# The Kervaire Invariant One Problem, Talk 3, Independent University of Moscow, Fall semester 2016

## 1 Motivation.

**Notation 1.1.** We will denote by $\mathcal{S}$ the $\infty$-category of spaces and by $\mathcal{S}_*$ the $\infty$-category of pointed spaces. We will denote by $\mathsf{Sp}$ the $\infty$-category of spectra which is defined as the limit

$$\ldots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

in the $\infty$-category $\mathsf{Cat}_\infty$ of $\infty$-categories. Recall that $\mathsf{Sp}$ is presentable, stable and symmetric monoidal. For two spectra $E$ and $X$ we will denote by $\mathsf{Map}(X, E) \in \mathsf{Sp}$ the mapping spectrum from $X$ to $E$, by $E_*X := \pi_*(X \otimes E)$ the $E$-homology of $X$ and by $E^*X := \pi_{-*}\,\mathsf{Map}(X, E)$ the $E$-cohomology of $X$. All the (co)limits are assumed to be homotopy (co)limits. We will frequently omit the symbol $\infty$ further.
In this lecture we will be interested in the homotopy type of various spectra and therefore will be mostly working in the homotopy category of spectra $\mathsf{h\,Sp}$ which we will consider as a triangluated category. By a homotopy commutative ring spectrum we will further mean an object $R \in \mathsf{CAlg}(\mathsf{h\,Sp})$ (not an $\mathbb{E}_\infty$-spectrum!). We also set $\mathsf{Mod}(R) := \mathsf{Mod}_{h\,\mathsf{Sp}}(R)$.

As a motivation, suppose we have an ordinary (discrete) ring $A$ and we wish to understand the category $\mathsf{Mod}(A)$ of its modules, or, in more geometric terms, the category $\mathsf{QCoh}(\mathsf{Spec}\,A)$ of quasi-coherent sheaves over the affine scheme $\mathsf{Spec}\,A$. One of the strategies to attack the problem is to cover the ring

$$\mathsf{Spec}\,B \xrightarrow{f} \mathsf{Spec}\,A$$

and try to use descent methods. Namely, the diagram of affine schemes

$$\ldots \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \mathsf{Spec}\,B \times_{\mathsf{Spec}\,A} \mathsf{Spec}\,B \rightrightarrows \mathsf{Spec}\,B$$

gives us a diagram of categories

$$\ldots \;\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}\; \mathsf{QCoh}(\mathsf{Spec}\,B \times_{\mathsf{Spec}\,A} \mathsf{Spec}\,B) \leftleftarrows \mathsf{QCoh}(\mathsf{Spec}\,B)$$

which in algebraic terms can be written as

$$\ldots \;\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}\; \mathsf{Mod}(B \otimes_A B) \leftleftarrows \mathsf{Mod}(B).$$

If the morphism $f$ was good enough (faithfully flat), the category $\mathsf{QCoh}(\mathsf{Spec}\,A) \simeq \mathsf{Mod}(A)$ can be recovered as the limit (totalization) of the diagram above. Geometrically, this means that a quasi-coherent sheaf on $\mathsf{Spec}\,A$ is precisely determined by its pullback to $\mathsf{Spec}\,B$ together with the descent data. Notice that in this 1-categorical situation it is sufficient to consider only the first three categories in the diagram (since the category of discrete categories is a 2-category).

Let now $Y \in \mathsf{h\,Sp}$ be an arbitrary spectrum we would like to understand. Notice that since the sphere spectrum $\mathbb{S} \in \mathsf{h\,Sp}$ is the monoidal unit, the category of modules over it is the homotopy category of the spectra itself, that is, there is an equivalence $\mathsf{Mod}(\mathbb{S}) \simeq \mathsf{h\,Sp}$ of triangulated categories. Consequently, we can canonically treat $Y$ as an $\mathbb{S}$-module. One of the things we may do is to try to emulate the situation above: namely, find an appropriate homotopy commutative ring spectrum $R$ so that the unique unit morphism

$$\mathbb{S} \xrightarrow{f} R$$

of homotopy commutative ring spectra would serve us as a covering of the sphere spectrum. We would be then able to understand the spectrum $Y$ by working with its pullback $Y \otimes R$ to the ring spectrum $R$ together with the descent data. To work with the problem more systematically, let us introduce the following

**Definition 1.2.** For a homotopy commutative ring spectrum $R$ we define a cosimplicial spectrum $\mathsf{Cobar}^\bullet_R(Y) \in \mathsf{Funct}(\Delta, \mathsf{Sp})$ simply as

$$\mathsf{Cobar}^\bullet_R(Y) := \left( Y \otimes R \rightrightarrows Y \otimes R \otimes R \rightrightarrows \ldots \right).$$

Now the question is: when the natural map $Y \xrightarrow{\varphi} \mathsf{Tot}(\mathsf{Cobar}^\bullet_R(Y))$ is an equivalence?

## 2 Bousfield Localization

To attack the question above, recall first the following

**Definition 2.1.**
1) A spectrum $A \in \mathsf{h\,Sp}$ is called **$R$-acyclic**, if $A \otimes R \simeq 0$. We will denote the full subcategory of $\mathsf{h\,Sp}$ spanned by $R$-acyclic spectra by $R - \mathsf{Acycl}$.
2) A spectrum $X \in \mathsf{h\,Sp}$ is called **$R$-local**, if $\mathsf{Map}(A, X) \simeq 0$ for any $R$-acyclic spectrum $A \in R - \mathsf{Acycl}$. We will denote the full subcategory of $\mathsf{h\,Sp}$ spanned by $R$-local spectra by $R - \mathsf{Loc}$.

**Examples 2.2.**
1) Since $A \otimes 0 \simeq 0$ for any spectrum $A \in \mathsf{h\,Sp}$ we see that we have $0 - \mathsf{Acycl} \simeq \mathsf{h\,Sp}$ and $0 - \mathsf{Loc} \simeq 0$.

