) \subset \Lambda \cap \Lambda(M) \subset \Lambda(M)$$

where these inclusions are of subgroups of finite order.

As in previous chapters, set $\Lambda = \Lambda(M_0) = M_0/M_0^\circ \subset \mathfrak{a}$. We have defined $\Lambda^+ = \{\lambda \in \Lambda \mid \operatorname{Ad}\lambda \text{ is a contraction on } U_0\}$. We could also simply define $\Lambda^+ = \Lambda \cap \mathfrak{a}^+$. Notice that a representation of M has a (strictly) negative central exponent if its central character is (strictly) negative on Λ^+. [?????????????????????????]

Remark. When the center of G is not compact, there is a nuance because then Λ^+ is not a lattice intersect a cone. Of course, we could always just consider the subgroup of finite index $G^\circ \times \Lambda_Z$ where $\Lambda_Z \subset Z(G)$ and G° has compact center. (See section II.2.1.)

2.2. Condition for Square Integrability.

Proposition 42. *Let G have compact center, and let π be a representation of G of finite length. Then π is square integrable if and only if, for any standard Levi subgroup $M \subset G$, all central exponents of $r_{M,G}(\pi)$ are strictly negative elements of $\mathfrak{a}(M)^*$.*

Remark. The proposition suggests that to find out if a representation is square integrable, we must know all Jacquet functors. In fact, it is equivalent to check the condition only for those associated to cuspidal components: if $\tau \in \mathrm{JH}(r_{M,G}(\pi))$ is not cuspidal, find $L \subset M$ a Levi subgroup such that $r_{L,M}(\tau)$ is cuspidal. Let $e(\rho)$ be an associated cuspidal central exponent. Then $\mathfrak{a}(M)^* \subset \mathfrak{a}(L)^*$ and it is not hard to see that $e(\tau)$ is the projection of $e(\rho)$ onto $\mathfrak{a}(M)^*$. As $\mathfrak{a}(M)^{*+}$ is a face of $\mathfrak{a}(L)^{*+}$, it is clearly sufficient to check the condition on L.

Proof. Fix a small open compact subgroup K as in Bruhat's theorem (section II.2.1). In particular, for any standard parabolic subgroup P, K and P are in good position. Let $e = x_1, \ldots, x_s$ be a list of elements of G which include representatives of $\Lambda(M)/\Lambda(Z(M))$ for each standard Levi subgroup M. We shall work with matrix coefficents of the form $m_{\tilde{\xi},\xi}(g) = \langle \tilde{\xi}, \pi(g^{-1})\xi \rangle$ where $x_i\xi$ and $x_i\tilde{\xi}$ are K invariant for each i. As K may be taken arbitrarily small, this is no real restriction.

By the Cartan decomposition, $G = K_0\Lambda^+K_0$, we have

$$\int_G |m_{\tilde{\xi},\xi}(g)|^2 d\mu = \sum_{\lambda\in\Lambda^+} \int_{K_0\lambda K_0} |m_{\tilde{\xi},\xi}(g)|^2 d\mu$$

Let k_i be representatives for K_0/K and let w_i be the action $\pi(k_i)$. Then

$$= \sum_{i,j} \sum_{\lambda\in\Lambda^+} |m_{w_i(\tilde{\xi}),w_j(\xi)}(\lambda)|^2 \mu(K\lambda K).$$

We are interested in when this sum converges. Hence, it is enough to look only at

$$\sum_{\lambda\in\Lambda^+} |\langle \tilde{\xi}, \pi(\lambda)\xi \rangle|^2 \mu(K\lambda K).$$

Now, each element of Λ^+ is in $\Lambda(M)^{++}$ for some standard Levi subgroup M. Thus, we may split the above sum into sums over $\Lambda(M)^{++} \cap \Lambda^+$ for the various M's. In other words, our goal is now to show that all central exponents are strictly negative if and only if all sums of the form

$$\sum_{\lambda\in\Lambda(M)^{++}\cap\Lambda^+} |\langle \tilde{\xi}, \pi(\lambda)\xi \rangle|^2 \mu(K\lambda K)$$

converge.

We know that

$$\Lambda(Z(M))^{++} \subset \Lambda(M)^{++} \cap \Lambda^+ \subset \Lambda(M)^{++}.$$

We would like to argue that we may reduce further to considering sums over only the $\Lambda(Z(M))^{++}$. This will follow from the following lemma.

Lemma 41. *For* $\eta \in \Lambda(M)^+ \cap \Lambda^+$,

$$\mu(K\eta K) = \mu(K)\Delta_M^{-1}(\eta).$$

Proof. Using the notation (and results) from Bruhat's theorem,

$$K\eta K = K_U K_M K_{\overline{U}} \eta K = K_U \eta K.$$

It follows that $\mu(K\eta K)/\mu(K)$ equals the index $[\eta^{-1}K_U\eta \colon K_U]$. But, $[\eta^{-1}K_U\eta \colon K_U] = [K_U \colon \eta K_U \eta^{-1}]$. Finally, by the remark at the end of section III.1.1, this equals $\Delta_M^{-1}(\eta)$. $\square$

Using this lemma and our assumptions about ξ and $\tilde{\xi}$, it is easy to see that it is enough to prove that all central exponents are strictly negative if and only if all sums of the form

$$\sum_{\Lambda(Z(M))^{++}} |\langle \tilde{\xi}, \pi(\lambda)\xi \rangle|^2 \Delta_M^{-1}(\lambda)$$

converge.

