Chromatic Homotopy Theory
Problem set 4. Adams-Novikov spectral sequence

Reminder

Let R be a commutative ring spectrum. For a spectrum $X \in \mathrm{Sp}$ define an **R-cobar complex of X** by

$$\mathrm{CB}^\bullet_R(X) := X \otimes R^{\otimes \bullet}.$$

This is a cosimplicial object of Sp, with the arrows induced by the adjunction $- \otimes R \colon \mathrm{Sp} \leftrightarrows \mathrm{Mod}_R \colon U_R$, where U_R is the forgetful functor. Recall that we proved:

Theorem. *Let R be a commutative ring spectrum.*

1. *Assume that R is connective and that $\pi_0 R$ is isomorphic either to $\mathbb{Z}/n$ or to a localization of $\mathbb{Z}$ in some set of primes. Then for any eventually connective spectrum $X \in \mathrm{Sp}$ the natural map $L_R X \to \mathrm{Tot}\,\mathrm{CB}^\bullet_R(X)$ is an equivalence, where $L_R \colon \mathrm{Sp} \to L_R\,\mathrm{Sp}$ denotes the Bousfield localization functor.*
2. *Assume that the ring R_*R is flat over R_*. Let X, Y be a pair of spectra such that R_*X is projective over R_*R. Then the spectral sequence associated to the cosimplicial spectrum $\mathrm{Map}(X, \mathrm{CB}^\bullet_R(Y))$ is naturally a spectral sequence in (R_*, R_*R)-comodules with the second page isomorphic to*

$$E_2^{s,t} \simeq \mathrm{Hom}_{R_*R}(R_*X, R_*Y)$$

and differentials of degree $|d_r| = (r, r-1)$. If additionally $\pi_0 R$ is as in the previous part, and X and Y are eventually connective, then the spectral sequence above converges strongly to $[\Sigma^{t-s}X, L_R Y]$.

The standard choices for R are $H\mathbb{F}_p$ (Adams spectral sequence), MU (Adams-Novikov spectral sequence), or BP (p-local Adams-Novikov spectral sequence).

1 Steenrod algebra

Problem 1. For $n \geq 0$, let $A(n) \subset \mathcal{A}^*_{(2)}$ be a subalgebra generated by Steenrod squares $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \dots, \mathrm{Sq}^{2^n}$

a) Prove that $A(n)$ is finite dimensional Hopf subalgebra of $\mathcal{A}^*_{(2)}$ for all n.

b) Deduce that all element of $\mathcal{A}^{>0}_{(2)}$ are nilpotent.

c) Deduce that the Steenrod algebra is injective as a module over itself.

Problem 2. Let B be a subalgebra of k-algebra A and $\varepsilon \colon B \to k$ an augmentation. Set

$$A//B := A \otimes_B k \simeq A/(A \cdot I(B)),$$

where $I(B) := \ker(\varepsilon)$ is an augmentation ideal.

a) Prove that if $A \cdot I(B) = I(B) \cdot A$, then $A//B$ is a k-algebra and the canonical map $A \to A//B$ is an algebra homomorphism.

b) Let A be a connected Hopf algebra and $B \hookrightarrow A$ a Hopf subalgebra. Prove that $A \simeq A//B \otimes_k B$ as right B-modules. In particular A is free as right B-module.

c) Let $B \hookrightarrow \mathcal{A}^*_{(2)}$ be a subalgebra such that $\mathcal{A}^*_{(2)}$ is flat as a right B-modules. Prove that if $H^*(E, \mathbb{F}_2) \simeq \mathcal{A}^*_{(2)}//B$, then for any spectrum X the second pages of the cohomological $\mathbb{F}_p$-Adams spectral sequence for $E_*(X)_{\widehat{p}}$ and $E^*(X)_{\widehat{p}}$ take the form

$$E_2^{*,*} \simeq \mathrm{Ext}^{*,*}_B(H^*(X, \mathbb{F}_2), \mathbb{F}_2) \quad \text{and} \quad E_2^{*,*} \simeq \mathrm{Ext}^{*,*}_B(\mathbb{F}_2, H^*(X, \mathbb{F}_2))$$

respectively.