2) Since $A \otimes \mathbb{S} \simeq A$ for any spectrum $A \in \mathsf{h\,Sp}$ we see that $\mathbb{S} - \mathsf{Acycl} \simeq 0$ and $\mathbb{S} - \mathsf{Loc} \simeq \mathsf{h\,Sp}$.

3) Notice that for any spectrum $A \in \mathsf{h\,Sp}$ we have $A \otimes H\mathbb{Q} = 0$ iff $\pi_*(A) \otimes \mathbb{Q} \simeq 0$, where $H\mathbb{Q}$ is the rational Eilenberg-Maclane spectrum. Consequently, we see that $H\mathbb{Q} - \mathsf{Acycl}$ is the category of spectra with torsion homotopy groups and therefore $H\mathbb{Q} - \mathsf{Loc}$ is the category of spectra with rational homotopy groups.

4) Let $R$ be a homotopy commutative ring spectrum. We then argue that any $M \in \mathsf{Mod}(R)$ is $R$-local. Indeed, since for any spectrum $A \in \mathsf{Sp}$ we have an adjunction

$$\mathsf{Map}(A, M) \simeq \mathsf{Map}_{\mathsf{Mod}(R)}(A \otimes R, M)$$

we see that any morphism $A \longrightarrow M$ factors as the composition

$$A \longrightarrow A \otimes R \longrightarrow M \otimes R \longrightarrow M.$$

If the spectrum $A$ was $R$-acyclic, then $A \otimes R \simeq 0$ and the result follows.

Now notice that directly from the construction the category $R - \mathsf{Loc}$ is closed under limits. Now since spectra of the form $Y \otimes R^{\otimes n}$ are $R$-modules spectra, due to the example above they are all $R$-local. Consequently, the spectrum $\mathsf{Tot}(\mathsf{Cobar}^\bullet_R(Y))$ is also $R$-local as the limit of $R$-local spectra. Therefore if the spectrum $Y$ is not $R$-local in the beginning, it is hopeless that the morphism $Y \xrightarrow{\varphi} \mathsf{Tot}(\mathsf{Cobar}^{\bullet}_R(Y))$ is an equivalence. We may, however, ask our question a bit more carefully. Recall the following result due to Bousfield

**Theorem 2.3.** ([Bou]) The natural inclusion functor

$$R - \mathsf{Loc} \longrightarrow \mathsf{h\,Sp}$$

admits a left adjoint functor $L_R$ which is called **Bousfield localization** with respect to $R$.

**Conventions.** By abuse of notation for $X \in \mathsf{h\,Sp}$ whenever it is convenient we will further also denote by $L_R(X)$ the composition of the Bousfield localization and the inclusion $R - \mathsf{Loc} \longrightarrow \mathsf{h\,Sp}$ .

**Remarks 2.4.**
1) In particular, it follows that any spectrum $X \in \mathsf{h\,Sp}$ sits in a distinguished triangle

$$G_R(X) \longrightarrow X \xrightarrow{\psi_X} L_R X,$$

where $L_R(X) \in R-\mathsf{Loc} \subseteq \mathsf{h\,Sp}$ and $G_R(X) \in R-\mathsf{Acycl} \subseteq \mathsf{h\,Sp}$. This gives the following convenient criterion: if a morphism $X \xrightarrow{f} Y$ in $\mathsf{h\,Sp}$ induces an equivalence on $R$-homology (such maps are called $R$-equivalences) and $Y$ is $R$-local, then $f$ exhibits $Y$ as the Bousfield localization of $X$ with respect to $R$. Indeed, since $Y$ is $R$-local, due to the universality of $L_R(X)$ the morphism $f$ factors as

$$X \xrightarrow{\psi_X} L_R(X) \xrightarrow{g} Y.$$

Now since $G_R(X) \in R - \mathsf{Acycl}$, we have $R_*(G_R(X)) = 0$ so that the morphism $\psi_X$ induces an isomorphism on $R$-homology. Since so does $f$, we see that the morphism $g$ induces an isomorphism on $R$-homology. Consequently, the fiber $F$ of the sequence

$$F \longrightarrow L_R(X) \xrightarrow{g} Y.$$

is $R$-acyclic. Now it is left to use that the functor $L_R$ transforms distinguished triangles to distinguished triangles so that after applying $L_R$ to the distinguished triangle above we get a distinguished triangle

$$L_R(F) \longrightarrow L_R(L_R(X)) \xrightarrow{L_R(g)} L_R(Y)$$

which since $L_R(X)$ and $Y$ are $R$-local can be rewritten as

$$L_R(F) \longrightarrow L_R(X) \xrightarrow{g} Y.$$

But since $F$ is $R$-acyclic, we have $L_R(F) \simeq 0$ and hence $g$ is an equivalence.
2) The Bousfield localization defined above can be actually lifted to a similar localization in the world of $\infty$-categories.

**Examples 2.5.**
1) Let $X$ be an arbitrary spectrum. Then since $\mathbb{S} - \mathsf{Loc} \simeq \mathsf{h\,Sp}$ we have $L_{\mathbb{S}}(X) \simeq X$.

2) For an arbitrary spectrum $X$ have $L_{H\mathbb{Q}}(X) \simeq X \otimes H\mathbb{Q}$. Indeed, since $H\mathbb{Q} \otimes H\mathbb{Q} \simeq H\mathbb{Q}$, the morphism $X \longrightarrow X \otimes H\mathbb{Q}$ induces an equivalence on $H\mathbb{Q}$-homology. Now since $X \otimes H\mathbb{Q}$ is a module over $H\mathbb{Q}$, it is $H\mathbb{Q}$-local. The result now follows from the remark 2.4.

