FREE GRADIENT DISCONTINUITY AND IMAGE SEGMENTATION

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Abstract:

This talk deals with free discontinuity problems related to contour enhancement in image segmentation, focussing on

the mathematical analysis of Blake & Zisserman functional,

precisely:

1. existence of strong solution
   under Dirichlet boundary condition is shown,
2. several extremal conditions on optimal segmentation are stated,
3. well-posedness of the problem is discussed,
4. non trivial local minimizers are analyzed.

The segmentation we look for provides
a cartoon of the given image satisfying some requirements:
the decomposition of the image is performed by choosing a pattern of lines of steepest discontinuity for light intensity,
and this pattern will be called segmentation of the image.

Eyjafjallajökull

rotoscope

A classic variational model for image segmentation has been proposed by **Mumford & Shah**, who introduced the functional

$$\int_{\Omega \setminus K} \left(|Du(x)|^2 + |u(\mathbf{x}) - g(\mathbf{x})|^2 \right) d\mathbf{x} + \gamma\, \mathcal{H}^{n-1}(K \cap \Omega) \qquad (1)$$

where

- $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open set,
- $K \subset \mathbb{R}^n$ is a closed set,
- u is a scalar function,
- Du denotes the distributional gradient of u,
- $g \in L^2(\Omega)$ is the datum (grey intensity levels of the given image),
- $\gamma > 0$ is a parameter related to the selected contrast threshold,
- $\mathcal{H}^{n-1}$ denotes $n-1$ dimensional Hausdorff measure.

According to this model
the segmentation of the given image is achieved by
minimizing (1) among admissible pairs (K, u),
say closed $K \subset \mathbb{R}^n$ and $u \in C^1(\Omega \setminus K)$.

This model led in a natural way to the study of
a new type of functional in Calculus of Variations:

free discontinuity problem.

Existence of minimizers of (1) was proven by

- De Giorgi, Carriero & Leaci (1989)

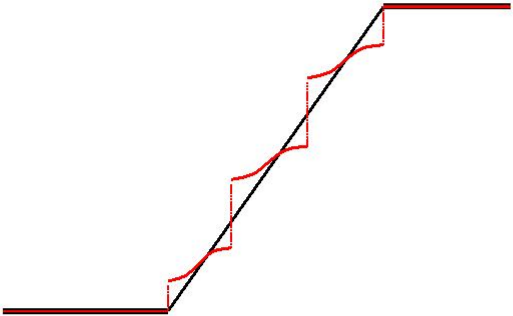
in the framework of bounded variation functions without Cantor part (space *SBV*) introduced in

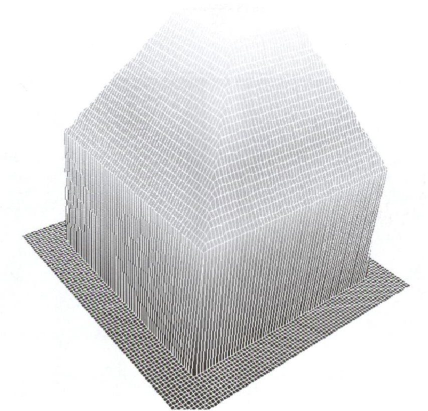
- De Giorgi & Ambrosio.

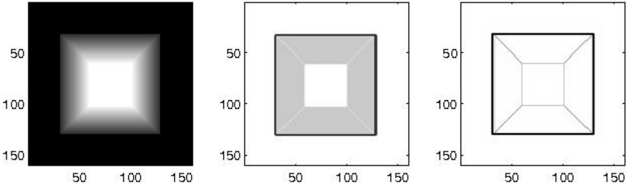
Further regularity properties of optimal segmentation
in Mumford & Shah model were shown by

- [Dal Maso, Morel & Solimini, (1992), $n = 2$,]
- [Ambrosio, Fusco & Pallara (2000)],
- [Lops, Maddalena, Solimini, (2001), $n = 2$,],
- [Bonnet & David (2003), $n = 2$].

stair-casing effect



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To overcome the problems and aiming to better description of stereoscopic images they proposed a different functional including second derivatives.

Blake & Zisserman variational principle faces segmentation as a minimum problem:

input is given by intensity levels of a monochromatic image,

output is given by

- meaningful boundaries whose length is penalized (correspond to discontinuity set of the given intensity and of its first derivatives)
- a piece-wise smooth intensity function (smoothed on each region in which the domain is splitted by such boundaries).

another problem with free discontinuity: Blake & Zisserman functional

$$F(K_0, K_1, v) = \\ = \int_{\Omega\setminus(K_0\cup K_1)} \left(\left|D^2 v(\mathbf{x})\right|^2 + |v(\mathbf{x}) - g(\mathbf{x})|^2\right) d\mathbf{x} + \\ + \alpha\,\mathcal{H}^{n-1}(K_0) + \beta\,\mathcal{H}^{n-1}(K_1 \setminus K_0) \tag{2}$$

to be **minimized among admissible triplets (K_0, K_1, v)**:

- K_0 , K_1 closed subsets of $\mathbb{R}^n$,
- $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ and continuous on $\Omega \setminus K_0$.

with data:

- $\Omega \subset \mathbb{R}^n$ open set, $n \geq 1$,
- $g \in L^2(\Omega)$ grey level intensity of the given image,
- α, β positive parameters
  (chosen accordingly to scale and contrast threshold),
- $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure.

Existence of minimizers for (2) has been proven by

- COSCIA $n = 1$
  (STRONG AND WEAK FORM. COINCIDE IFF $n = 1$!), AND BY
- [CARRIERO, LEACI & T.] $n = 2$,

via direct method in calculus of variations:
solution of a weak formulation of minimum problem
(performed for any dimension $n \geq 2$)
and subsequently proving additional regularity
of weak minimizers under Neumann bdry condition ($n = 2$)
[C-L-T, Ann.S.N.S., Pisa (1997)]

Since we looked for a weak formulation
of a free discontinuity problem,
we wrote a suitable relaxed form relaxed version of BZ functional;
this form depends only on u (not on triplets!):
optimal segmentation ($K_0 \cup K_1$) has to be recovered through
jumps (u discontinuity set) and creases (Du discontinuity set)
[C-L-T, in PNLDE, 25 (1996)]

We proved also several density estimates
for minimizers energy and optimal segmentation:

- [C-L-T, Nonconvex Optim. Appl.55 (2001)],
- [C-L-T, C.R.Acad.Sci.(2002)],
- [C-L-T J. Physiol.(2003)];

by exploiting this estimates, via Gamma-convergence techniques,

- [Ambrosio, Faina & March, SIAM J.Math.An. (2002)] obtained an approximation of Blake & Zisserman functional with elliptic functionals,

and numerical implementation was performed by

- [R.March]
- [M.Carriero, A.Farina, I.Sgura].

