The Kervaire Invariant One Problem, Talk 1
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1 Arf invariant

Let k be a field and $b\colon V \otimes V \to k$ a nondegenerate antisymmetric bilinear form. Recall the following definition

Definition 1.1. For $r \in V$ and $\lambda \in k$ the *transvection* $T_{r,\lambda} \in \mathrm{GL}(V)$ with respect to r and λ is defined to be

$$T_{r,\lambda}(x) = x - \lambda b(x,r)r$$

One can easily see that $T_{r,\lambda}$ is always symplectic with respect to b. Moreover

Theorem 1.2. *Every symplectomorphism $V \to V$ can be decomposed into a composition of transvections.*

Let $q \in \mathrm{Sym}^2_k(V^\vee)$ be a quadratic form over the field k of characteristic 2, and let

$$b_q(x,y) = q(x+y) - q(x) - q(y)$$

be the associated bilinear form. For the rest of this section we will assume that q is nondegenerate, in the sense that the associated bilinear form b_q is nondegenerate. In this case one can choose a symplectic basis e_i, f_i of V with respect to b_q.

Definition 1.3. The *Arf invariant* of q with respect to the basis $\{e_i, f_i\}$ is defined to be

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(e_i)q(f_i) \in k/\wp(k)$$

where $\wp\colon k \to k$ is the Lang's isogeny for $\mathbb{G}_a/\mathbb{F}_2$, i.e. the map $x \mapsto x^2 - x$.

Assume for a moment that the Arf invariant of q is well defined (i.e. does not depend on the choice of symplectic basis). Then we have the following basic properties

Proposition 1.4. *1. If q' is equivalent to q, then* $\mathrm{Arf}(q') = \mathrm{Arf}(q)$.

2. For a pair of quadratic forms q_1, q_2

$$\mathrm{Arf}(q_1 \oplus q_2) = \mathrm{Arf}(q_1) + \mathrm{Arf}(q_2)$$

Proof. 1. Let q' be equivalent to q. By definition it means that there is exists a map $A \in \mathrm{GL}(V)$ such that $q'(x) = q(Ax)$. Then $b_{q'}(x,y) = b_q(Ax, Ay)$ and $\{A^{-1}e_i, A^{-1}f_i\}$ is the symplectic basis for $q_{b'}$ if and only if $\{e_i, f_i\}$ is symplectic basis for b_q. But then by definition

$$\mathrm{Arf}(q') = \sum_{i=1}^{n} q'(A^{-1}e_i)q'(A^{-1}f_i) = \sum_{i=1}^{n} q(AA^{-1}e_i)q(AA^{-1}f_i) = \sum_{i=1}^{n} q(e_i)q(f_i) = \mathrm{Arf}(q)$$

2. If $\{e_i^{(1)}, f_i^{(1)}\}$ and $\{e_i^{(2)}, f_i^{(2)}\}$ are symplectic bases for q_1 and q_2 respectively then

$$\{e_i^{(1)}, f_i^{(1)}, e_i^{(2)}, f_i^{(2)}\}$$

is symplectic basis for $q_1 \oplus q_2$ and the statement follows by definition.

□

Now following [Dye78] we will prove

Theorem 1.5. *Let q be a quadratic form over the field k of characteristic 2. Then*

1. *The Arf invariant does not depend on the choice of symplectic basis.*
2. *If k is perfect and $\mathrm{Arf}(q_1) = \mathrm{Arf}(q_2)$ then q_1 is equivalent to q_2.*