3) The Bousfield localization at the Moore spectrum $\mathbb{S}_{(p)}$ is given by $L_{\mathbb{S}_{(p)}} X = X \otimes S_{(p)}$, where $S_{(p)}$ is defined as the colimit

$$\mathbb{S}_{(p)} := \mathsf{colim}(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} ...)$$

Indeed, since smash product commutes with colimits, it follows that $\mathbb{S}_{(p)} \otimes \mathbb{S}_{(p)} \simeq \mathbb{S}_{(p)}$ and therefore the natural map $X \simeq X \otimes \mathbb{S} \longrightarrow X \otimes \mathbb{S}_{(p)}$ is an $\mathbb{S}_{(p)}$-equivalence. Now since $X \otimes \mathbb{S}_{(p)}$ is a module spectrum over $\mathbb{S}_{(p)}$, it is $\mathbb{S}_{(p)}$-local. The result now follows from the remark 2.4.

4) The Bousfield localization at the Moore spectrum $\mathbb{S}/p$ is given by the $p$-completion $L_{\mathbb{S}/p}X \simeq \widehat{X}_p$, where $\widehat{X}_p$ is defined as the limit

$$\widehat{X}_p := \mathsf{lim}(... \longrightarrow X/p^3 \longrightarrow X/p^2 \longrightarrow X/p).$$

Indeed, the diagram of distinguished triangles

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] \\
X \arrow[r, "p^3"'] \arrow[d, "p"'] & X \arrow[r] \arrow[d, "\mathsf{Id}_X"] & X/p^3 \arrow[d] \\
X \arrow[r, "p^2"'] \arrow[d, "p"'] & X \arrow[r] \arrow[d, "\mathsf{Id}_X"] & X/p^2 \arrow[d] \\
X \arrow[r, "p"'] & X \arrow[r] & X/p
\end{tikzcd}$$

by taking limits vertically gives a distinguished triangle

$$\mathsf{Map}(\mathbb{S}_{(p)}, X) \longrightarrow X \xrightarrow{\psi} \widehat{X}_p$$

Now since the map $\mathbb{S}_{(p)} \xrightarrow[\sim]{p} \mathbb{S}_{(p)}$ is an equivalence, the induced map $\mathsf{Map}(\mathbb{S}_{(p)}, X) \xrightarrow[\sim]{p} \mathsf{Map}(\mathbb{S}_{(p)}, X)$ is an equivalence so that

$$\mathsf{Map}(\mathbb{S}_{(p)}, X) \otimes \mathbb{S}/p \simeq \mathsf{Map}(\mathbb{S}_{(p)}, X)/p \simeq 0$$

and therefore $\psi$ is a $\mathbb{S}/p$-equivalence. Now to see that $\widehat{X}_p$ is $\mathbb{S}/p$-local, let $A \in \mathsf{h\,Sp}$ be such that $A \otimes \mathbb{S}/p \simeq 0$. From the distinguished triangle $A \xrightarrow{p} A \longrightarrow A/p$ we see that the morphism $A \xrightarrow[\sim]{p} A$ is an equivalence, so that $A \otimes \mathbb{S}_{(p)} \simeq A$. Consequently, we get

$$\mathsf{Map}(A, \mathsf{Map}(\mathbb{S}_{(p)}, X)) \simeq \mathsf{Map}(A \otimes \mathbb{S}_{(p)}, X) \simeq \mathsf{Map}(A, X)$$

and so that the first morphism in the distinguished triangle

$$\mathsf{Map}(A, \mathsf{Map}(\mathbb{S}_{(p)}, X)) \longrightarrow \mathsf{Map}(A, X) \longrightarrow \mathsf{Map}(A, \widehat{X}_p)$$

is an equivalence showing that $\mathsf{Map}(A, \widehat{X}_p) \simeq 0$. It follows that $\widehat{X}_p$ is $\mathbb{S}/p$-local and it is left to use the remark 2.4.
It is also useful to calculate the homotopy groups of the spectrum $\widehat{X}_p$. Recall that for any abelian group $G$ the free resolution

$$\bigoplus_R \mathbb{Z} \longrightarrow \bigoplus_F \mathbb{Z} \longrightarrow G$$

induces the sequence

$$\bigoplus_R \mathbb{S} \longrightarrow \bigoplus_F \mathbb{S} \longrightarrow \mathbb{S}G \longrightarrow \bigoplus_R \Sigma\mathbb{S} \longrightarrow \bigoplus_F \Sigma\mathbb{S}$$

which in fact can be served as a concrete construction of the Moore spectrum $\mathbb{S}G \in \mathsf{h}\,\mathsf{Sp}$. Consequently, for any spectrum $X$ we get a sequence

$$[\bigoplus_F \Sigma\mathbb{S}, X] \longrightarrow [\bigoplus_R \Sigma\mathbb{S}, X] \longrightarrow [\mathbb{S}G, X] \longrightarrow [\bigoplus_F \mathbb{S}, X] \longrightarrow [\bigoplus_R \mathbb{S}, X]$$

which we can rewrite as

$$\mathsf{Hom}_{\mathsf{Ab}}(\bigoplus_F \mathbb{Z}, \pi_1(X)) \longrightarrow \mathsf{Hom}_{\mathsf{Ab}}(\bigoplus_R \mathbb{Z}, \pi_1(X)) \longrightarrow [\mathbb{S}G, X] \longrightarrow$$

$$\longrightarrow \mathsf{Hom}_{\mathsf{Ab}}(\bigoplus_F \mathbb{Z}, \pi_0(X)) \longrightarrow \mathsf{Hom}_{\mathsf{Ab}}(\bigoplus_R \mathbb{Z}, \pi_0(X))$$

giving a short exact sequence

$$\mathsf{Ext}^1_{\mathsf{Ab}}(G, \pi_1(X)) \longrightarrow [SG, X] \longrightarrow \mathsf{Hom}_{\mathsf{Ab}}(G, \pi_0(X))$$

and similar for higher homotopy groups. It is direct to show that this sequence non-canonically splits. Now let $\mathbb{Z}/p^\infty \in \mathsf{Ab}$ be the cokernel of the morphism $\mathbb{Z} \longrightarrow \mathbb{Z}_{(p)}$ so that we get a distinguished triangle

$$\Omega\mathbb{S}/p^\infty \longrightarrow \mathbb{S} \longrightarrow \mathbb{S}_{(p)}.$$