No uniqueness due to nonconvexity,

nevertheless generic uniqueness olds true in 1-D.

About uniqueness and well-posedness:

- [T.Boccellari, F.T., Ist.Lombardo Rend.Sci 2008, **142** 237-266] ($n \geq 1$),
- [T.Boccellari, F.T.] QDD Dip.Mat.Polit.MI 2010] ($n = 1$),

Stime a priori e continuità del valore di minimo

Theorem - Minimizing triplets (K_0, K_1, u) of Blake & Zisserman $F^g_{\alpha,\beta}$ functional fulfil (in any dimension n):

- $\|u\|_{L^2} \leq 2\,\|g\|_{L^2}$,
- $0 \leq m^g(\alpha,\beta) \leq \|g\|^2_{L^2}$,
- $$\left|m^g(\alpha,\beta) - m^h(a,b)\right| \;\leq\; 5(\|g\|_{L^2} + \|h\|_{L^2})\,\|g-h\|_{L^2} + \frac{\min\left\{\|g\|^2_{L^2}, \|h\|^2_{L^2}\right\}}{\min\{\alpha,a\}}\,|\alpha - a| + \frac{\min\left\{\|g\|^2_{L^2}, \|h\|^2_{L^2}\right\}}{\min\{\beta,b\}}\,|\beta - b| ,$$

$$\left\{ \begin{array}{l} \mathcal{H}^{n-1}(K_0) \leq \dfrac{2}{\alpha}\left(\|g\|^2 + \eta^2\right), \quad \mathcal{H}^{n-1}(K_1 \setminus K_0) \leq \dfrac{2}{\beta}\left(\|g\|^2 + \eta^2\right) \\ \qquad \text{per ogni terna } (u, K_0, K_1) \text{ minimizzante } F^h_{\alpha,\beta} \text{ con } \|h-g\|_{L^2} < \eta . \end{array} \right.$$

notice that 1-dimensional case fits very well to a short presentation, since (only in 1-d) strong and weak functional coincide.

1-d Blake & Zisserman 1-d functional

Given $g \in L^2(0,1)$, $\alpha, \beta \in \mathbb{R}$ we set $F^g_{\alpha,\beta}$:

$$F^g_{\alpha,\beta}(u) = \int_0^1 |\ddot{u}(x)|^2 \, dx + \int_0^1 |u(x) - g(x)|^2 \, dx + \alpha \sharp (S_u) + \beta \sharp (S_{\dot{u}} \setminus S_u) \tag{3}$$

to be minimized among $u \in L^2(0,1)$ t.c. $\sharp (S_u \cup S_{\dot{u}}) < +\infty$ t.c. $u', u'' \in L^2(I)$ for every interval $I \subseteq (0,1) \setminus (S_u \cup S_{\dot{u}})$

Notation:
$\dot{u}$ denotes the absolutely continuous part of u',
$\ddot{u}$ the absolutely continuous part of $(\dot{u})' = u''$,
$S_u \subseteq (0,1)$ the set of jump points of u,
$S_{\dot{u}} \subseteq (0,1)$ the set of jump points of $\dot{u}$,
$\sharp$ the counting measure.

$n = 1$

Summary of analytic results:

- Euler equations for local minimizers,
- compliance identity for local minimizers,
- a priori estimates on minimum value and minimizers,
- continuous dependence of minimum value $m^g(\alpha, \beta)$ with respect to g, α, β.

Theorem

$F^g_{\alpha,\beta}$ achieves its minimum provided
the following conditions are fulfilled:

$$0 < \beta \leq \alpha \leq 2\beta < +\infty \tag{4}$$

$$g \in L^2. \tag{5}$$

Uniqueness fails

There are many kinds of uniqueness failure:

precisely, even considering the simple 1-d case:

- if g has a jump,
  then there $\exists\, \alpha > 0\, s.t.\;\; F^{g}_{\alpha,\alpha}$ has exactly two minimizers;
- there are $\alpha > 0$ and $g \in L^2(0,1)$ s.t. uniqueness fail for every β in a non empty interval $(\alpha - \varepsilon, \alpha]$;
- for every α and β fulfilling $0 < \beta \leq \alpha < 2\beta$ there is $g \in L^2(0,1)$ s.t.
  $$\sharp\,(\operatorname{argmin} F^{g}_{\alpha,\beta}) \geq 2.$$
- Eventually we can show an example of a set $\mathcal{N} \subseteq L^2(0,1)$ with non empty interior part in $L^2(0,1)$ s.t. for every $g \in \mathcal{N}$ there are α and β satisfying (4) and
  $$\sharp\,(\operatorname{argmin} F^{g}_{\alpha,\beta}) \geq 2.$$

Generic uniqueness

Euler eqs are an over-determined system
(singular set is an unknown)

Nevertheless we can prove

Theorem ([T.Boccellari & F.T])

For any α, β s.t.

$$0 < \beta \leq \alpha \leq 2\beta \,, \qquad \alpha/\beta \notin \mathbb{Q} \,,$$

there is a G_δ (countable intersection of dense open sets) set $E_{\alpha,\beta} \subset L^2(0,1)$ such that

$$\sharp \left(\operatorname{argmin} F^g_{\alpha,\beta}\right) \;=\; 1 \,.$$

Idea of the proof:

- we show analytic dependence of (absolutely continuous part of) energy with respect to variations of open cells of CW-complex structure of partitions of $(0,1)$ induced by singular set of piecewise affine data g
- the set of all piece-wise affine data (related to suitably refined partitions of $(0,1)$) and exhibiting non uniqueness of minimizer with different quality" (ordering of jump and creases) and same prescribed cardinality of singular set has null m dimensional Lebesgue measure (here m is the dimension of the space of continuous piece-wise functions in $(0,1)$ affine with at most m creases)
- technical density argument

The whole picture is coherent with the presence of instable patterns, each of them corresponding to a bifurcation of optimal segmentation under variation of parameters α e β, related to:

- contrast threshold ($\sqrt{\alpha}$),
- “luminance sensitivity”,
- resistance to noise,
- crease detection ($\sqrt{\beta}$),
- double edge detection.