Problem 3.

a) Prove that $H^*(H\mathbb{Z}, \mathbb{F}_2) \simeq \mathcal{A}^*_{(2)}//A(0)$.

b) Prove that $H^*(ko, \mathbb{F}_2) \simeq \mathcal{A}^*_{(2)}//A(1)$.

*c) Prove that there does not exist a spectrum X such that $H^*(X, \mathbb{F}_2) \simeq \mathcal{A}^*_{(2)}//A(n)$ for $n > 2$.

2 Adams spectral sequence

Problem 4.

a) Let A be an augmented associative k-algebra with the augmentation ideal $I := \ker(A \to k)$. Prove that $\mathrm{Ext}^1_A(k, k)$ is canonically isomorphic to the space of indecompossable elements I/I^2.

b) Prove that

$$\mathrm{Ext}^1_{\mathcal{A}^{(2)}_*}(\mathbb{F}_2, \Sigma^t\mathbb{F}_2) \simeq \begin{cases} \mathbb{F}_2, & t = 2^i \\ 0, & t \neq 2^i \end{cases}$$

c) Prove that h_0 detects $2 \in \mathbb{Z}_2 \simeq \pi_0(\mathbb{S})_{\hat{2}}$. Using Adams spectral sequence for $\pi_*(\mathbb{S}/2)_{\hat{2}}$ deduce that $\pi_2(\mathbb{S}/2) \simeq \mathbb{Z}/4\mathbb{Z}$.

Definition 2.1. Generators of $\mathrm{Ext}^1_{\mathcal{A}^{(2)}_*}(\mathbb{F}_2, \Sigma^{2^i}\mathbb{F}_2)$ are usually denoted by h_i.

Problem 5. (Division algebras, H-space structures on $\mathbb{S}^{2n-1}$ and Hopf invariant one)

a) Let X_φ be the cofiber of a map $\varphi\colon S^{2n-1} \to S^n$. By excision $H_*(X_\varphi, \mathbb{F}_2) \simeq \mathbb{F}_2 \oplus \Sigma^{-n}\mathbb{F}_2 \oplus \Sigma^{-2n}\mathbb{F}_2$. Let x, y be generators of $H^n(X_\varphi, \mathbb{F}_2)$ and $H^{2n}(X_\alpha, \mathbb{F}_2)$ respectively. *Hopf invariant of the map* φ is the number $h(\varphi) \in \mathbb{F}_2$, such that $x^2 = h(\varphi)y$. Prove that if $\mathbb{S}^{2n-1}$ admits a structure of H-space, then there exists a map $\mathbb{S}^{2n-1} \to \mathbb{S}^n$ with Hopf invariant 1. *(Hint: Use generalized Hopf fibration).*

b) Prove that the map $\varphi\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ has Hopf invariant one, then n is a power of 2. *(Hint: interpret $h(\varphi)$ in terms of Steenrod squares and use relations in Steenrod algebra).*

c) For $n = 2^i$, prove that if $\varphi \in \pi_{2n-1}(\mathbb{S}^n)$ has Hopf invariant one, then h_i is a permanent cycle (i.e. survives to $E^{*,*}_\infty$) in the cohomological Adams spectral sequence $E^{*,*}_2 = \mathrm{Ext}^*_{\mathcal{A}^*_{(2)}}(\mathbb{F}_2, \Sigma^*\mathbb{F}_2) \Rightarrow \pi_*(\mathbb{S})_{\hat{2}}$.

Remark 2.2. In the next problem set You will prove that h_i are not infinite cycles for $i > 3$. Hence the only unital division algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

Problem 6. (Vanishing line in the Adams Spectral Sequence) Let

$$\varepsilon'(s) = \begin{cases} 0, & s = 0 \mod 4, \\ 1, & s = 1 \mod 4, \\ 2, & s = 2, 3 \mod 4. \end{cases} \quad \text{and} \quad \varepsilon(s) = \begin{cases} 1, & s = 1 \mod 4, \\ 2, & s = 2 \mod 4, \\ 3, & s = 0, 3 \mod 4. \end{cases}$$

a) Let M be a connective $\mathcal{A}^*_{(2)}$-module (i.e. $M^{<0} \simeq 0$) free as $A(0)$-module. Prove that