The key observation at this stage is that $\Lambda(Z(M))^{++} \subset \Lambda(M,K)^{++}$ (i.e. λ strictly dominant with respect to (P,K), see section II.2.1). But, the results of section III.3.2 imply that, for $\lambda \in \Lambda(M,K)^{++}$

$$\langle \tilde{\xi}, \pi(\lambda)\xi \rangle = \langle \overline{r}_{M,G}(\tilde{\xi}), J(\pi)(\lambda) r_{M,G}(\xi) \rangle.$$

Furthermore, $\Delta_M^{-1}(\lambda)$ provides exactly the usual normalization for the Jacquet restriction functor.

Thus, for $\lambda \in \Lambda(Z(M))^{++} \subset \Lambda(M,K)^{++}$,

$$|\langle \tilde{\xi}, \pi(\lambda)\xi \rangle|^2 \Delta_M^{-1}(\lambda) = |\langle \tilde{\xi}, \tau(\lambda)\xi \rangle|^2$$

where $\tau = r_{M,G}(\pi)$ and the ξ and $\tilde{\xi}$ on the right hand side are to be interpreted as elements of $r_{M,G}(\pi)$ and $\overline{r}_{M,G}(\pi)$.

We are reduced to showing that all central exponent are strictly negative if and only if each sum of the form

$$\sum_{\Lambda(Z(M))^{++}} |\langle \tilde{\xi}, \tau(\lambda)\xi \rangle|^2$$

converges. But restricted to $Z(M)$, τ is just the sum of the central characters of the various Jordan-Holder components of τ. Consequently, the previous sum converges if and only if

$$\sum_{\Lambda(Z(M))^{++}} |\chi(\lambda)|^2$$

converges for all central characters χ. [EASY FROM HERE – JUST HAVE TO ARGUE THAT SUM OVER CONE (POINT NOT AT ORIGIN) OF CHARACTER CONVERGES IFF STRICTLY NEGATIVE CENTRAL EXPONENTS.]

For this to happen, it is obviously necessary that all $|\chi(\lambda)| < 1$. It is now easy to see that all central exponents must be strictly negative. On the other hand, it is well known that the sum of a positive $\square$

Dropping the assumption that G has compact center, observe that $\mathfrak{a}(G) = \mathfrak{a}(Z)$ and $\mathfrak{a} = \mathfrak{a}(G) \bigoplus \mathfrak{a}(G)^{\perp}$. Also, $\mathfrak{a}^{+} = \mathfrak{a}(G) \oplus (\mathfrak{a}(G)^{\perp})^{+}$. We will say that an element of $\mathfrak{a}^*$ is *striclty negative modulo center* if it is strictly negative on $\Lambda \cap (\mathfrak{a}(G)^{\perp})^{+}$. It is not hard to show

PROPOSITION 43. *Let π be a representation of G of finite length with unitary central character. Then, π is square integrable modulo center if and only if for any standard Levi subgroup $M \subset G$, all central exponents of $r_{M,G}(\pi)$ are strictly negative modulo center.*

2.3. Tempered Representations. In this section, we introduce tempered representations and state the Langlands classification. Although we will not need it in these notes, this topic does play an important role in the representation theory of p-adic groups.

Suppose that G has compact center.

DEFINITION 31. A representation π is *tempered* if all central exponents of π are (not strictly) negative.

PROPOSITION 44. *Irreducible tempered representations are unitary.*

REMARKS. 1. Square integrable representations may be embedded into $L^2(G)$ and so are obviously unitary. However, tempered is slightly weaker than square integrable so this is not so clear. In particular, the proposition is false without the irreducibility hypothesis.

PROOF. Let π be an irreducible tempered representation of G. If π is square integrable then it is unitary. Otherwise, the idea of the proof is to find a Levi subgroup M and an irreducible unitary representation, σ, of M so that $r_{M,G}(\pi) \to \sigma$. Then, by adjunction, $\pi \hookrightarrow i_{G,M}(\sigma)$ and so π is a subrepresenation of a unitary module and so is unitary.

To find such a σ we must get a representation which has a unitary central character and is square integrable modulo center. By proposition 43, this second condition is equivalent to having all central exponents strictly negative modulo $\mathfrak{a}(M)$.

Let N be a Levi subgroup and ρ an irreducible cuspidal representation so that $r_{N,G}(\pi) \to \rho$. As cuspidal representations are compactly supported modulo center, if ρ has unitary central character we are done. If this is not the case, then, since π is not square integrable, we may assume that $e(\rho)$ is on the boundary of $-\mathfrak{a}(N)^{+*}$. Therefore, there exists $M \supset N$ and $\sigma \in \mathrm{JH}(r_{M,G}(\pi))$ so that $e(\sigma) = 0$. That is, σ has unitary central character. Moreover, if we choose M maximal with this property, then, for $M' \subset M$, the central exponents of $r_{M',M}(\sigma)$ are strictly negative modulo $\mathfrak{a}(M)$. $\square$

The next result we state without proof.