Dirichlet problem for BZ functional, $n = 2$

Image InPainting refers to the filling in of missing or partially occluded regions of an image.

Minimizing Blake & Zisserman functional is useful
to achieve contour continuation in the whole image region $\widetilde{\Omega}$
when occlusion or local damage occur in $\widetilde{\Omega} \setminus \Omega$
e.g. blotches in a fresco or a movie film.

Dirichlet problem:
minimize the energy $F(K_0, K_1, v)$ in $\widetilde{\Omega} \subset \mathbb{R}^2$:

$$\begin{aligned} F(K_0, K_1, v) &= \\ &= \int_{\widetilde{\Omega}\setminus(K_0 \cup K_1)} \left(\left| D^2 v(\mathbf{x}) \right|^2 + \mu \left| v(\mathbf{x}) - g(\mathbf{x}) \right|^2 \right) d\mathbf{x} + \\ &\qquad + \alpha\, \mathcal{H}^1(K_0) + \beta\, \mathcal{H}^1(K_1 \setminus K_0) \end{aligned} \tag{6}$$

among triplets which assume prescribed data w on $\widetilde{\Omega} \setminus \Omega$:
say $v = w$ a.e. $\widetilde{\Omega} \setminus \Omega$

Weak formulation of Dirichlet pb for BZ functional

Minimize $\mathcal{F} : X \to [0, +\infty]$ defined by

$$\mathcal{F}(v) = \int_{\widetilde{\Omega}} (|\nabla^2 v|^2 + \mu |v - g|^2)\, d\mathbf{x} + \alpha \mathcal{H}^1(S_v) + \beta \mathcal{H}^1(S_{\nabla v} \setminus S_v) \quad (7)$$

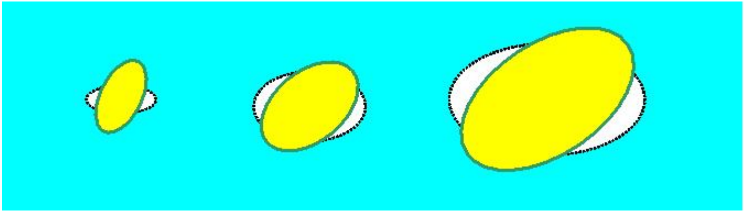
where $\Omega \subset\subset \widetilde{\Omega} \subset \mathbb{R}^2$ are open sets, $\mathbf{x} = (x, y) \in$
Omega*and*
$X = GSBV^2(\widetilde{\Omega}) \cap L^2(\widetilde{\Omega}) \cap \left\{ v = w \ a.e. \widetilde{\Omega} \setminus \Omega \right\}$.

Theorem

If $g \in L^2(\widetilde{\Omega}),\ w \in X$ and $\beta \leq \alpha \leq 2\beta$
then $\mathcal{F}$ has at least one minimizer in X.

The main part $\mathcal{F}$ is denoted by $\mathcal{E}$:

$$\mathcal{E}(v) = \int_{\widetilde{\Omega}} |\nabla^2 v|^2\, d\mathbf{x} + \alpha \mathcal{H}^1(S_v) + \beta \mathcal{H}^1(S_{\nabla v} \setminus S_v) \quad (8)$$



the example with cut and tilted disks tells that

1. sublevels of functional $\mathcal{E}$ are not compact on
$$\left\{ v \in SBV(\Omega) \ : \ \nabla v \in SBV(\Omega)^2 \right\}$$
2. by letting untilted some of the big disks
we find functions with unbounded gradient
with arbitrarily small energy $\mathcal{E}$

We recall the definitions of some function spaces related to first derivatives which are special measures in the sense of De Giorgi

$SBV(\Omega)$ denotes the class of functions $v \in BV(\Omega)$ s.t.

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, dy + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1 .$$

$$SBV_{loc}(\Omega) \;=\; \{v \in SBV(\Omega') : \; \forall \Omega' \subset\subset \Omega\} ,$$

$$GSBV(\Omega) = \big\{v : \Omega \to \mathbb{R}\, Borel ;\; -k \vee v \wedge k \in SBV_{loc}(\Omega)\; \forall k\big\}$$

$$GSBV^2(\Omega) \;=\; \big\{v \in GSBV(\Omega),\; \nabla v \in \big(GSBV(\Omega)\big)^2\big\}$$

We emphasize that

$GSBV(\Omega)$, $GSBV^2(\Omega)$ are neither vector spaces nor subsets of distributions in Ω.

Nevertheless
smooth variations of a function in $GSBV^2(\Omega)$ still belong to the same class.

