Chromatic Homotopy Theory
Problem set 1b. More on Spectra

1 Steenrod algebra

Definition 1.1 (Mod 2 Steenrod algebra)**.** Recall there exists a unique family of natural transformations $\mathrm{Sq}^n \colon H^*(-, \mathbb{Z}/2) \to H^{*+n}(-, \mathbb{Z}/2)$ satisfying the following properties:

- $\mathrm{Sq}^0(x) = x$.
- $\mathrm{Sq}^n(x) = x^2$ for $x \in H^n(X, \mathbb{Z}/2)$.
- $\mathrm{Sq}^n(x) = 0$ for $x \in H^{<n}(X, \mathbb{Z}/2)$.
- (Cartan's product formula). If we denote $Sq(x) := \sum_{n=0}^{\infty} Sq^n(x)$ then $Sq(x \cdot y) = \mathrm{Sq}(x) \cdot \mathrm{Sq}(y)$.

Since by Brown representability the map $\mathcal{A}^*_{(2)} := [H\mathbb{Z}/2, \Sigma^* H\mathbb{Z}/2] \to \mathrm{End}^*(H^*(-, \mathbb{Z}/2))$ is surjective, Sq^n lifts to a map of spectra $H\mathbb{Z}/2 \to \Sigma^n H\mathbb{Z}/2$. One can prove that Sq^n generate $\mathcal{A}^*_{(2)}$ as an algebra.

Problem 1.

a) By applying Elinberg-MacLane functor H to a short exact sequence of abelian groups $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ we obtain a fiber sequence of spectra

$$H\mathbb{Z}/2 \to H\mathbb{Z}/4 \to H\mathbb{Z}/2 \xrightarrow{\beta} \Sigma H\mathbb{Z}/2$$

where β is the so-called **Bockstein morphism**. Prove that $\mathrm{Sq}^1 = \beta$.

b) For $n \in \mathbb{Z}_{\geq 0}$ let us define $\mathrm{Sq}^{2n+1}_{\mathbb{Z}}$ as the following composite

$$\mathrm{Sq}^{2n+1}_{\mathbb{Z}} \colon H\mathbb{Z} \to H\mathbb{Z}/2 \xrightarrow{\mathrm{Sq}^{2n}} \Sigma^{2n} H\mathbb{Z}/2 \xrightarrow{\Sigma^{2n}(\beta_2)} \Sigma^{2n+1} H\mathbb{Z}$$

where β_2 is the connecting morphism in the fiber sequence $H\mathbb{Z} \xrightarrow{\cdot 2} H\mathbb{Z} \to H\mathbb{Z}/2 \xrightarrow{\beta_2} \Sigma H\mathbb{Z}$. Prove that $\mathrm{Sq}^{2n+1}_{\mathbb{Z}}$ is a lift of Sq^{2n+1} in the sense that the diagram below is commutative

$$\begin{tikzcd}
H\mathbb{Z} \arrow[r, "\mathrm{Sq}^{2n+1}_{\mathbb{Z}}"] \arrow[d] & \Sigma^{2n+1} H\mathbb{Z} \arrow[d] \\
H\mathbb{Z}/2 \arrow[r, "\mathrm{Sq}^{2n+1}"] & \Sigma^{2n+1} H\mathbb{Z}/2
\end{tikzcd}$$

Problem 2. Prove that $H^*(H\mathbb{Z}, \mathbb{F}_2) \simeq \mathcal{A}^*_{(2)}/(\mathrm{Sq}^1)$ as a module over the Steenrod algebra.

Problem 3. Prove that for $n > 0$ spectra $\Sigma^\infty \mathbb{RP}^\infty / \mathbb{RP}^n$ and $\Sigma^\infty \Sigma^n \mathbb{RP}^\infty$ are not homotopy equivalent.

Problem 4. Prove that there does not exist a pointed space X such that $\Sigma^\infty X \simeq H\mathbb{F}_2$.