$$\mathrm{Ext}^s_{\mathcal{A}^*_{(2)}}(M, \Sigma^t\mathbb{F}_2) \simeq 0$$

for $0 < t - s < 2s - \varepsilon'(s)$. *(Hint: first prove the statement for $s \leq 4$. Then show that there exists an $A(0)$-free resolution of M, hence $\mathrm{Ext}^{s+4}_{\mathcal{A}^*_{(2)}}(M, \Sigma^t\mathbb{F}_2) \simeq \mathrm{Ext}^s_{\mathcal{A}^*_{(2)}}(M', \Sigma^t\mathbb{F}_2)$ for some other $A(0)$-free module M').*

b) Show that $\ker(\mathcal{A}^*_{(2)}//A(0) \to \mathbb{F}_2)$ is free as an $A(0)$-module.

c) Deduce that $\mathrm{Ext}^s_{\mathcal{A}^*_{(2)}}(\mathbb{F}_2, \Sigma^t\mathbb{F}_2) \simeq 0$ for $0 < t - s < 2s - \varepsilon(s)$.

3 Adams-Novikov spectral sequence

Problem 7. (Few low dimensional stable homotopy groups)

a) Prove that

$$\pi_i(\mathbb{S})_{\widehat{p}} \simeq \begin{cases} 0 & i < 2p-3 \\ \mathbb{Z}/(p) & i = 2p-3 \end{cases}$$

b) Using Adams spectral sequence compute $\pi_i(\mathbb{S})_{\widehat{2}}$ for $i \leq 4$.

c) Deduce that

i	0	1	2	3	4
$\pi_i(\mathbb{S})$	$\mathbb{Z}$	$\mathbb{Z}/(2)\langle\eta\rangle$	$\mathbb{Z}/(2)\langle\eta^2\rangle$	$\mathbb{Z}/(8)\langle\nu\rangle \oplus \mathbb{Z}/(3)\langle x\rangle$	0

where x is a generator of 3-torsion in $\pi_3(\mathbb{S})$ and ν representes the generalized Hopf fibration corresponding to $\mathbb{H}$ and is detected in ASS by h_2. Using multiplicative structure of ASS prove that $\eta^3 = 4\nu \neq 0$ and $\eta^4 = 0$,

Problem 8.

a) Prove that

$$H_*(MU, \mathbb{F}_p) \simeq P \otimes_{\mathbb{F}_p} \mathbb{F}_p[\{y_i\}_{i+1 \neq p^k}]$$

as a co-module over the dual Steenrod algebra, where P is sub-Hopf algebra of $\mathcal{A}_*^\vee$ defined as

$$P = \begin{cases} \mathbb{F}_p[\xi_1, \xi_2, \ldots], & p \neq 2 \\ \mathbb{F}_p[\xi_1^2, \xi_2^2, \ldots], & p = 2 \end{cases}$$

b) Prove that $\mathcal{A}_*^\vee // \mathbb{F}_p[\xi_1, \xi_2, \ldots, \xi_n]$ is projective as a co-module over itself.

c) Deduce that $\mathrm{Ext}^{*,*}_{\mathcal{A}_p^\vee}(\mathbb{F}_p[\xi_1, \xi_2, \ldots, \xi_n], \mathbb{F}_p) \simeq 0$. Passing to co-limit deduce that $\mathrm{Ext}^{*,*}_{\mathcal{A}_p^\vee}(P, \mathbb{F}_p) \simeq 0$

d) Deduce that $\mathrm{Map}(MU, \mathbb{S}_p) \simeq 0$. Deduce that $\mathrm{Map}(MU, \mathbb{S}) \simeq 0$.

Problem 9.

a) Let Y be a finite spectrum. For X

- Any MU-module (in particular any complex oriented cohomology theory R),
- KU, HA for any complex of abelian groups A,
- Any bounded from above spectrum X

prove that $\mathrm{Map}(X, Y) \simeq 0$.

b) Let X be bounded above spectrum. Deduce that X is dualizable if and only if $X \simeq 0$. (In particular HA is not dualizable for $A \not\simeq 0$).