Notice that,

- if $v \in GSBV(\Omega)$, then S_v is countably $(\mathcal{H}^1, 1)$ rectifiable and ∇v exists a.e. in Ω.
- $Dv \neq \nabla v$ in $GSBV^2(\Omega)$
- $S_{\nabla v} = \bigcup_{i=1}^{2} S_{\nabla_i v}$

Remark

1. $v \in BV \cap L^\infty\,, \quad P(E) < +\infty \quad \Rightarrow \quad v\,\chi_E \,\in\, BV$
2. $v \in BV\,, \quad P(E) < +\infty \qquad\qquad \not\Rightarrow^{\,*} \quad v\,\chi_E \,\in\, BV$
3. $v \in BV\,, \quad P(E) < +\infty \qquad\qquad \Rightarrow \quad v\,\chi_E \,\in\, GBV$

$*$ the trace of v could be not integrable, e.g.:

$$n = 2 \quad \Omega = B_1 \quad v = \varrho^{-1/2} \in W^{1,1}(B_1)$$

$$E \;=\; \left\{ \mathbf{x} = \{x, y\} \;\; \frac{1}{k^2+1} < \varrho < \frac{1}{k^2}\,,\; k \in \mathbf{N} \right\}$$

Theorem [CLT, Adv.Math.Sci.Appl., 2010] existence of strong minimizer

Assume

$$0 < \beta \le \alpha \le 2\beta,\ \mu > 0,\ \ g \in L^2(\widetilde{\Omega}) \cap L^4_{loc}(\widetilde{\Omega})\,,\ w \in L^2(\widetilde{\Omega})\,,$$

Ω is a bounded open set with C^2 boundary $\partial\Omega$,

$$w \in C^2\left(\widetilde{\Omega}\right)\,, \qquad D^2 w \in L^\infty(\widetilde{\Omega})\,.$$

Then there is at least one triplet (C_0, C_1, u) minimizing functional

$$F(K_0, K_1, v) \ = \ \int_{\Omega \setminus (K_0 \cup K_1)} \left(\left|D^2 v(\mathbf{x})\right|^2 \ + \ |v(\mathbf{x}) - g(\mathbf{x})|^2 \right) \, d\mathbf{x} \ +$$

$$+\ \alpha\, \mathcal{H}^1(K_0) \ + \ \beta\, \mathcal{H}^1(K_1 \setminus K_0)$$

with finite energy, among admissible triplets (K_0, K_1, v):

$$\begin{cases} K_0\,,\ K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0 \cup K_1 \text{ closed,} \\ v \in C^2\left(\Omega_\varepsilon \setminus (K_0 \cup K_1)\right),\ v \text{ approximately continuous in } \left(\Omega_\varepsilon \setminus K_0\right), \\ v = w \text{ a.e. in } \Omega_\varepsilon \setminus \Omega\,. \end{cases}$$

Moreover, any minimizing triplet (K_0, K_1, v) fulfils:

$$K_0 \cap \widetilde{\Omega} \text{ and } K_1 \cap \widetilde{\Omega} \text{ are } (\mathcal{H}^1, 1) \text{ rectifiable sets,}$$

$$\mathcal{H}^1(K_0 \cap \widetilde{\Omega}) = \mathcal{H}^1(\overline{S_v}), \quad \mathcal{H}^1(K_1 \cap \widetilde{\Omega}) = \mathcal{H}^1(\overline{S_{\nabla v}} \setminus S_v),$$

$$\begin{cases} v \in GSBV^2(\widetilde{\Omega}), \text{ hence} \\ v \ \nabla v \text{ have well defined two-sided traces, } \mathcal{H}^1 \, a.e. \text{ finite on } K_0 \cup K_1, \end{cases}$$

the function v is also a minimizer of the weak functional $\mathcal{F}$

$$\mathcal{F}(z) = \int_{\Omega} (|\nabla^2 z|^2 + \mu |z - g|^2) \, d\mathbf{x} + \alpha \mathcal{H}^1(S_z) + \beta \mathcal{H}^1(S_{\nabla z} \setminus S_z)$$

over $z \in L^2(\Omega) \cap GSBV(\Omega) : \nabla z \in (GSBV(\Omega))^2$, $z = w$ a.e. $\widetilde{\Omega} \setminus \Omega$.

Eventually, the third element v of any minimizing triplet (K_0, K_1, v) fulfils

$$\mathcal{F}(v) = F(K_0, K_1, v). \qquad \square$$

Steps of the proof

Existence of minimizing triplets is achieved by showing partial regularity of the weak solution with penalized Dirichlet datum. The novelty consists in the regularization at the boundary for a free gradient discontinuity problem;

regularity is proven at points with 2-dimensional energy density by:

1. **blow-up technique**
2. **suitable joining along lunulae filling half-disk**
3. **a decay estimate for weak minimizers**

In the blow-up procedure, two refinements of relevant tools are

- **hessian decay of a function which is bi-harmonic in half-disk and vanishes together with normal derivative on the diameter**
- **a Poincaré-Wirtinger inequality for $GSBV$ functions vanishing in a sector**

Theorem (Biharmonic extension and L^2 decay of Hessian)

Set $B_R^+ = B_R(\mathbf{0}) \cap \{y > 0\} \subset \mathbb{R}^2\,,\ R > 0\,.$

Assume $\ z \in H^2(B_R^+)\,,\ \ \Delta^2 z \equiv 0\ B_R^+\,,\ \ z = z_y \equiv 0$ on $\{y = 0\}$.)

Then there exists an (obviously unique) extension Z of z in whole B_R such that $\quad \Delta^2 Z \equiv 0\ B_R\ .$

This extension may increase a lot L^2 hessian norm of $D^2 Z$ nevertheless it implies nice decay on half-ball:

$$\|D^2 Z\|^2_{L^2(B^+_{\eta R})} \le \eta^2 \|D^2 z\|^2_{L^2(B^+_R)}\,.$$

Such extimate is not a straightforward consequence of classical Schwarz reflection principle for harmonic functions vanishing on the diameter, since the Almansi decomposition on the half-disk B_R^+ may neither respect the vanishing value on the diameter:
e.g. $\varrho^3\big(\cos\vartheta - \cos(3\vartheta)\big) = \varrho^2\varphi + \psi$ where $\varphi = x,\ \psi = 3x^2y - x^3$
are both harmonic but do not vanish on the diameter $\{y = 0\}$,
nor preserves orthogonality in L^2 or H^2:
cancelation of big norms
may take place in one half-disk and not in the other: see Fig. (39)

Duffin extension formula

Assume
$z \in H^2(B_1^+)$,
z is bi-harmonic in B_1^+
$z = \partial z/\partial y = 0$ on $B_1(\mathbf{0}) \cap \{y = 0\}$.

Then
z has a bi-harmonic extension Z in B_1 defined by

$$\begin{cases} Z(x,y) = z(x,y) & \forall\,(x,y) \in B_1^+\,, \\ Z(x,-y) = -z(x,y) + 2yz_y(x,y) - y^2\Delta z(x,y) & \forall\,(x,-y) \in B_1^-\,. \end{cases}$$

Almansi-type decomposition (revisited)

Let $u \in H^2(B_R \setminus \Gamma)$.