2 Toda brackets

Definition 2.1. Let

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$$

be such that $f_{i+1} \circ f_i \simeq 0$. A choice of nullhomotopies $\alpha\colon f_2 \circ f_1 \simeq 0$ and $\beta\colon f_3 \circ f_2 \simeq 0$ determines a loop $\gamma(\alpha, \beta)$ in a mapping space $\operatorname{Hom}(X_0, X_3)$ at a zero morphism given by the composition

$$\gamma(\alpha,\beta)\colon\ 0 \simeq f_3 \circ 0 \xrightarrow[\sim]{f_3 \circ \alpha} f_3 \circ f_2 \circ f_1 \xrightarrow[\sim]{\beta^{-1} \circ f_1} 0 \circ f_1 \simeq 0$$

Note that

$$\gamma(\alpha,\beta) \in \pi_1 \operatorname{Hom}(X_0, X_3) \simeq \pi_0(\Omega \operatorname{Hom}(X_0, X_3)) \simeq \pi_0 \operatorname{Hom}(\Sigma X_0, X_3) = [\Sigma X_0, X_3]$$

Toda bracket $\langle f_1, f_2, f_3\rangle$ is defined to be the **subset of** $[\Sigma X_0, X_3]$ consisting of $\gamma(\alpha,\beta)$ for all possible choices of α, β. Note that the set of nullhomotopies $f_{i+1} \circ f_i \simeq 0$ is a torsor for $[\Sigma X_{i-1}, X_{i+1}]$, hence the Toda bracket $\langle f_1, f_2, f_3\rangle$ is a torsor for a subgroup $f_3 \circ [\Sigma X_0, X_2] - [\Sigma X_1, X_3] \circ f_1 \subseteq [\Sigma X_0, X_3]$.

Remark 2.2. You will see below that $\langle f_1, f_2, f_3\rangle$ itself is **not** a subgroup of $[\Sigma X_0, X_3]$ in general.

Problem 5.

a) Let $x, y \in \pi_n \mathbb{S}$. Since the addition on the sphere spectrum is commutative there exists a homotopy $x + y \sim y + x$. Restricting to the diagonal we obtain a loop $\sigma\colon x + x \sim x + x$ living in $\pi_{n+1}(\mathbb{S}, 2x) \simeq \pi_{n+1}\mathbb{S}$. Prove that $\sigma = \eta x$. (*Hint: you may use that the multiplication map on a circle* $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ *induces a Hopf map on top cells after stabilization*).

b) (Toda's relation) Let $x \in \pi_*(\mathbb{S})$ be a 2-torsion. Deduce that $\eta x \in \langle 2, x, 2\rangle$.

c) Deduce that $\langle 2, \eta, 2\rangle = \{\eta^2\}$.

Problem 6.

(Alternative description of the Toda brackets) Let

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$$

be as above. Since $f_3 \circ f_2 \simeq 0$ there exists a map $\widetilde{f_2}\colon X_1 \to \mathrm{fib}(f_3)$ lifting f_2

$$\begin{tikzcd}
 & \Sigma^{-1} X_3 \arrow[r] & \mathrm{fib}(f_3) \arrow[d, "i"] & \\
X_0 \arrow[r, "f_1"] \arrow[ur, squiggly, dashed, "\gamma"] & X_1 \arrow[r, "f_2"] \arrow[ur, dashed, "\widetilde{f_2}"] & X_2 \arrow[r, "f_3"] & X_3
\end{tikzcd}$$

and analogously since the composition $i \circ \widetilde{f_2} \circ f_1 \simeq f_2 \circ f_1$ is nullhomotopic, there exists a map γ lifting $\widetilde{f_2} \circ f_1$. Prove that $\gamma \in \langle f_1, f_2, f_3\rangle$ and that all elements of $\langle f_1, f_2, f_3\rangle$ can be obtained this way. Deduce that $\langle f_1, f_2, f_3\rangle$ depends only on triangulated structure of $h\,\mathrm{Sp}$ and not of its ∞-categorical enhancement.

Problem 7.

a) Let X be a spectrum and $f\colon \Sigma^{|f|}X \to X$ a self-map. There exists a short exact sequence of graded abelian groups

$$0 \to \pi_*(X)/f_*\pi_{*-|f|}(X) \xrightarrow{\alpha_*} \pi_*(X/f) \xrightarrow{\beta_*} \pi_{*-|f|-1}(X)[f] \to 0$$

Let $x \in \pi_*\mathbb{S}$, $y, w \in \pi_*(X)$ and $\widetilde{y} \in \pi_*(X/f)$ be such that

- $f_*(y) = 0$ and $xy = 0$.
- $\beta_*(\widetilde{y}) = y$, i.e. $\widetilde{y}$ is a lift of y.
- $\alpha_*(w) = x\widetilde{y}$.