Applying tha functor $\mathsf{Map}(\bullet, X)$ we get a triangle

$$\mathsf{Map}(\mathbb{S}_{(p)}, X) \longrightarrow X \longrightarrow \mathsf{Map}(\Omega\mathbb{S}/p^\infty, X)$$

which gives an equivalence $\widehat{X}_{(p)} \simeq \mathsf{Map}(\Omega\mathbb{S}/p^\infty, X) \simeq \mathsf{Map}(\mathbb{S}/p^\infty, \Sigma X)$. Now applying the short exact sequence above in the special case $G := \mathbb{Z}/p^\infty$ for each $n \in \mathbb{N}$ we get a short exact sequence

$$\mathsf{Ext}^1_{\mathsf{Ab}}(\mathbb{Z}/p^\infty, \pi_n(X)) \longrightarrow \pi_n(\widehat{X}_p) \longrightarrow \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}/p^\infty, \pi_{n-1}(X))$$

which non-canonically splits. In particular, we see that for those $n$ for which $\pi_n X$ is finitely generated we have $\pi_n \widehat{X}_p \simeq \widehat{\pi_n(X)}_p$.

5) Let $X$ be connective spectrum. We argue that then $L_{H\mathbb{F}_p} X \simeq L_{\mathbb{S}/p} X$ (for non-connective spectrum this may be false). Indeed, notice that since $X$ connective, it admits a Postnikov tower which is bounded below. Since all of its slices are Eilenberg-Maclane spectra, they are all $H$-local, and hence $X$ is also $H$-local as a limit of $H$-local spectra. Consequently, we get $L_H X \simeq X$ so that $L_{\mathbb{S}/p}(X) \simeq L_{\mathbb{S}/p}(L_H X)$. We now argue that for an arbitrary spectrum $E$ we have $L_{\mathbb{S}/p}(L_E(X)) \simeq L_{E/p}(X)$ and the result will follows by setting $E := H$. In order to see this, consider the composition

$$X \xrightarrow{\varphi} L_E X \xrightarrow{\psi} L_{\mathbb{S}/p}(L_E(X)).$$

Since

$$X \otimes E/p \simeq (X \otimes E)/p \simeq (L_E X \otimes E)/p \simeq L_E X \otimes E/p$$

we see that the morphism $\varphi$ is a $E/p$-equivalence. Similarly, since

$$L_E X \otimes E/p \simeq L_E X \otimes \mathbb{S}/p \otimes E \simeq L_{S/p}(L_E(X)) \otimes \mathbb{S}/p \otimes E \simeq L_{\mathbb{S}/p}(L_E(X)) \otimes E/p$$

we see that the morphism $\psi$ is a $E/p$-equivalence. Consequently, the composition $\psi \circ \varphi$ is also a $E/p$-equivalence. It is left to show that $L_{\mathbb{S}/p}(L_E(X))$ is $E/p$-local. Just as in the example 4 above, using the triangle

$$\mathsf{Hom}_{\mathsf{Sp}}(\mathbb{S}_{(p)}, L_E(X)) \longrightarrow L_E(X) \longrightarrow L_{\mathbb{S}/p}(L_E(X))$$

it is sufficient to prove that for any $E/p$-acyclic $A$ the induced map

$$\mathsf{Map}(A_{(p)}, L_E(X)) \simeq \mathsf{Map}(A, \mathsf{Hom}_{\mathsf{Sp}}(\mathbb{S}_{(p)}, L_E(X))) \longrightarrow \mathsf{Map}(A, L_E(X))$$

is an equivalence. But we have $\mathsf{Map}(A_{(p)}, L_E(X)) \simeq \mathsf{Map}(L_E(A_{(p)}), L_E(X))$ and $\mathsf{Map}(A, L_E(X)) \simeq \mathsf{Map}(L_E(A), L_E(X))$ so that it is sufficient to prove that the natural map $A \longrightarrow A_{(p)}$ induces an equivalence $L_E(A) \longrightarrow L_E(A_{(p)})$. Since $E/p \otimes A \simeq 0$, we see that the morphism $A \otimes E \xrightarrow[\sim]{p} A \otimes E$ is an equivalence, so that $A \otimes E \simeq A_{(p)} \otimes E$ showing that $L_E(A) \simeq L_E(A_{(p)})$ as desired. The result now follows from the remark 2.4.

6) We have $L_{KU} \simeq L_{KO}$. Indeed, since $KU \simeq KO \otimes \Sigma^\infty \mathbb{CP}^2$ we get a triangle

$$\Sigma KO \xrightarrow{\eta} KO \longrightarrow KU,$$

where $\eta \in \pi_1 \mathbb{S}$ is the generator. Hence $X \otimes KO \simeq 0$ implies $X \otimes KU \simeq 0$. Conversely, if $X \otimes KU \simeq 0$, then $KO_*(X) \xrightarrow[\sim]{\eta_*} KO_*(X)$ is an isomorphism. But since $\eta$ is nilpotent (in fact, $\eta^4 = 0$), it follows that $KO_* X = 0$. Consequently, we see that $KO - \mathsf{Acycl} = KU - \mathsf{Acycl}$ so that $L_{KU} \simeq L_{KO}$ as desied.

We therefore see that the map

$$Y \xrightarrow{\varphi} \mathsf{Tot}(\mathsf{Cobar}^\bullet_R(Y))$$

naturally factors as

$$Y \longrightarrow L_R Y \xrightarrow{\psi} \mathsf{Tot}(\mathsf{Cobar}^\bullet_R(Y)) \ .$$

If the spectra $R$ and $Y$ were good enough, the localization $L_R Y$ is not that far from the initial spectrum $Y$ so that the natural question is when the morphism $\psi$ is an equivalence. In order to answer the question, we need the following

**Definition 2.6.** For an ordinary (discrete) ring $S$ define its **core** $cS$ as an equalizer

$$cS \longrightarrow S \rightrightarrows S \otimes S.$$

We now have a result due to Bousfield

**Theorem 2.7.** ([Bou]) Let $R$ be a homotopy commutative ring spectrum such that the core $\pi_0 R$ is either $\mathbb{Z}/n$, $n \geq 2$, or localization of integers away from some nonempty set of primes $I$, that is, $c\pi_0 R = \mathbb{Z}[I^{-1}]$.
Then if $R$ and $Y$ are both connective, the natural map

$$L_R Y \xrightarrow{\psi} \mathsf{Tot}(\mathsf{Cobar}^\bullet_R(Y))$$

is an equivalence.