Then

$$\Delta_{\mathbf{x}}{}^2 u = 0 \ \ B_R \setminus \Gamma \tag{9}$$

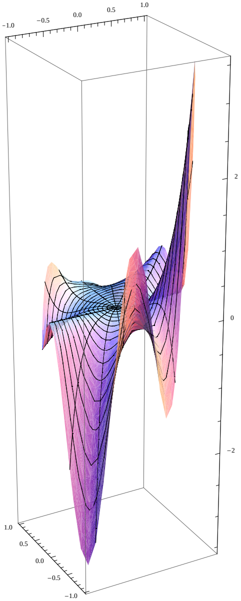
iff

$$\exists\, \varphi, \psi : \ u(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2\, \varphi(\mathbf{x}), \quad \Delta_{\mathbf{x}}\, \varphi(\mathbf{x}) = \Delta_{\mathbf{x}}\, \psi(\mathbf{x}) \equiv 0, \ B_R \setminus \Gamma. \tag{10}$$

Moreover decomposition (10) is unique
up to possible linear terms in ψ:

say $A\varrho\cos\vartheta = Ax$ and $B\varrho\sin\vartheta = By$
that can switch independently to
respectively $A\varrho^{-1}\cos\vartheta$ and $B\varrho^{-1}\sin\vartheta$ in φ.

Back to hessian decay estimate (36)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0.0
−0.5
−1.0

We use a new Poincaré-Wirtinger type inequality in the class $GSBV$ which allows surgical truncations of non integrable functions of several variables with the aim of taming blow-up at boundary points in case of functions vanishing in a full sector.

Notice that $v \in GSBV^2(\Omega)$ does not even entail that either v or ∇v belongs to $L^1_{loc}(\Omega)$.

DECAY

Theorem - Decay of functional $\mathcal{F}$ at boundary points

There are constants $C_1,\ C_2$ (dep. on $\alpha,\ g,\ w$) s.t.,

$$\begin{cases} \forall k > 2,\ \forall \eta, \sigma \in (0,1) \text{ with } \eta < C_2 \\ \exists \varepsilon_0 > 0,\ \vartheta_0 > 0 : \forall \varepsilon \in (0, \varepsilon_0], \end{cases}$$

$\forall \mathbf{x} \in \partial\Omega$ and any local minimizer u of $\mathcal{F}$ in $\Omega \cap B_\varrho(\mathbf{x})$, s.t.

$0 < \varrho \leq \left(\varepsilon^k \wedge C_1\right)$, $\quad \int_{B_\varrho(\mathbf{x})} |g|^4 \leq \varepsilon^k$ and

$$\alpha\, \mathcal{H}^1 \left(S_u \cap \overline{\Omega \cap B_\varrho(\mathbf{x})} \right) + \beta\, \mathcal{H}^1 \left((S_{\nabla u} \setminus S_u) \cap \overline{\Omega \cap B_\varrho(\mathbf{x})} \right) < \varepsilon\, \varrho\,,$$

we have

$$\mathcal{F}_{B_{\eta\varrho}(\mathbf{x})}(u) \leq$$

$$\leq \eta^{2-\sigma} \max \left\{ \mathcal{F}_{B_\varrho(\mathbf{x})}(u)\,,\ \varrho^2\, \vartheta_0 \left(\left(\mathrm{Lip}(\gamma_{\partial\Omega}{}')\right)^2 + \left(\mathrm{Lip}(Dw)\right)^2 \right) \right\}.$$

Admissible triplets and localization

Admissible triplets: (K_0, K_1, v) is an admissible triplet if

$$\begin{cases} K_0\,,\, K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0 \cup K_1 \text{ closed,} \\ v \in C^2\Big(\widetilde{\Omega} \setminus (K_0 \cup K_1)\Big),\ v \text{ approximately continuous in } \widetilde{\Omega} \setminus K_0, \\ v = w \text{ a.e. in } \widetilde{\Omega} \setminus \Omega\,. \end{cases}$$

(Localization)

We will use the symbols F_A, $\mathcal{F}_A$ to denote respectively functionals F, $\mathcal{F}$ when $\widetilde{\Omega}$ is substituted by a Borel set $A \subset \widetilde{\Omega}$, (resp. $E_A, \mathcal{F}_A$ for $E, \mathcal{F}$)

(Locally minimizing triplet of F (6))

Admissible triplet (K_0, K_1, u), is a locally minimizing triplet of F if

$$F_A(K_0, K_1, u) < +\infty$$

$$F_A(K_0, K_1, u) \le F_A(T_0, T_1, v)$$

$\forall$ smooth open $A \subset\subset \Omega$ and any admissible triplet (T_0, T_1, v) s.t.

$$\mathrm{spt}(v-u) \quad \text{and} \quad \overline{(T_0 \cup T_1) \triangle (K_0 \cup K_1)} \quad \text{are subsets of } A.$$

(Essential locally minimizing triplet of F, resp. E)

Given a locally minimizing triplet (T_0, T_1, v) of the functional F (resp. E),
there is another triplet (K_0, K_1, u) , called
essential locally minimizing triplet, which is uniquely defined by

$$\begin{aligned} u &= \widetilde{v} \\ K_0 &= \overline{T_0 \cap K} \setminus (T_1 \setminus T_0) \\ K_1 &= \overline{T_1 \cap K} \setminus T_0 \end{aligned}$$

where $\widetilde{v}$ is the approximate limit of v, a.e. defined by

$$g(\widetilde{v}(\mathbf{x})) = \lim_{\varrho \downarrow 0} \fint_{B_\rho(0)} g(v(\mathbf{x}+\mathbf{y}))d\mathbf{y} \qquad \forall g \in C^0(\overline{\mathbb{R}})$$

and K is
the smallest closed subset of $T_0 \cup T_1$ such that $\widetilde{v} \in C^2(\Omega \setminus K)$.