Langlands Classification. *Let $G \supset M \supset N$ be a tower of Levi subgroups. Suppose that σ is a representation of N which is square integrable modulo center, and let ψ be an unramified character of M which takes positive real values and is strictly positive (in the Weyl chamber).*

(1) *$\tau = i_{M,N}(\sigma)$ can be decomposed as a direct sum of tempered representations, $\bigoplus \tau_i$.*
(2) *$i_{G,M}(\psi\tau_i)$ has a unique irreducible submodule. Moreover, there is a partial order on the quotients so that the irreducible submodule is strictly maximal.*
(3) *any irreducible representation of G may be realized as in (2).*

Let $R(G)$ be the Grothendieck group of representations of G of finite length. $R(G)$ is a free group generated be $\operatorname{Irr} G$.

Corollary. *The set of $i_{G,M}(\psi\tau)$ with ψ a positive real character and τ a tempered representation form a basis in $R(G)$.*

Remark. Usually the corollary is the form of this result that is used.

3. Uniqueness of $\Pi(\Omega)$

3.1. Preliminaries on Corank 1. Before proving the uniqueness theorem, we need some preliminaries on corank 1 Levi subgroups. We will say $M \subset G$ is of *corank 1* if the associated parabolic subgroup P is a maximal proper parabolic subgroup in G. For example, if $G = \mathrm{GL}(n)$, then the corank 1 Levi subgroups are those which consist of two blocks. That is, $M = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$.

Recall that $W(M) = \{w \in G \mid wMw^{-1}$ is a standard Levi subgroup in $G\}$ modulo M. When M is corank 1, $W(M) = \{1, \sigma\}$. When $G = \mathrm{GL}(n)$, σ switches n_1 and n_2. In general, suppose that $\sigma\colon M \xrightarrow{\sim} N$ and $P = MU$, $Q = NV$. Then one could also take the point of view that σ is the map that switches P and $\overline{Q}$.

Our goal is the following theorem.

Theorem 28. *Suppose that G has compact center and $M \subset G$ is a corank 1 Levi subgroup with $W(M) = \{1, \sigma\}$, $\sigma\colon M \xrightarrow{\sim} N$. Let ρ be a cuspidal irreducible representation of M and set $\sigma(\rho) = \rho'$. Suppose that $\pi = i_{G,M}(\rho)$ is reducible. Then $N = M$ and $\rho' = \phi\rho$, where ϕ is a positive real-valued character.*

PROOF. Define $r(\pi) = \{r_{L,G}(\pi)\}$. The basic geometric lemma implies that

$$r(\pi) = \begin{cases} \{(M,\rho),(N,\rho')\} & \text{if } N \neq M \\ \{(M,\rho\oplus\rho')\} & \text{if } N = M. \end{cases}$$

In either case, the fact that Jacquet restriction is exact proves that $|\operatorname{JH}(\pi)| \leq 2$, and similarly for $\pi' = i_{G,M}(\rho')$. Moreover, it follows that

$$\operatorname{Hom}(\pi,\pi') = \operatorname{Hom}(r_{N,G}i_{G,M}(\rho),\rho')$$

is one dimensional, and similarly for $\operatorname{Hom}(\pi',\pi)$. Of course, if π is irreducible, $\pi \cong \pi'$. But here π is reducible. Hence,

$$0 \to \pi_1 \to \pi \to \pi_2 \to 0$$

with π_1 and π_2 irreducible. It is easy to see that $r(\pi_1) = (M,\rho)$, and $r(\pi_2) = (N,\rho')$. It follows that

$$0 \to \pi_2 \to \pi' \to \pi_1 \to 0.$$

We know that $\mathfrak{a}(M)$ is one dimensional; this is equivalent to corank 1. Thus, there are exactly three cases for the central exponent, $e = e(\rho)$: $e = 0$, $e > 0$, and $e < 0$. In the first case, ρ is unitary. Therefore, π is also unitary and so completely reducible: $\pi = \pi_1 \oplus \pi_2$. But this leads to a map $\pi_2 \hookrightarrow \pi$ and so implies that $r_{M,G}(\pi_2) = \rho$ which is false. The second case is easily reduced to the third by taking contragredients, and so $e < 0$ is the situation that we consider.

If $e < 0$, then the only central exponent of π_1 is strictly negative. Thus, by proposition 42, π_1 is square integrable and hence unitary. Therefore, its contragredient, $\tilde{\pi}_1$, is the same as its complex conjugate, ${}^{\mathrm{c}}(\pi_1)$. So,

$$\begin{aligned} {}^{\mathrm{c}}(\overline{r}_{M,G}(\pi_1)) &= \overline{r}_{M,G}({}^{\mathrm{c}}(\pi_1)) \\ &= \overline{r}_{M,G}(\tilde{\pi}_1) \end{aligned}$$

using theorem 21,

$$\begin{aligned} &= \widetilde{r_{M,G}(\pi_1)} \\ &= \tilde{\rho}. \end{aligned}$$

If $N \neq M$, $r_{N,G}(\pi_1) = 0$. But, it is easy to see that σ switches the functors $\overline{r}_{M,G}$ and $r_{N,G}$. Thus, $\overline{r}_{M,G}(\pi_1)$ is zero also, and so by what we have just seen, $\rho = 0$. This contradiction implies that $N = M$.