We instantly get the following

**Corollary 2.8.** Suppose we are in the situation as in the theorem above and let $X$ be any spectrum. Since the functor $\mathsf{Hom}_{\mathsf{Sp}}(X, \bullet)$ commutes with limits, we get an equivalence

$$\mathsf{Hom}^\bullet_{\mathsf{h\,Sp}}(X, L_R Y) \xrightarrow[\sim]{\psi_*} \mathsf{Hom}^\bullet_{\mathsf{h\,Sp}}(X, \mathsf{Tot}(\mathsf{Cobar}^\bullet_R(Y))) \simeq \mathsf{Tot}(\mathsf{Hom}^\bullet_{\mathsf{h\,Sp}}(X, \mathsf{Cobar}^\bullet_R(Y))) \ .$$

The Bousfield-Kan spectral sequence of a cosimplicial space then gives us a spectral sequence

$$\mathsf{E}_2^{s,t} = \pi^s \pi_t \, \mathsf{Hom}_{\mathsf{h\,Sp}}(X, \mathsf{Cobar}^\bullet_R(Y)) \simeq \pi^s[\Sigma^t X, \mathsf{Cobar}^\bullet_R(Y)] \Rightarrow [\Sigma^{t-s} X, L_R Y].$$

with the differential $|d_r| = (r, r-1)$.

## 3 From homotopy theory to homological algebra.

Now the spectral sequence above is quite useful itself, but we would like to have a bit more pleasant representation of the second page- currenly, our second page has some morphisms in the homotopy category of spectra which are in many cases quite hard to calculate. Nevertheless, while computations in homotopy theory are hard, computations in homological algebra are much easier. Therefore it is a good idea to try to somehow find a representation of the second page in terms of the homological algebra.

The first idea is as following: recall that if $R$ is a homotopy commutative ring spectrum, then $R_* := \pi_* R$ has a natural structure of a commutative graded ring. Moreover, for every $M \in \mathsf{Mod}(R)$ the graded abelian group $M_*$ has a natural structure of a module over $R_*$: concretely, given $r \in R_p$ and $m \in M_q$, the element $rm \in M_{p+q}$ can be defined as the smash product

$$\mathbb{S}^{p+q} \simeq \mathbb{S}^p \otimes \mathbb{S}^q \xrightarrow{r \otimes m} R \otimes M \longrightarrow M,$$

where the last morphism uses the $R$-module structure on $M$. In some cases this simple observation indeed gives a way to transfer to homological algebra:

**Proposition 3.1.** Let $R$ be a homotopy commutative ring spectrum and $M, N \in \mathsf{Mod}(R)$ be $R$-modules. Then if $M_*$ is a projective $R_*$-module, then the natural morphism

$$\mathsf{Hom}^\bullet_{\mathsf{Mod}(R)}(M, N) \longrightarrow \mathsf{Hom}^\bullet_{\mathsf{Mod}(R_*)}(M_*, N_*)$$

is an isomorphism.

*Proof.* In the case when $M_*$ is a free $R_*$-module (and hence $M$ is a free $R$-module), the statement is true by the formal nonsense. Suppose now that $M_*$ is projective so that it is a retract of a free $R_*$-module. Another words, we have a diagram

$$M_* \xrightarrow{i} F_* \xrightarrow{r} M_*$$

such that $r \circ i \simeq \mathsf{Id}_{M_*}$, where $F_*$ is free. Consequently, the graded module $M_* \in \mathsf{Mod}(R_*)$ can be obtained as the colimit

$$F_* \xrightarrow{i o r} F_* \xrightarrow{i o r} F_* \xrightarrow{i o r} \ldots$$

in $\mathsf{Mod}(R_*)$. But then $M \in \mathsf{Mod}(R)$ itself can be obtained as the colimit of the corresponding diagram

$$F \longrightarrow F \longrightarrow F \longrightarrow \ldots$$

in $\mathsf{Mod}(R)$ so that $M \in \mathsf{Mod}(R)$ is a retract of a free $R$-module $F \in \mathsf{Mod}(R)$. Now the desired statement follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{Hom}^\bullet_{\mathsf{Mod}(R)}(M, N) \arrow[r] \arrow[d] & \mathsf{Hom}^\bullet_{M_*}(M_*, N_*) \arrow[d] \\
\mathsf{Hom}^\bullet_{\mathsf{Mod}(R)}(F, N) \arrow[r, "\sim"] \arrow[d] & \mathsf{Hom}^\bullet_{R_*}(F_*, N_*) \arrow[d] \\
\mathsf{Hom}^\bullet_{\mathsf{Mod}(R)}(M, N) \arrow[r] & \mathsf{Hom}^\bullet_{R_*}(M_*, N_*).
\end{tikzcd}$$

$\square$

Recall now that our initial motivation was to understand the second page of the spectral sequence

$$\mathsf{E}_2^{s,t} = \pi^s[\Sigma^t X, \mathsf{Cobar}^\bullet_R(Y)] \simeq \pi^s \, \mathsf{Hom}^t_{\mathsf{Mod}(R)}(X \otimes R, \mathsf{Cobar}^\bullet_R(Y))$$

Though in the case when $(X \otimes R)_* = R_*X$ is projective $R_*$-module the proposition above works fine, in our special case when $M = X \otimes R$ is the free $R$-module on the spectrum $X$, there is even a richer structure on $R_*X$. In order to formulate it, recall first the following