Some Euler equations in 2 dimensional case
[C.L.T] Calc.Var.Part.Diff.Eq, 2008
[C.L.T] Prepr.24 Dip.Mat.Univ.Salento, 2008

- $\Delta^2 u + \mu u = \mu g \qquad \Omega \setminus (K_0 \cup K_1)$
- Neumann boundary operators (plate-type bending moments) vanishing in $K_0 \cup K_1$
- $\left[\!\left[|D^2 u|^2 + \mu |u - g|^2 \right]\!\right] = \alpha\, \mathcal{K}(K_0)$
- $\left[\!\left[|D^2 u|^2 \right]\!\right] = \beta\, \mathcal{K}(K_1 \setminus K_0)$
- Integral and geometric conditions
  at the "boundary" of singular set: crack-tip and crease-tip.

Euler equations

From now on, for sake of simplicity,
we examine only the main part E of functional F:

$$E(K_0, K_1, v) = \tag{11}$$
$$= \int_{\Omega \setminus (K_0 \cup K_1)} \left| D^2 v(x) \right|^2 dx \; + \; \alpha \, \mathcal{H}^1(K_0) \; + \; \beta \, \mathcal{H}^1(K_1 \setminus K_0)$$

and the structural assumption $\beta \leq \alpha \leq 2\beta$ will be always understood.

Euler equations I : smooth variations

Theorem

Any essential locally minimizing triplet (K_0, K_1, u) for functional F fulfils

$$\Delta^2 u + \mu (u - g) = 0 \qquad in\ \Omega \setminus (K_0 \cup K_1)\ .$$

Any essential locally minimizing triplet (K_0, K_1, u) for the functional E fulfils

$$\Delta^2 u = 0 \qquad in\ \Omega \setminus (K_0 \cup K_1)\ .$$

Euler equations II : boundary-type conditions on singular set

Necessary conditions on jump discontinuity set K_0 for natural boundary operators

Assume (K_0, K_1, u) is an essential locally minimizing triplet for the functional E, $B \subset\subset \Omega$ is an open disk such that $K_0 \cap B$ is a diameter of the disk and $(K_1 \setminus K_0) \cap B = \emptyset$. Then

$$\left(\frac{\partial^2 u}{\partial N^2}\right)^{\pm} = 0 \qquad \text{on } K_0 \cap B\,,$$

$$\left(\frac{\partial^3 u}{\partial N^3} + 2\,\frac{\partial}{\partial N}\left(\frac{\partial^2 u}{\partial \tau^2}\right)\right)^{\pm} = 0 \qquad \text{on } K_0 \cap B$$

where B^+, B^- are the connected components of $B \setminus K_0$, N is the unit normal to K_0 pointing toward B^+, v^+, v^- the traces of any v on K_0 respectively from B^+ and B^-,$\tau = (\tau_1, \tau_2) = (-N_2, N_1)$ the choice of the unit tangent vector to K_0 .

Euler equations III : singular set variations

Next we evaluate the first variation of the energy
around a local minimizer u,
under compactly supported smooth deformation of K_0 and K_1

Integral Euler equation

If (K_0, K_1, u) is a locally minimizing triplet of E. Then $\forall \eta \in C_0^2(\Omega, \mathbb{R}^2)$

$$\int_{\Omega\setminus(K_0\cup K_1)} \Big(|D^2 u|^2 \operatorname{div} \eta - 2\big(D\eta\, D^2 u + (D\eta)^t\, D^2 u + Du\, D^2 \eta \big) : D^2 u \Big) d\mathbf{x}$$

$$+ \alpha \int_{K_0} \operatorname{div}^\tau_{\mathrm{K}_0} \eta\, d\mathcal{H}^1 + \beta \int_{K_1\setminus K_0} \operatorname{div}^\tau_{\mathrm{K}_1\setminus\mathrm{K}_0} \eta\, d\mathcal{H}^1 \;\; = \;\; 0\,,$$

where $\operatorname{div}^\tau_S$ denotes the tangential (to set S) divergence and

$$\big(D\eta D^2 u + (D\eta)^t D^2 u + Du D^2 \eta \big)_{ij} \; =$$

$$= \sum_k \Big(D_k \eta_i D^2_{kj} u + D_i \eta_k D^2_{kj} u + D_k u D^2_{ij} \eta_k \Big)$$

Curvature of jump set K_0 and squared hessian jump

If (K_0, K_1, u) is an essential locally minimizing triplet for functional E, $B \subset\subset U \subset \widetilde{\Omega}$ two open disks, s.t. $K_0 \cap U$ is the graph of a C^4 function, B^+ (resp. B^-) the open connected epigraph (resp. subgraph) of such function in B,.
$K_1 \cap U = \emptyset$, and u in $W^{4,r}(B^+) \cap W^{4,r}(B^-)$, $r > 1$.
Then

$$\left[\!\left[|D^2 u|^2 \right]\!\right] \;=\; \alpha\, \mathcal{K}(K_0) \qquad \text{on } K_0 \cap B\,.$$

where we denote
by $\mathcal{K}$ the curvature and by $[\![w]\!]$ the jump of a function w on K_0

Analogous results holds true for crease set $K_1 \setminus K_0$

Both results follows by plugging
(normal to singular set) vector fields in Integral Euler equation

Crack-tip

Now we perform a qualitative analysis of the “boundary” of the singular set, by assuming it is manifold as smooth as required by the computation of boundary operators.
The strategy is a new choice of the test functions in Euler equation: a vector field η tangential to K_0 (or K_1).