When $N = M$, $\sigma(\overline{r}_{M,G}(\pi_1)) = r_{M,G}(\pi_1) = \rho$. Hence,

$$\rho' = \overline{r}_{M,G}(\pi_1) = {}^{\mathrm{c}}(\tilde{\rho}).$$

By step 1 in the proof of theorem 27, there is a real character, ψ, and a representation ρ_0 with unitary central character, so that

$$\rho = \psi\rho_0.$$

As ρ_0 is cuspidal, it is actually unitary. Hence,

$$\rho' = {}^{\mathrm{c}}(\tilde{\rho}) = {}^{\mathrm{c}}(\psi^{-1}\tilde{\rho}_0) = \psi^{-1}\ {}^{\mathrm{c}}(\tilde{\rho}_0)) = \psi^{-1}\rho_0.$$

Thus,

$$\rho = \psi^2\rho'$$

□

COROLLARY. *If $i_{G,M}(\rho)$ is reducible, then (M, D) is invariant with respect to σ.*

The first non-trivial case of this result is $M = \mathrm{GL}_1 \times \mathrm{GL}_2 \subset \mathrm{GL}_3$. Here we conclude that induced representations of cuspidal representations are *always* irreducible. Although there are other ways to prove it for GL_n, this is a hard result. The only general proof is the one that we gave which used results on unitary representations.

The next corollary is the key result of this section.

COROLLARY. *Let $M \subset G$ be of corank 1, (M, D) a cuspidal component, and Ω the associated component of $\Omega(G)$. Suppose that $\sigma(M, D) \neq (M, D)$. Then*

(1) *$i_{G,M}$ maps irreducible representations in $\mathcal{M}(D)$ into irreducible representations in $\mathcal{M}(\Omega)$.*
(2) *Set $r_{D,G}(\pi) = r_{M,G}(\pi)_D \in \mathcal{M}(D)$. Then $r_{D,M}$ is a faithful functor (i.e. maps nonzero modules to nonzero modules).*
(3) *For $\pi \in \mathcal{M}(\Omega)$, the natural morphism $\pi \to i_{G,M}r_{D,G}(\pi)$ is an isomorphism. Therefore,*

$$r_{D,G} \colon \mathcal{M}(\Omega) \to \mathcal{M}(D)$$

is an equivalence of categories.

PROOF. (1) is clear. (2) Let $\sigma(D) = D'$. Then by definition of Ω, at least one of $r_{D,G}(\pi)$ and $r_{D',G}(\pi)$ is non-zero. But σ maps one to the other. Hence they are both non-zero. (3) follows easily. □

3.2. Proof of Uniqueness. The first step is to prove the uniqueness of $\Pi(\Omega)$ in the special case M of corank 1.

PROPOSITION 45. *If $M \subset G$ is of corank 1 with D a cuspidal component, and $\sigma(M, D) = (N, D')$, then*

$$i_{G,M}(\Pi(D)) \cong i_{G,N}(\Pi(D')).$$

REMARK. This isomorphism is NOT canonical.

PROOF. If $(N, D') = (M, D)$ then this is trivial. Otherwise, by the equivalence of categories in the second corollary to theorem 28, it is enough to show

$$r_{D,G} i_{G,N}(\Pi(D')) = \Pi(D).$$

But this is true by the basic geometric lemma. $\square$

Finally, we can prove the theorem:

UNIQUENESS THEOREM. *Let $M \subset G$ be any Levi subgroup, D a cuspidal component of M. If there is $w \in W(M)$ so that $w(M, D) = (N, D')$, then*

$$i_{G,M}(\Pi(D)) \cong i_{G,N}(\Pi(D')).$$

PROOF. First, consider the case of $w = \sigma$, an elementary transformation; i.e. there is a Levi subgroup L, $M \subset L \subset G$, so that M is corank 1 in L and $\sigma \in L$. By the provious proposition,

$$i_{L,M}(\Pi(D)) \cong i_{L,N}(\Pi(D')).$$

Hence,

$$i_{G,M}(\Pi(D)) = i_{G,L}(i_{L,M}(\Pi(D))) \cong i_{G,L}(i_{L,N}(\Pi(D'))) = i_{G,N}(\Pi(D')).$$

The general case now follows from the following geometric lemma

LEMMA 42. *Any transformation $w \colon M \to N$ may be written as a composition of elementary reflections.*

PROOF. Omitted. But here is an example: $G = \mathrm{GL}(n)$, $M = \mathrm{GL}(n_1) \times \ldots \times \mathrm{GL}(n_k)$. A transformation is just a permutation of the blocks. The statement is that such a permutation is just a product of transpositions. $\square$

$\square$

Combined with the results of section III.4.1, we have shown that, if Ω is a component of $\Omega(G)$, and (M, D) is any pair which yields Ω, then $\Pi(\Omega) = i_{G,M}(\Pi(D))$ is a projective generator for $\mathcal{M}(\Omega)$. Observe that, already in section III.4.1, we could have taken

$$\bigoplus_{M,D} i_{G,M}(\Pi(D))$$

as a projective generator and so avoided needing the result above. However, the sum is more complicated and so less usefull for illuminating the structure of $\mathcal{M}(\Omega)$.

4. Cohomological Dimension

We know that $\mathcal{M}(G)$ has enough projectives (section I.2.2). Thus for $M \in \mathrm{Ob}(\mathcal{M}(G))$, there is a projective resolution

$$\cdots \to P_1 \to P_0 \to M \to 0.$$

Note that if M is Noetherean, this resolution may be chosen to be by Noetherean projectives.