**Definition 3.2.** A (graded) **Hopf Algebroid** is an internal cogropoud in the category of graded rings.

**Remark 3.3.** In concrete terms, a Hopf algebroid $(A, \Gamma)$ is a pair of graded algebras $A$ and $\Gamma$ together with maps
**Left unit/source**

$$\nu_L : A \longrightarrow \Gamma$$

**Right unit/target**

$$\nu_R : A \longrightarrow \Gamma$$

**Augmentation/identity**

$$\epsilon : \Gamma \longrightarrow A$$

**Coproduct/composition**

$$\Delta : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$$

**Conjugation/inverse**

$$c : \Gamma \longrightarrow \Gamma$$

which satisfy various equations so that the functors $\mathsf{Hom}(A, \bullet)$ and $\mathsf{Hom}(\Gamma, \bullet)$ together with $(\nu_L, \nu_R, \epsilon, \Delta, c)$ determine a functor to (small) groupoids, where $\mathsf{Hom}(A, \bullet)$ represents the set of objects and $\mathsf{Hom}(\Gamma, \bullet)$ represents the set of morphisms.

**Example 3.4.** If $\nu_R = \nu_L$ then $\Gamma$ is simply a graded commutative Hopf algebra over $A$.

To understand how the definition above is related to our problem, we first need the following

**Proposition 3.5.** Let $Y, Q$ be two spectra such that $R_*Q$ is flat as a module over $R_*$. Then the natural map

$$R_*Y \otimes_{R_*} R_*Q \longrightarrow R_*(Y \otimes Q)$$

is an isomorphism.

*Proof.* In the case when $Y \simeq \mathbb{S}$ is the sphere spectrum, the statement is obviously true. Now it is left to notice that since $R_*Q$ is flat over $R_*$ both sides preserve exact sequences and filtered colimits by $Y$. ☐

**Corollary 3.6.** Suppose that $R$ is a homotopy commutative ring spectrum such that $R_*R$ is flat over $R_*$ (this is true in the cases $R = H\mathbb{F}_p, MU, BP, KO, KU$ and false in the cases $R = H\mathbb{Z}, MSU$). Then the pair $(R_*, R_*R)$ admits the structure of a Hopf algebroid. Left and right units maps are induced by

$$S \otimes R \xrightarrow{\epsilon \otimes 1} R \otimes R$$

and

$$R \otimes S \xrightarrow{1 \otimes \epsilon} R \otimes R,$$

augmentation map is induced by the multiplication

$$R \otimes R \xrightarrow{\mu} R,$$

coproduct map is induced by the map

$$R \otimes R \simeq R \otimes \mathbb{S} \otimes R \longrightarrow R \otimes R \otimes R$$

and the conjugation map is induced by the twist

$$R \otimes R \xrightarrow{\mathsf{T}} R \otimes R.$$

Similar to modules over rings, for Hopf algebroids we can introduce the following

**Definition 3.7.** Let $(A, \Gamma)$ be a Hopf algebroud. A (left) $(A, \Gamma)$-comodule is a left $A$-module $M$ together with the coation map

$$M \longrightarrow \Gamma \otimes_A M$$

which is counital and coassociative. Given two $(A, \Gamma)$-comodules $M$ and $N$ we will frequently denote by $\mathsf{Hom}_\Gamma(M, N)$ the abelian group of $(A, \Gamma)$-comodule maps between $M$ and $N$.

**Remarks 3.8.**
1) Notice that the fogetful functor $(A, \Gamma)$-comodules to $A$-modules admits a right adjoint, which sends an $A$-module $M \in \mathsf{Mod}(A)$ to the comodule $\Gamma \otimes_A M$.

2) It is quite pleasant to reformulate the homological algebra of Hopf algebroids in geometric terms. Namely, just as any ring gives an affine scheme, any Hopf algebroid gives a prestack. Moreover, the quasi-coherent sheaves over that prestack are precisely comodules over the initial Hopf algebroid. In particular, the adjunction above can be then understood simply as the pull-back/pushforward adjunction. From here one can then stackificate this prestack in an appropriate topology (say, flat topology) to try to describe the category of its quasi-coherent sheaves using geometric methods. A very interesting special case of the phenomena above is the case when we consider the Hopf algebroid $(MU_*, MU_*MU)$ built from the complex cobordisms spectrum: the resulting stack happens to be closely related to the stack of formal groups which has a deep connection to number theory.

**Example 3.9.** Given a commutative ring spectrum $R$ with $R_*R$ being flat over $R_*$ for any other spectrum $X \in \mathsf{Sp}$ the $R$-homology $R_*X$ of $X$ admit the structure of a $(R_*, R_*R)$-comodule with the coaction map induced by

$$R \otimes \mathbb{S} \otimes X \longrightarrow R \otimes R \otimes X.$$

In particular, because of the adjunction discussed in 3.8 we see that for $X, Y \in \mathsf{h\,Sp}$ there is an equivalence

$$\mathsf{Hom}_{R_*R}(R_*X, R_*(R \otimes X)) \simeq \mathsf{Hom}_{R_*R}(R_*X, R_*R \otimes_{R_*} R_*X) \xrightarrow[\sim]{} \mathsf{Hom}^\bullet_{\mathsf{Mod}(R_*)}(R_*X, R_*Y).$$

The language of Hopf Algebroids provides us with the appropriate language to shift from the stable homotopy theory problem we had to homological algebra:

**Proposition 3.10.** Let $R$ be a homotopy commutative ring spectrum such that $R_*R$ is flat over $R_*$ and $X, Y$ be arbitrary spectra. Then the natural map

$$[X, Y \otimes R]^\bullet \longrightarrow \mathsf{Hom}^\bullet_{R_*R}(R_*X, R_*(Y \otimes R))$$

is an isomorphism if $R_*X$ is projective over $R_*R$.