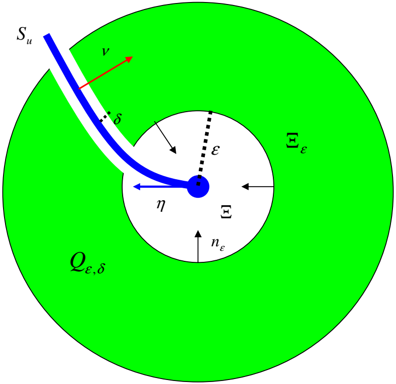
Crack-tip Theorem

Assume (K_0, K_1, u) is an essential locally minimizing triplet of E, $B = B(\mathbf{x}_0) \subset \Omega$ an open disk with center at $\mathbf{x}_0$ s.t. $(K_1 \setminus K_0) \cap B = \emptyset$, $K_0 \cap B = \overline{S_u} \cap B$ is a is a smooth curve from center to bdry of B and

$$\exists\, r > 1 \;:\quad u \in W^{4,r}\big(U \setminus (K_0 \cup B_\varepsilon(\mathbf{x}_0))\big) \qquad \forall\, \varepsilon > 0$$

Then u fulfils, for every $\eta \in C_0^3(B, \mathbb{R}^2)$ s.t. $\eta = \zeta\tau$
($\zeta \in C_0^\infty(B)$, $\tau \in C^3(B, S^1)$ and
η vector field tangent to K_0 pointing toward K_0 at $\mathbf{x}_0$)

$$\lim_{\varepsilon \to 0_+} \int_{\partial B_\varepsilon(\mathbf{x}_0) \setminus K_0} \mathcal{L}^\eta(u)\, d\mathcal{H}^1 \;=\; \alpha\, \zeta(\mathbf{x}_0)$$


S_u
ν
δ
ε
Ξ_ε
η
Ξ
n_ε
$Q_{\varepsilon,\delta}$

Summarizing:

- By performing suitable smooth variations we found Euler equation in $\Omega \setminus \overline{K_0 \cup K_1)}$ and jump conditions for u and for Du in $K_0 \cup K_1$;
- by performing smooth variations of jump and crease sets $K_0, K_1 \setminus K_0$ around a minimizer we found integral and geometric conditions on optimal segmentation sets.

In addition we proved:

1. Caccioppoli inequality: as a consequence
   any locally minimizing triplet of $\mathcal{E}$ in $\mathbb{R}^2$ with finite energy and compact segmentation set $K_0 \cup K_1$
   actually must have empty segmentation;
2. Liouville property:
   $(\emptyset, \emptyset, u)$ with bi-harmonic u is locally minimizing triplet of $\mathcal{E}$ in $\mathbb{R}^n$
   iff u is affine;
3. neither a straight infinite wedge
   nor a straight 1–dimensional uniform jump
   are locally minimizing triplets of E in $\mathbb{R}^2$;
4. 3/2 homogeneity: .
   any locally minimizing triplet (K_0, K_1, u) is transformed in another locally minimizing triplet by all
   natural re-scaling centered at $\mathbf{x}_0 \in \Omega$, which maps

$$u(\mathbf{x}) \qquad \text{to} \quad \varrho^{-3/2} u(\mathbf{x}_0 + \varrho \mathbf{x}) \,,$$

$$K_j \qquad \text{to} \quad \varrho^{-1}(K_j - \mathbf{x}_0), \qquad \varrho > 0,\ j = 0, 1 \,.$$

Mode 1 (JUMP) :

$$\varrho^{3/2}\,\omega(\theta) \;=\; \varrho^{3/2}\left(\sin\frac{\theta}{2} \;-\; \frac{5}{3}\,\sin\left(\frac{3}{2}\theta\right)\right) \qquad -\pi < \theta < \pi$$

Mode 2 (CREASE) :

$$\varrho^{3/2}\,\mathrm{w}(\theta) \;=\; \varrho^{3/2}\left(\cos\frac{\theta}{2} \;-\; \frac{7}{3}\,\cos\left(\frac{3}{2}\theta\right)\right) \qquad -\pi < \theta < \pi$$

CANDIDATE:

$$W \;=\; \pm\sqrt{\frac{\alpha}{193\pi}}\;\varrho^{3/2}\left(\sqrt{21}\,\omega(\theta) \pm \mathrm{w}(\theta)\right) \qquad -\pi < \theta < \pi$$

W fulfils all Euler equations,
all constraints on jump and curvature of singular set and

Energy equipartition: $$\int_{B_\varrho(0)} |\nabla^2 u|^2\,d\mathbf{x} \;=\; \alpha\varrho$$

Candidate conjecture

Assume $0 < \beta \leq \alpha \leq 2\beta < +\infty$.

Then triplet

$$(\, K_0 = \text{ negative real axis}\,,\, K_1 = \emptyset\,,\ \text{function } W\,)\,)$$

is a locally minimizing triplet for E in $\mathbb{R}^2$.

Moreover we conjecture that
there are no other nontrivial locally minimizing triplets
with non empty jump set and different from triplets

$$(K_0 = \text{ closed negative real axis, } K_1 = \emptyset, \Phi)$$

$$\Phi = (A\,\omega(\vartheta) \,+\, B\,w(\vartheta))\, r^{3/2}, \quad 35\,A^2 + 37\,B^2 = \frac{4\,\alpha}{\pi}\,, \quad A \neq 0$$

possibly swayed by rigid motions of $\mathbb{R}^2$ co-ordinates
and/or addition of affine functions.

(69)

Proving the minimality of a given candidate for a free discontinuity problem is a difficult task in general.

As far as we know,
neither the calibration techniques [Alberti, Bouchitte, DalMaso], nor the method used by [Bonnet, David] (both successfully applied to Mumford & Shah functional to test non trivial minimizers) seem to apply to the present context of second order functionals.

Even the *excess identity* approach of [Percivale & T.],
which succeeds with second order functionals related to
elasto-plastic plates,
does not apply to the present context since Blake & Zisserman functional do not control $\int_{S_{Dv}} |[Dv]| d\mathcal{H}^1$.