Proof. 1. Let e_i', f_i' be some other symplectic bases of V. Consider the symplectomorphism $e_i \mapsto e_i', f_i \mapsto f_i'$. By theorem 1.2 it can be decomposed as a composition of transvections. By direct calculations

$$q(Tx) = q(x) + (\lambda^2 q(r) + \lambda) b_q(x,r)^2$$

Write $r = \sum_{i=1}^{n} s_i e_i + t_i f_i$. For any $\mu \in k$ we have

$$\begin{aligned}\mathrm{Arf}(q(Tx)) &= \sum_{i=1}^{n}(q(e_i) + \mu t_i^2)(q(f_i) + \mu s_i^2) = \mathrm{Arf}(q) + \mu\left(\sum_{i=1}^{n} q(e_i)t_i + q(f_i)s_i\right) + \\ &+ \sum_{i=1}^{n} \mu^2 s_i^2 t_i^2 = \mathrm{Arf}(q) + \mu\left(q(r) + \sum_{i=1}^{n} s_i t_i\right) + \left(\mu \sum_{i=1}^{n} s_i t_i\right)^2 = \\ &= \mathrm{Arf}(q) + \mu q(r) + \wp(\mu \sum_{i=1}^{n} s_i t_i)\end{aligned}$$

Hence $\mathrm{Arf}(q) - \mathrm{Arf}(q') \in \wp(k)$ if and only if $\mu q(r) \in \wp(k)$. Which is our case, because

$$\mu q(r) = (\lambda^2 q(r) + \lambda) q(r) = (\lambda q(r))^2 + \lambda q(r)$$

2. Let $\mathrm{Arf}(q) = \mathrm{Arf}(q')$. By proposition 1.4 we can assume $b_q = b_{q'}$. Hence

$$q(x) + q'(x) = \sum_{i=1}^{n} a_i x_i^2 + b_i y_i^2$$

The field k is perfect, so we can find $s_i, t_i \in k$ such that $a_i = t_i^2$ and $b_i = s_i^2$. If we define $r = \sum_{i=1}^{n} s_i e_i + t_i f_i$, then $q'(x) = q(x) + b_q(r,x)^2$ and hence there is some $v \in k$ such that $q(r) = v^2 + v$. Let $\lambda = v^{-1}$. Then $q'(x) = q(T_{r,\lambda}x)$ because

$$\lambda^2 q(r) + \lambda = \lambda^2(\lambda^{-2} + \lambda^{-1}) + \lambda = 1 + 2\lambda = 1$$

$\square$

Corollary 1.6. *Let q be a nondegenerate quadratic form over a perfect field k. Then in some basis $\{e_i, f_i\}$ it has a form*

$$q_v\left(\sum_i x_i e_i + y_i f_i\right) = \sum_{i=1}^{n} x_i y_i + v(x_n^2 + y_n^2)$$

for some $v \in k$.

Proof. The form q_v may be decomposed as a direct sum

$$q_v \simeq q_{0,0}(x,y)^{\oplus n-1} \oplus q_{v,v}$$

where $q_{a,b}$ is a quadratic form of rank 2 defined by $q_{a,b}(x,y) = ax^2 + xy + by^2$. Note, that by definition $\mathrm{Arf}(q_{a,b}) = ab$.

So by proposition 1.4 the Arf invariant of q_v is v^2. Since k is perfect we conclude by theorem 1.5. $\square$

Remark 1.7. We have the Kummer short exact sequence of Gal_k-representations

$$0 \to \mathbb{F}_2 \to k_+^{sep} \xrightarrow{\wp} k_+^{sep} \to 0$$

which induces

$$k \xrightarrow{\wp} k \to H^1(k, \mathbb{F}_2) \to 0$$

where $H^1(k, k_+^{sep}) \simeq 0$ by additive Hilbert 90'th theorem.

So for any quadratic form q the Arf invariant $\mathrm{Arf}(q)$ is canonically an element of $H^1(k, \mathbb{F}_2)$, which classify some quadric field extension l of k. One can describe l explicitly as $k(\alpha)$, where α is a root of the equation $x^2 + x = \mathrm{Arf}(q)$. By definition the form q has zero Arf invariant in l, hence $q = q_0$ in notation of the previous corollary over l.

Other way around, one can define the Arf invariant of q to be the class in $H^1(k, \mathbb{Z}/2)$, which classifies the smallest extension l of k, such that q is equivalent to q_0 over l (but it is not completely obvious that such an l is of degree 2 over k).

References

[Dye78] R. H. Dye, *On the Arf invariant*, Journal of Algebra **53** (1978), 36-39, available at http://www.maths.ed.ac.uk/~aar/papers/dye.pdf.