In this section we prove

THEOREM 29. *$\mathcal{M}(G)$ has bounded cohomological dimension. More precisely,*

$$\text{cohom.}\dim \mathcal{M}(G) \leq \operatorname{rank}(G).$$

One would hope to be able to establish this directly. However, it seems as though an entirely new ingredient is needed: the Tits Building.

4.1. Tits Building.

THEOREM 30. *There is a simplicial complex X with a G-action, satisfying*

1. $\dim X = \operatorname{rank} G$.
2. *X is contractible.*
3. *G preserves the simplicial structure and acts linearly on every simplex.*
4. *X has a finite number of simplices modulo G.*
5. *The stabilizer of any simplex $\sigma \subset X$, $\mathrm{Stab}(\sigma)$, is a compact open subgroup of G.*

For example, $G = \mathrm{PGL}(2, F)$ acts on the plane F^2. The vertices of X are the open compact $\mathcal{O}$ submodules (i.e. lattices) $\Lambda \subset F^2$, up to scaler. Two vertices Λ and Λ' are joined by an interval if $\Lambda \cap \Lambda'$ has index q in Λ and Λ'. (Here q is the residue characteristic of F.) Thus, X is a tree with $q+1$ branches at every vertex. Putting the standard unit metric on each interval gives a metric of hyperbolic type; X is an analog of a symmetric space.

PROOF. Omitted. But here are some remarks: first consider the case G split. In this case, one can give an explicit construction of X. In general, consider a Galois field extension $E \supset F$ so that G splits. The Galois group Γ acts on the building X_E and we define X to be $(X_E)^\Gamma$. Then deduce properties of X from those of X_E. The hardest part is showing that X is contractible. $\square$

4.2. Finiteness. Using the building, we may prove that $\mathcal{M}(G)$ has cohomological dimension $\leq \operatorname{rank} G$. We will use the properties of X repeatedly. Let $l = \operatorname{rank} G$ and let C be the chain complex of X. That is,

$$C : 0 \to C_l \xrightarrow{\partial} C_{l-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

where C_i is the free abelean group on the i simplices. As X is contractible, this is an acyclic complex; i.e. $H^i(C) = 0$ for $i \neq 0$, and $H^0(C) = \mathbb{C}$. On the other hand, each C_i is a representation of G. Hence, C may be viewed as a G-module resolution of the trivial G-module $\mathbb{C}$.

PROPOSITION 46. *The trivial module $\mathbb{C}$ has a projective resolution of length* $\operatorname{rank} G$*:*

$$P_l \to P_{l-1} \to \cdots \to P_0 \to \mathbb{C}.$$

Moreover, the P_i are finitley generated.

PROOF. Just take $P_i = C_i$. It only remains to check that the C_i are projective, finitely generated G-modules. Let $\sigma_1, \ldots, \sigma_r$ be representatives for the set of i simplices of X, modulo the action of G. Elements of $\operatorname{Stab}(\sigma_j)$ may preserve or reverse the orientation of σ_j. We will denote by ε_j the corresponding one dimensional representation of $\operatorname{Stab}(\sigma_j)$. Then, the G module C_i is equivalent to

$$\bigoplus_j \operatorname{Ind}^G_{\operatorname{Stab}(\sigma_j)}(\varepsilon_j)$$

which is finitely generated and projective. $\square$

Let V be a G module. We try to construct a projective resolution of V by

$$V \otimes P_l \to V \otimes P_{l-1} \to \cdots \to V \otimes P_0 \to V \otimes \mathbb{C} = V$$

where we give $V \otimes P_i$ the diagonal action of G. Note that even if V is finitely generated, $V \otimes P_i$ need not be. However,

CLAIM. *$V \otimes P_i$ is projective.*

PROOF. Recall that, in the notation of the last proof,

$$P_i = \bigoplus_j \operatorname{Ind}^G_{\operatorname{Stab}(\sigma_j)}(\varepsilon_j).$$

Thus, it is enough to show that $V_j = V \otimes \mathrm{Ind}_{\mathrm{Stab}(\sigma_j)}^G(\varepsilon_j)$ is projective. Here we have the diagonal action of G. We may view V_j as a set of functions $f\colon G \to V$ satisfying some properties.

We also define $W_j = V \otimes \mathrm{Ind}_{\mathrm{Stab}(\sigma_j)}^G(\varepsilon_j)$ but with the action of G only on the second component. W_j is obviously projective. Once again, we view W_j as a set of functions on G. It is easy to see that the map

$$f(g) \mapsto g^{-1}f(g)$$

defines a G-module isomorphism from V_j to W_j. Thus, V_j is projective. $\square$

For any object in $\mathcal{M}(G)$, we have given an explicit construction of a projective resolution no longer than the rank of G. This proves theorem 29.

Remark. The resolution that we have explicitly constructed is by infinitely generated modules. When V is finitely generated, there is a resolution by finitely generated projective modules. (Note that $\mathcal{M}(G)$ is a Noetherean category; that is, any finitely generated object is Noetherean. See section III.2.2.) However, we have no explicit construction of a finite resolution using only finitely generated modules.

5. Duality.

5.1. Cohomological Duality.

Definition 32. For a finitely generated, projective (left) G-module P, set

$$P^* = \mathrm{Hom}_G(P, \mathcal{H}(G)).$$

Since $\mathcal{H}(G)$ is both a right and a left G-module, P^* is a right G module. P finitely generated projective implies P^* smooth finitely generated projective.