Problem 10. Let E be a spectrum such that $\pi_* E$ and $H_*(E, \mathbb{Z})$ are torsion free. Prove that the restriction map $\pi_* \mathrm{End}_{\mathrm{Sp}}(E, E) \to \mathrm{End}^*_{\mathbb{Z}}(E^*)$ is injective. *(Hint: Atiyah-Hirzebruch spectral sequence).*

Problem 11.

a) (MU-acyclic spectra) Let X be such that $\mathrm{Map}(X, \mathbb{S}) \simeq 0$ and Y be such that the Brown-Comenetz dual spectrum IY is a finite spectrum. Prove that $X \wedge Y \simeq 0$. Deduce that there exists a non-zero MU-acyclic spectrum.

b) For a spectrum X let $\widehat{X} := \varprojlim_n X/n$. Prove that for a profinite spectrum X (i.e. $X \simeq \widehat{X}$) the following conditions are equivalent: $X \wedge I \simeq 0 \Leftrightarrow \mathrm{Map}_{\mathrm{Sp}}(X, \widehat{\mathbb{S}}) \simeq 0$.

c) Deduce that that $I \wedge I \simeq 0$ (equivalently $L_I I \simeq 0$).

Problem 12. (Adams tower) Let R be a commutative ring spectrum. For any $X \in \mathrm{Sp}$ the **Adams tower** $\mathrm{A}^R_\bullet(X)$ **of** X

$$\ldots \to \mathrm{A}^R_1(X) \to \mathrm{A}^R_0(X) \to \mathrm{A}^R_{-1}(X) \simeq X$$

defined inductively as follows: $\mathrm{A}^R_{-1}(X) := X$ and $\mathrm{A}^R_{n+1}(X)$ is defined as the fiber of the R-Hurewicz morphism of $\mathrm{A}^R_n(X)$

$$\mathrm{A}^R_{n+1}(X) := \mathrm{fib}\left(\mathrm{A}^R_n(X) \simeq \mathrm{A}^R_n(X) \wedge \mathbb{S} \xrightarrow{1\wedge\eta} \mathrm{A}^R_n(X) \wedge R\right)$$

where $\eta\colon \mathbb{S} \to R$ is the unit map.

a) (Total homotopy fiber vs. iterated homotopy fiber) Let S be a finite set and denote by $\mathcal{P}(S)$ the lattice of subsets of S. Given a cube $X_\bullet\colon \mathcal{P}(S) \to \mathcal{C}$ in a category $\mathcal{C}$ which admits finite limits let us denote *total homotopy fiber of* $X_\bullet$ as

$$\mathrm{tfib}(X_\bullet) \simeq \mathrm{fib}\left(X_\varnothing \to \lim_{T\in\mathcal{P}(S)\setminus\varnothing} X_T\right)$$

Prove that total homotopy fiber can be computed iteratively by restricting on smaller subcubes: for any $s \in S$

$$\mathrm{tfib}(X_\bullet) \simeq \mathrm{fib}\left(\mathrm{tfib}(X_{|\mathcal{P}(S\setminus s)}) \to \mathrm{tfib}(X_{|\mathcal{P}(S\setminus s)\cup\{s\}})\right)$$

b) Deduce that if R admits a structure of the homotopy commutative E_1-ring spectrum, there exists a natural equivalence of towers over X

$$\mathrm{fib}\left(X \to \mathrm{Tot}^{\leq n}\,\mathrm{CB}^\bullet_R(X)\right) \simeq \mathrm{A}^R_n(X)$$

(Hint: use the fact that the natural functor $\mathcal{P}(n) \to \Delta_{\leq n}$ is final).

c) Prove that each successive map $\mathrm{A}^R_{n+1}(M) \to \mathrm{A}^R_n(M)$ in the Adams tower becomes nullhomotopic after tensoring with R. Deduce that $f \in \mathrm{Map}(X,Y)$ has filtration n in the R-based Adams-Novikov spectral sequence if and only if there exists a decomposition $f = f_n \circ f_{n-1} \circ \ldots \circ f_1$ such that each $f_i \wedge R, 1 \leq i \leq n$ is nullhomotopic.

d) Let $R \to S$ be a ring morphism. Prove that $F^s_R\,\mathrm{Map}_{\mathrm{Sp}}(X,Y) \subseteq F^s_S\,\mathrm{Map}_{\mathrm{Sp}}(X,Y)$, i.e. an element $x \in \mathrm{Map}_{\mathrm{Sp}}(X,Y)$ is detected in R-based Adams-Novikov SS at $s \leq s'$.