*Proof.* Observe that there is a commutative diagram

$$\begin{tikzcd}
{[X, Y \otimes R]^\bullet} \arrow[rr, "(1)"] \arrow[dr, "(2)"'] & & \mathsf{Hom}^\bullet_{R_*R}(R_*X, R_*(Y \otimes R)) \arrow[dl, "(3)"] \\
& \mathsf{Hom}^\bullet_{\mathsf{Mod}(R_*)}(R_*X, R_*Y) &
\end{tikzcd}$$

Now the morphism 2 is an isomorphism by the adjunction

$$[X, Y \otimes R]^\bullet \simeq \mathsf{Hom}^\bullet_{\mathsf{Mod}(R)}(X \otimes R, Y \otimes R)$$

and Proposition 3.1 in the special case $M := X \otimes R$, $N := Y \otimes R$. Since the morphism (3) is an isomorphism by the example 3.9, the morphism (1) is also an isomorphism as desired. ∎

The proposition above allows us to prove the main

**Theorem 3.11.** Let $R$ be a commutative ring spectrum and $X, Y$ be two arbitrary spectra. Suppose also that
1) $R$ and $Y$ are connective.
2) The core $\pi_0 R$ is either $\mathbb{Z}/n,\ n \geq 2$, or localization of integers away from the set of primes $I$, that is, $c\pi_0 R = \mathbb{Z}[I^{-1}]$.
3) $R_*R$ is flat over $R_*$ and $R_*X$ is projective over $R_*R$.
Then there is there is a strongly convergent spectral sequence

$$\mathsf{E}_2^{s,t} = \mathsf{Ext}^{s,t}_{R_*R}(R_*X, R_*Y) \Rightarrow [\Sigma^{t-s}X, L_R Y]$$

with $|d_r| = (r, r-1)$.

*Proof.* We have

$$\mathsf{E}_2^{s,t} = \pi^s[\Sigma^t X, \mathsf{Cobar}^\bullet_R(Y)] = \pi^s([\Sigma^t X, Y \otimes R] \rightrightarrows [\Sigma^t X, Y \otimes R \otimes R] \Rrightarrow \ldots) \simeq$$

$$\simeq \pi^s(\mathsf{Hom}_{R_*R}(R_*\Sigma^t X, R_*(Y \otimes R)) \rightrightarrows \mathsf{Hom}_{R_*R}(R_*\Sigma^t X, R_*(Y \otimes R \otimes R)) \Rrightarrow \ldots) \simeq$$

$$\simeq \pi^s(\mathsf{Hom}_{R_*R}(\Sigma^t R_*X, R_*Y \otimes_{R_*} R_*R) \rightrightarrows \mathsf{Hom}_{R_*R}(\Sigma^t R_*X, R_*Y \otimes_{R_*} R_*R \otimes_{R_*} R_*R) \Rrightarrow \ldots) \simeq$$

$$\simeq \mathsf{Ext}^s_{R_*R}(\Sigma^t R_*X, R_*Y) = \mathsf{Ext}^{s,t}_{R_*R}(R_*X, R_*Y)$$

as desired. □

**Remarks 3.12.**
1) In fact, it is not hard to show that the spectral sequence above is multiplicative.

2) Similarly, there is a cohomology version of the spectral sequence with

$$\mathsf{E}_2^{s,t} = \mathsf{Ext}^{R^*R}_{s,t}(R^*Y, R^*X) \Rightarrow [\Sigma^{t-s}X, L_R Y]$$

with $|d_r| = (r, r-1)$. However, this tends to be less managable since in most of the cases there are no simple finite hypotheses.

3) From the geometric perspective, let $\mathcal{M}$ be the prestack which corresponds to the Hopf algebroid $(R_*, R_*R)$ and $\mathcal{F}_X, \mathcal{F}_Y$ be the quasi-coherent sheaves which correspond to the $(R_*, R_*R)$-comodules $R_*X$ and $R_*Y$. Then the spectral sequence above can be rewritten as

$$\mathsf{Ext}^{\bullet,\bullet}_{\mathsf{QCoh}(\mathcal{M})}(\mathcal{F}_X, \mathcal{F}_Y) \Rightarrow [\Sigma^\bullet X, L_R Y].$$

In particular, in the special case when $X := \mathbb{S}$ is the sphere spectrum we have $\mathcal{F}_X \simeq \mathcal{O}_\mathcal{M}$ so that we get a spectral sequence

$$\mathsf{H}^\bullet(\mathcal{F}_Y; \mathcal{M}) \Rightarrow \pi_\bullet(L_R Y)$$

**Example 3.13.** Consider the special case $R := H\mathbb{F}_p$ and $X, Y := \mathbb{S}$. Then by the example 2.5 we have $\mathbb{L}_{H\mathbb{F}_p}(\mathbb{S}) \simeq \widehat{\mathbb{S}}_p$ and therefore we get two spectral sequences

$$\mathsf{E}_2^{s,t} = \mathsf{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}(\widehat{\mathbb{S}}_p)$$

and

$$\mathsf{E}_2^{s,t} = \mathsf{Ext}^{\mathcal{A}_p}_{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}(\widehat{\mathbb{S}}_p).$$

Notice that since $\pi_0(\mathbb{S}) \simeq \mathbb{Z}$ and $\pi_i(\mathbb{S})$ are finite groups for all $i \neq 0$ by the explicit description of the homotopy groups of $p$-completion we get

$$\pi_{t-s}(\widehat{\mathbb{S}}_p) = \begin{cases} \text{the p primary part of } \pi_{t-s}(\mathbb{S}),\ t-s \neq 0 \\ \widehat{\mathbb{Z}}_p,\ t-s=0. \end{cases}$$