For example, take $K \subset G$ open compact and $P_K = \mathcal{H}(G)e_K$. P_K is projective and generated by e_K. Then, $P_K^* = \mathrm{Hom}_G(P_K, \mathcal{H}(G)) = e_K\mathcal{H}(G)$. Thus we have a duality between left and right G modules.

Consider any (not necessarily exact) complex of finitely generated projective objects

$$0 \to A_k \to \cdots \to A_0 \to 0.$$

Then we may form

$$0 \to A_0^* \to \cdots \to A_k^* \to 0.$$

In this way, we define a duality in the associated derived category. Of course, given any finitely generated G-module π, there is a resolution by finitely generated projectives,

$$0 \to P_l \to \cdots \to P_0 \to \pi \to 0,$$

and so may we view π as an element of the derived category. It is natural to hope that the dual object,

$$0 \to P_0^* \to \cdots \to P_l^* \to 0,$$

has homology in only one dimension. This would lead to a duality on finitely generated $\mathcal{H}$-modules. Ideally, this would take irreducible modules to irreducible modules, or more generally, modules of finite length to modules of finite length, and have other nice properties. In this section, we show that this is, in fact, the case.

For π and P_i as above, set

$$E^i(\pi) = \mathrm{Ext}^i(\pi, \mathcal{H}(G)) = H^i(P_0^* \to \cdots \to P_l^*).$$

Let $\mathcal{M}^f(\Omega)$ be the subcategory of $\mathcal{M}(\Omega)$ consisting of representations of finite length.

Theorem 31. *Fix a component $\Omega \subset \Omega(G)$ and let $d = \dim \Omega$. Suppose that $\pi \in \mathcal{M}^f(\Omega)$. Then $E^i(\pi) = 0$ unless $i = d$, and $D(\pi) \overset{def}{=} E^d(\pi)$ is a representation of finite length. Converting $D(\pi)$ from a right module to a left module in the standard way gives a map*

$$D \colon \mathcal{M}^f(\Omega) \to \mathcal{M}^f(\Omega).$$

The map D is called cohomological duality. *It has the following properties:*

(1) *D is exact.*
(2) *D^2 is the identity.*
(3) *If ρ is irreducible cuspidal, then $D(\rho) \cong \tilde{\rho}$.*
(4) *$D \circ i_{G,M} = \bar{i}_{G,M} \circ D$, where $\bar{i}_{G,M}$ is taken with the oposite parabolic.*
(5) *$D \circ r_{M,G} = r_{M,G} \circ D$.*

Remark. It follows that π irreducible imples $D(\pi)$ irreducible.

For example, take $G = \mathrm{GL}(2)$. Let $\mathbf{1}$ and S be the trivial and Steinberg representations, respectively. Set $M = \mathrm{GL}(1) \times \mathrm{GL}(1)$. Then there are characters ρ_1 and ρ_2 of $\mathrm{GL}(1)$ so that $R = i_{G,M}(\rho_1, \rho_2)$ satisfies

$$0 \to \mathbf{1} \to R \to S \to 0.$$

Using the properties of duality and our knowledge of corank 1 Levi subgroups, we get

$$0 \to D(S) \to D(R) \to D(\mathbf{1}) \to 0$$

and can conclude that

$$D(\mathbf{1}) = S$$
$$D(S) = \mathbf{1}.$$

In terms of Ext groups, this means $\mathrm{Ext}^1(\mathbf{1}, \mathcal{H}(G)) = S$, etc.

For the proof of the theorem, we will need a result from commutative algebra which we state without proof. Let A be a regular, finitely generated algebra over a field. Let M be a finitely generated A-module. We should think of A as the algebra of functions on a non-singular algebraic variety Y; M corresponds to a sheaf on Y. $\mathrm{Supp}\, M \subset Y$ is a closed subset. Then

THEOREM 32 (SERRE). (1) *The category of modules over A, $\mathcal{M}(A)$, has cohomological dimension equal* $\dim A$ *(*$= \dim Y$*).*

(2) $\mathrm{Ext}^i_{\mathcal{M}(A)}(M, A) = 0$ *for $i <$ codimension of* $\mathrm{Supp}\, M$*.*

(3) *The codimension of* $\mathrm{Supp}\, \mathrm{Ext}^i_{\mathcal{M}(A)}(M, A)$ *is $\geq i$.*

REMARK. Part (1) is trivial. We will not need part (3).

The proof of theorem 31 will occupy the rest of this section. The proof has two parts. First we show that $E^i(\pi) = 0$ for $i \neq d$. Then we prove properties (1) to (5). (The fact that $E^d(\pi)$ has finite length follows from the properties.)