Prove that $w \in \langle x, y, f\rangle$.

b) Deduce that $\pi_2(\mathbb{S}/2) \simeq \mathbb{Z}/4\mathbb{Z}$.

c) Deduce that $\mathbb{S}/2$ does not admit a structure of a homotopy associative ring spectrum.

Problem 8.

a) Prove that for an object X of a $\mathbb{Z}$-linear category $\mathcal{C}$ the multiplication by n map is nullhomotopic on X/n.

b) Deduce that the category of spectra does not admit dg-enhancement.

3 Phantom maps

By Brown representability the map from the homotopy category of spectra to the category of cohomology theories is essentially surjective and full, but as we will see later is not faithful in general.

Definition 3.1. We will call the map of spectra $f\colon E \to F$ **cohomological phantom** if its restriction to the subcategory of eventually connective spectra $\mathrm{Sp}_{>-\infty}$ is nullhomotopic. In particular the map of associated cohomology theories $E^* \to F^*$ is zero.

Problem 9. Prove that there are no cohomological phantom maps from eventually connective spectrum to any spectrum. Deduce that the natural surjection $\mathcal{A}^*_{(2)} \twoheadrightarrow \mathrm{End}(H^*(-,\mathbb{Z}/2))$ is an isomorphism.

Problem 10. (Example of cohomological phantoms)

a) Prove that $KU \wedge H\mathbb{Z} \simeq \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Q}$. Deduce that $[KU, \Sigma H\mathbb{Z}] \not\simeq 0$. (*Hint: You may use without proof Snaith theorem: let $\beta\colon S^2 \to \mathbb{CP}^\infty$ be a generator of π_2. Under the canonical map $\Sigma^\infty_+\mathbb{CP}^\infty \to KU$ (classifying trivial bundle of rank 1) β maps to the Bott element. Since β is invertible in the target the map above admits a canonical factorization $\Sigma^\infty_+\mathbb{CP}^\infty[\beta^{-1}] \to KU$. Snaith theorem asserts this is an equivalence*).

b) Prove that all $\alpha \in [KU, \Sigma H\mathbb{Z}]$ are cohomological phantoms.

A map of spectra $f\colon E \to F$ is called **homological phantom** or just **phantom** if one of the following equivalent conditions holds

- The induced map of homology theories $f_*\colon E_* \to F_*$ is zero.
- The restriction of f to the category of finite spectra Sp^{fin} is nullhomotopic. I.e. for any map $V \to E$ from a finite spectrum V the composite map $V \to E \xrightarrow{f} F$ is nullhomotopic.

Problem 11. Let X be a spectrum represented as a filtered colimit $X \simeq \varinjlim X_\alpha$ of finite spectra X_α

a) Prove that the map $f\colon X \to Y$ is phantom if and only if $f_{|X_\alpha}\colon X_\alpha \to X \xrightarrow{f} Y$ vanish for all α.

b) Deduce there exists a nontrivial phantom map $X \to Y$ if and only if the $\lim^1$-term of the Milnor exact sequence

$$0 \to \varprojlim{}^1[\Sigma X_\alpha, Y] \to [X,Y] \to \varprojlim[X_\alpha, Y] \to 0$$

is non-trivial.

Problem 12.

a) Let X be a spectrum. Prove there does not exist a nontrivial phantom map out of X if and only if X is a retract of a sum of finite spectra.

b) Deduce there exits a spectrum $\widetilde{X}$ and phantom map $p\colon X \to \widetilde{X}$ universal in the following sense

  - Any phantom map out of X factors through p.
  - There are no nontrivial phantoms out of $\widetilde{X}$.

c) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be phantoms. Deduce that $g \circ f \simeq 0$.

Problem 13. (Phantoms and Brown-Comenetz duality)

a) Let X, Y be a pair of spectra. Prove that any phantom map $X \to IY$ is nullhomotopic.

b) Prove that the map $X \to Y$ is phantom if and only if the composite $X \to Y \to IV$ vanish for all finite spectra V.

c) Prove that the map of spectra $X \to Y$ is phantom if and only if the composition $X \to Y \to I^2Y$ vanish.

d) Assume that all $\pi_n Y$ are finitely generated abelian groups. Deduce there exists a nontrivial phantom map in Y if and only if $H_*(Y, \mathbb{Q}) \not\simeq 0$.