# 4 Steenrod algebra.

Now in the next lecture we will need some first values of $\mathsf{Ext}_s^{\mathcal{A}_2}(\mathbb{F}_2, \Sigma^t\mathbb{F}_2)$ which is the second page of the cohomological spectral sequence with $R = H\mathbb{F}_2$, $X = Y = \mathbb{S}$. Recall that the mod 2 Steenrod algebra $\mathcal{A}_2$ is defined simply as $H\mathbb{F}_2^*(H\mathbb{F}_2)$ and can be described explicitly as follows:
1) As an $\mathbb{F}_2$-module it is generated by the elements of the form

$$\mathsf{Sq}^{i_1,i_2,\dots,i_n} := \mathsf{Sq}^{i_1}\mathsf{Sq}^{i_2}\dots\mathsf{Sq}^{i_n},$$

where $i_k \geq 2i_{k+1}$ with the degree $|\mathsf{Sq}^{i_n}| = i_n$. Such elements are called **admissible**.
2) The ideal of relations in $\mathcal{A}_2$ is generated by the Adem relation

$$\mathsf{Sq}^a\mathsf{Sq}^b = \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i} \mathsf{Sq}^{a+b-i}\mathsf{Sq}^i$$

where $a < 2b$. Here we use the convention that $\binom{p}{q} = 0$ if $p < q$ of if $q < 0$. If follows that as an algebra $\mathcal{A}_2$ is generated by the elements of the form $\mathsf{Sq}^{2^i}$ with $i \geq 0$.

**Example 4.1.**
0) In degree 0 we have an only element $\mathsf{Sq}^0 = 1$.
1) In degree 1 we have an only element $\mathsf{Sq}^1$.
2) In degree 2 we have elements $\mathsf{Sq}^2$ and $\mathsf{Sq}^{1,1} = 0$ due to the Adem relation.
3) In degree 3 we have elements $\mathsf{Sq}^3$, $\mathsf{Sq}^{2,1}$ and $\mathsf{Sq}^{1,2} = \mathsf{Sq}^3$.
4) In degree 4 we have elements $\mathsf{Sq}^4$, $\mathsf{Sq}^{3,1}$, $\mathsf{Sq}^{1,3} = 0$ and $\mathsf{Sq}^{2,2} = \mathsf{Sq}^{3,1}$.
5) In degree 5 we have elements $\mathsf{Sq}^5$, $\mathsf{Sq}^{4,1}$,$\mathsf{Sq}^{1,4} = \mathsf{Sq}^5$, $\mathsf{Sq}^{2,3} = \mathsf{Sq}^5 + \mathsf{Sq}^{4,1}$ and $\mathsf{Sq}^{3,2} = 0$.

Now to calculate the first values of the ext groups $\mathsf{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_p, \mathbb{F}_p)$ let us write a free resolution

$$\dots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{F}_2$$

of $\mathbb{F}_2$ as an $\mathcal{A}_2$ module.
0) The easiest choice is to set $P_0 := \mathcal{A}_2\langle g_{0,0}\rangle$ a free module on a generator called $g_{0,0}$ which has degree 0. The map $\epsilon$ is then simply the augmentation map.
1) The augmentation ideal of $\mathcal{A}_2$ as a module is generated by admissible elements $\mathsf{Sq}^{i_1,i_2,\dots,i_n}$ with $i_1 + \dots + i_n \geq 1$. As $\mathcal{A}_2$ is generated as an algebra by the elements of the form $\mathsf{Sq}^{2^i}$ it follows that we can set $P_1 := \bigoplus_{i\geq 0}\mathcal{A}\langle g_{1,i}\rangle$ where $g_{1,i}$ sits in degree $2^i$ with $d_1(g_{1,i}) := \mathsf{Sq}^{2^i} g_{0,0}$.

2) The kernel of the map $\bigoplus_{i\geq 0}\mathcal{A}\langle g_{1,i}\rangle \xrightarrow{d_1} \mathcal{A}_2\langle g_{0,0}\rangle$ encodes relations in the Steenrod algebra. In the range $t \leq 11$ we can generate $P_2$ by the elements $g_{2,1}$,$g_{2,2}$,$g_{2,3}$,$g_{2,4}$ and $g_{2,5}$ with

$$d_2(g_{2,1}) = \mathsf{Sq}^3 g_{1,0} + \mathsf{Sq}^2 g_{1,1}$$

$$d_2(g_{2,2}) = \mathsf{Sq}^4 g_{1,0} + \mathsf{Sq}^2\mathsf{Sq}^1 g_{1,1} + \mathsf{Sq}^1 g_{1,2}$$
$$d_2(g_{2,3}) = \mathsf{Sq}^7 g_{1,0} + \mathsf{Sq}^6 g_{1,1} + \mathsf{Sq}^4 g_{1,4}$$
$$d_2(g_{2,4}) = \mathsf{Sq}^8 g_{1,0} + \mathsf{Sq}^7 g_{1,1} + \mathsf{Sq}^4\mathsf{Sq}^1 g_{1,2} + \mathsf{Sq}^1 g_{1,3}$$
$$d_2(g_{2,5}) = \mathsf{Sq}^7\mathsf{Sq}^2 g_{1,0} + \mathsf{Sq}^8 g_{1,1} + \mathsf{Sq}^4\mathsf{Sq}^2 g_{1,2} + \mathsf{Sq}^2 g_{1,3}.$$

**Remark 4.2.** For every $s$ we therefore set $P_s := \bigoplus_j \mathcal{A}_2\langle g_{s,j}\rangle$. We then have $\mathsf{Hom}_{\mathcal{A}_2}(\bigoplus_j \mathcal{A}_2\langle g_{s,j}\rangle, \mathbb{F}_2) \simeq \prod_j \mathcal{A}_2\langle\gamma_{s,j}\rangle$ with $\gamma_{s,j}(g_{s,i}) = \delta_{ij}$. In fact, the product is finite in each degree. Then since $\gamma_{s,i}\circ\partial_{s+1} = 0$ we see that $\mathsf{Ext}^s_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \simeq \mathsf{Hom}_{\mathcal{A}_2}(P_s, \mathbb{F}_2) \simeq \mathbb{F}_2\langle\gamma_{s,i}\rangle$.

**Remark 4.3.** One of the many important results in this are is the so-called Adams vanishing: we have $\mathsf{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $t - s < 0$ or $t - s < s$ and $\mathsf{Ext}^{s,s}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\langle\gamma_{s,0}\rangle$, where $\gamma_{s,0}$ is the dual to $g_{s,0}$.

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