We want to study $\mathrm{Ext}^i(\pi, \mathcal{H}(G))$. Write $\mathcal{H}(G) = \bigoplus_j \mathcal{H}(\Omega_j)$. For us, the only term that matters is $\mathcal{H}(\Omega)$. Furthermore, $\mathcal{H}(\Omega)$ is a quotient of $\bigoplus \Pi(\Omega)$. As $\mathcal{H}(\Omega)$ is projective, it is a direct summand. Thus, to show that $\mathrm{Ext}^i(\pi, \mathcal{H}(G)) = \mathrm{Ext}^i(\pi, \mathcal{H}(\Omega)) = 0$, it is enough to show that $\mathrm{Ext}^i(\pi, \bigoplus \Pi(\Omega)) = 0$. This, in turn, would follow from

$$\mathrm{Ext}^i(\pi, \Pi(\Omega)) = 0.$$

As $\Pi(\Omega) = i_{G,M}(\Pi(D))$ for some cuspidal component D of the Levi subgroup M, we would like to apply Frobenius reciprocity to conclude

$$\mathrm{Ext}^i(\pi, \Pi(\Omega)) = \mathrm{Ext}^i(r_{M,G}(\pi), \Pi(D)).$$

To see that this is correct, write everything out in terms of resolutions and use the fact that $i_{G,M}$ and $r_{M,G}$ are exact and map projective objects to projective objects. (In fact, it is sufficient to know this only for $r_{M,G}$.)

Recall from section II.3.3 that the category $\mathcal{M}(D)$ is equivalent to the category ${}^r\mathcal{M}(\Lambda(D))$ of right modules over $\Lambda(D) = \mathrm{End}(\Pi(D))$. Under this equivalence $\Pi(D)$ maps to $\Lambda(D)$ and $r_{M,G}(\pi)$ maps to some finitely generated $\Lambda(D)$ module, say σ. We will be done if we show that

$$\mathrm{Ext}^i_{\mathcal{M}(\Lambda(D))}(\sigma, \Lambda(D)) = 0.$$

Suppose that D is the cuspidal component consisting of $\psi\rho$ where $\psi \in \Psi(M)$ and ρ is a fixed cuspidal representation of M. The first case to consider is when $\psi\rho \neq \rho$ unless $\psi = 1$. In this situation, $\Lambda(D) = F$, the ring of regular functions on Ψ. Now apply part theorem 32 with $A = F$, $Y = \Psi$ and $M = \sigma$. By (1), $\mathrm{Ext}^i_{\mathcal{M}(F)}(\sigma, F) = 0$ for $i > d$. Moreover, as $r_{M,G}(\pi)$ has finite length, $\dim(\mathrm{Supp}\, \sigma) = 0$, (2) implies $\mathrm{Ext}^i_{\mathcal{M}(F)}(\sigma, F) = 0$ for $i < d$.

Now consider the general case. $\Lambda(D) \supset F$ so we have the change of rings functor $u\colon \mathcal{M}(\Lambda(D)) \to \mathcal{M}(F)$. The functor u has a right adjoint $v\colon \mathcal{M}(F) \to \mathcal{M}(\Lambda(D))$ given by $v\colon M \mapsto \mathrm{Hom}_F(\Lambda(D), M)$. More precisely, for $M \in \mathcal{M}(F)$ and $N \in \mathcal{M}(\Lambda(D))$, there is a functorial isomorphism

$$\alpha\colon \mathrm{Hom}_{\mathcal{M}(F)}(u(N), M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}(\Lambda(D))}(N, v(M)).$$

As $\Lambda(D)$ is finitely generated and free over F (and hence projective), v is exact. Thus by proposition 8, u maps projectives to projectives. Furthermore, it is obvious that that u is exact. It is easy to check that this is sufficient for α to extend to a functorial isomorphism

$$\alpha\colon \mathrm{Ext}^i_{\mathcal{M}(F)}(u(N), M) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{M}(\Lambda(D))}(N, v(M)).$$

Observe that $v(F) = \Lambda(D)$. Hence,

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{M}(\Lambda(D))}(\sigma, \Lambda(D)) &= \mathrm{Ext}^i_{\mathcal{M}(\Lambda(D))}(\sigma, v(F))\\ &= \mathrm{Ext}^i_{\mathcal{M}(F)}(u(\sigma), F)\\ &= 0 \qquad \text{for } i < d\end{aligned}$$

by the previous case.

Remark. Under the equivalence of categories $\mathcal{M}(D) \sim {}^r\mathcal{M}(\Lambda(D))$, we may consider $v\colon \mathcal{M}(F) \to \mathcal{M}(D)$. It is not hard to see that v may be given by $v(M) \mapsto M \otimes \rho$. In particular, $v(F) = F \otimes \rho = \Pi(D)$.

We now turn to the proof of the properties (1) to (5).

To prove (1), suppose

$$0 \to \pi' \to \pi \to \pi'' \to 0$$

is exact. Then by the properties of Ext, there is a long exact sequence

$$\cdots \to E^{d-1}(\pi') \to E^d(\pi'') \to E^d(\pi) \to E^d(\pi') \to E^{d+1}(\pi'') \to \cdots.$$

But $E^i = 0$ unless $i = d$, so this gives

$$0 \to D(\pi'') \to D(\pi) \to D(\pi') \to 0$$

as needed.

(2) We know that there is a finite projective resolution of π,

$$0 \to P_d l \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \pi \to 0.$$

Since $E^i(\pi)$ vanishes for $i \neq d$,

$$0 \to P_0^* \to P_1^* \to \cdots \to P_{d-1}^* \to P_d^* \to P_{d+1}^* \to \cdots \to p_l^* \to 0$$

is a projective complex which is exact except at P_d. Next, "retract" the right tail of this sequence as follows: let Q^*_{l-1} be the kernel of $P^*_{l-1} \to P^*_l \to 0$; then $P^*_{l-1} = Q^*_{l-1} \oplus P^*_l$, and Q^*_{l-1} is projective. Consequently,

$$0 \to P^*_0 \to P^*_1 \to \cdots \to P^*_{d-1} \to P^*_d \to P^*_{d+1} \to \cdots \to p^*_{l-2} \to Q^*_{l-2} \to 0$$

is still a projective complex which is exact except at P_d. Continuing in this way gives

$$0 \to P^*_0 \to P^*_1 \to \cdots \to P^*_{d-1} \to Q^*_d \to 0.$$

Clearly,

$$0 \to P^*_0 \to P^*_1 \to \cdots \to P^*_{d-1} \to Q^*_d \to D(\pi) \to 0$$

is a projective resolution of $D(\pi)$. It is now obvious that $D^2(\pi) = \pi$.

Remarks. 1. It is at this stage in the proof that the finite cohomological dimension of $\mathcal{M}(\Omega)$ is really essential. With an infinite complex, we would be faced with an element of the derived category with homology in only one dimension, but no way of relating it to a resolution for $D(\pi)$. 2. Among other things, this argument implies that $\mathcal{M}(\Omega)$ has cohomological dimension exactly d. More generally, our arguments so far may be modified to show the following: let R and S be rings and $\alpha\colon R \to S$ a homomorphism which makes S into a projective R-module. Suppose that $\mathcal{M}(S)$ has finite cohomological dimension and $\operatorname{cohom.dim} \mathcal{M}(R) = d$. Then $\operatorname{cohom.dim} \mathcal{M}(S) = d$.

(3) Let $G^* = G^\circ \times \Lambda(Z(G))$. Suppose that we could show $D(\rho)|_{G^*} \cong \tilde{\rho}|_{G^*}$. Then, since G^* has finite index in G, it would follow that $D(\rho) \cong \tilde{\rho}$. As $G^* = G^\circ \times \mathbb{Z}^n$, the result follows from:

Claim. (1) *If π is a compact representation of $G = G^\circ$, then $D(\pi) = \tilde{\pi}$.*
(2) *If π is any representation of $G = \mathbb{Z}$, then $D(\pi) = \tilde{\pi}$.*

Proof. For π a compact (and hence cuspidal) representation of G°, then $d = 0$. Thus we are reduced to studying $\operatorname{Hom}(\pi, \mathcal{H}(G))$. As compact representations are splitting, it is enough to consider π irreducible. We showed in setion I.4.2 that $\operatorname{Hom}_{G\times G}(\pi \otimes \tilde{\pi}, \mathcal{H}(G)) = \mathbb{C}$. It follows that $D(\pi) = \tilde{\pi}$.

When $G = \mathbb{Z}$, $d = 1$. It is clearly enough to consider $\pi = \psi$ a character. In this case, $\mathcal{H}(G) \cong \mathbb{C}[t, t^{-1}]$, the ring of functions on $\Psi(G) = \mathbb{C}^*$. Thus, it is a standard exercise to show $\operatorname{Ext}^1(\psi, \mathcal{H}(G)) = \psi$. Passing from right to left modules gives $D(\pi) = \psi^{-1} = \tilde{\psi}$. $\square$

(4) and (5). It is clear that these statements reduce to the analogous statements for $\operatorname{Hom}(\cdot, \mathcal{H})$. More precisely, we must prove that there are functorial isomorphisms

$$\operatorname{Hom}_G(i_{G,M}P, \mathcal{H}(G)) \cong \bar{i}_{G,M}(\operatorname{Hom}_M(P, \mathcal{H}(M)))$$

and

$$\mathrm{Hom}_G(r_{M,G}Q, \mathcal{H}(M)) \cong r_{M,G}(\mathrm{Hom}_G(Q, \mathcal{H}(G)))$$

for projective objects $P \in \mathcal{M}(M)$ and $Q \in \mathcal{M}(G)$. By the second adjunction and Frobenius reciprocity, this is equivalent to

$$\mathrm{Hom}_M(P, \overline{r}_{M,G}\mathcal{H}(G)) \cong \overline{i}_{G,M}(\mathrm{Hom}_M(P, \mathcal{H}(M)))$$

and

$$\mathrm{Hom}_G(Q, i_{G,M}\mathcal{H}(M)) \cong r_{M,G}(\mathrm{Hom}_G(Q, \mathcal{H}(G))).$$

These follow immediately from:

CLAIM. *Let G be a group, $P \subset G$ a parabolic subgroup with $P = MU$. Think of $\mathcal{H}(G)$ and $\mathcal{H}(M)$ as $G \times G$ and $M \times M$ modules, respectively. Then there is an isomorphism of $G \times M$ modules*

$$(i_{G,M} \times \mathbf{1})\mathcal{H}(M) \cong (\mathbf{1} \times r_{M,G})\mathcal{H}(G).$$

Here $\mathbf{1}$ *is the trivial functor.*

PROOF. It is simplest to identify $\mathcal{H}(G)$ with locally constant compactly supported functions on G. Then $(\mathbf{1} \times r_{M,G})\mathcal{H}(G)$ is functions on G/U and $(i_{G,M} \times \mathbf{1})\mathcal{H}(M)$ is functions on $G \times_P M \cong G/U$. $\square$

This completes the proof of theorem 31.

5.2. Cohen-Macualey Duality. [NOT FINISHED]

List of Notations

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