Mumford-Shah functional

Theorem [M.Carriero, A.Leaci, D.Pallara, E.Pascali]

If $(\mathbb{R}^-, u)$ is a local minimizer of

$$\int_{B_1} |\nabla v|^2 + \alpha \mathcal{H}^1(S_v)$$

then

$$u(\rho,\theta) = a_0 \pm u^S(\rho,\theta) + u^R(\rho,\theta))$$

where

$$u^S(\rho,\theta) = \sqrt{\frac{2\alpha}{\pi}}\rho^{1/2} \sin\frac{\theta}{2}\,, \qquad u^R(\rho,\theta) = o(\rho^{1-\varepsilon})$$

CONJECTURE (E.De Giorgi)

$$\psi(\rho,\theta) \;=\; \sqrt{\frac{2\,\alpha}{\pi}}\,\rho^{1/2}\,\sin\frac{\theta}{2}$$

is a local minimizer of Mumford-Shah functional in $\mathbb{R}^2$.
ψ **is the only non trivial local minimizer in $\mathbb{R}^2$**
(up to the sign and/or a rigid motion and constant addition)

where local minimizer of M–S functional refers to
compactly-supported variation
(without topological restrictions)

With a slightly different definition
competitor for (u,K)**:** any pair (w,H) s.t. .... and
if $x,y \in \mathbb{R}^2 \setminus (K \cup B_R)$ are separated by K,
then also H separates them,

A.Bonnet & G.David proved the conjecture in a weak form.
(the difference does not play any role for candidate ψ.)

Theorem - Uniform density estimates up to the bdry [C-L-T, Pure Math.Appl., 2009]

(**Density upper bound for the functional** F)
Let (K_0, K_1, u) be an essential locally minimizing triplet for the functional F under structural assumptions, $g \in L^4_{loc}(\Omega)$, and

$$\exists \bar{\varrho} > 0 \ : \ \mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \partial\Omega\,,\ \forall \varrho \le \bar{\varrho}\,.$$

Then for every $0 < \varrho \le (\bar{\varrho} \wedge 1)$ and for every $x \in \overline{\Omega}$ such that $\overline{B}_\varrho(x) \subset \widetilde{\Omega}$ we have

$$F_{\overline{B}_\varrho(x) \cap \overline{\Omega}}(K_0, K_1, u) \le c_0 \varrho$$

where $c_0 = C^2\pi + 2\pi^{\frac{1}{2}}\mu\big(\|w\|^2_{L^4(B_\varrho(\mathbf{x}))} + \|g\|^2_{L^4(B_\varrho(\mathbf{x}))}\big) + (2\pi + C)\alpha$.

Theorem - Uniform density estimates up to the bdry [C-L-T, Comm.Pure Appl.Anal. 2010]

Let (K_0, K_1, u) be an essential locally minimizing triplet for the functional F under structural assumptions, $g \in L^4_{loc}(\Omega)$, and

$$\exists \bar{\varrho} > 0 \ : \ \mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \partial\Omega\,,\ \forall \varrho \leq \bar{\varrho}\,.$$

(**Density lower bound for the functional** F)
Then there exist $\varepsilon_0 > 0$, $\varrho_0 > 0$ such that, for every $0 < \varrho \leq (\bar{\varrho} \wedge 1)$ and for every $x \in \overline{\Omega}$ such that $\overline{B}_\varrho(x) \subset \widetilde{\Omega}$ we have

$$F_{B_\varrho(\mathbf{x})}(K_0, K_1, u) \ \geq \ \varepsilon_0 \varrho \qquad \forall x \ \in \ (K_0 \cup K_1) \cap \overline{\Omega}, \quad \forall \varrho \leq \varrho_0$$

(**Density lower bound for the segmentation length**)
and there exist $\varepsilon_1 > 0$, $\varrho_1 > 0$ such that

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x})\right) \ \geq \ \varepsilon_1 \varrho \qquad \forall x \in (K_0 \cup K_1) \cap \overline{\Omega}, \ \forall \varrho \leq \varrho_1.$$

Theorem - Uniform density estimates up to the bdry [C-L-T, Comm.Pure Appl.Anal. 2010]

(**Elimination property**) Let (K_0, K_1, u) be an essential locally minimizing triplet for the functional F under structural assumptions, $g \in L^4_{loc}(\Omega)$, and

$$\exists \bar{\varrho} > 0 \ : \ \mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \partial\Omega\,,\ \forall \varrho \leq \bar{\varrho}\,.$$

Then and let $\varepsilon_1 > 0$, $\varrho_1 > 0$ as above. If $x \in \overline{\Omega}$ and

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x})\right) \ < \ \frac{\varepsilon_1}{2}\varrho$$

then

$$(K_0 \cup K_1) \cap B_{\varrho/2}(\mathbf{x}) = \emptyset\,.$$

Theorem - Uniform density estimates up to the bdry [C-L-T, Comm.Pure Appl.Anal. 2010]

(**Minkowski content of the segmentation**)
Let (K_0, K_1, u) be an essential locally minimizing triplet for the functional F under structural assumptions and $g \in L^4(\widetilde{\Omega})$.

Then $K_0 \cup K_1$ is $(\mathcal{H}^1, 1)$ rectifiable and

$$\lim_{\varrho \downarrow 0} \frac{|\{\mathbf{x} \in \Omega_\varepsilon \,;\, \text{dist}(\mathbf{x}, (K_0 \cup K_1) \cap \overline{\Omega}) < \varrho \,\}|}{2\varrho} = \mathcal{H}^1 (K_0 \cup K_1) \ .$$

Numerical experiments

- [F.Doveri] proved the Γ convergence and implemented the GNC algorithm proposed by Blake & Zisserman.
- [G.Bellettini, A.Coscia] (n=1) approximation by elliptic functionals.
- [L.Ambrosio, L.Faina, R.March] (n=2) variational approximation of B& Z fctl:

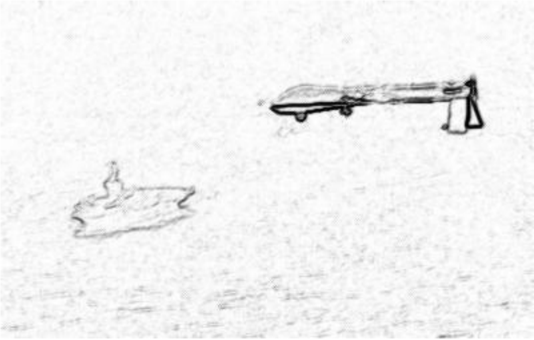
$$\mathcal{F}_\varepsilon(u,s,\sigma) = \int_\Omega \left((\sigma^2+\kappa_\varepsilon)|\nabla^2 u|^2\right) + \mu|u-g|^2 d\,\mathbf{x} +$$

$$+(\alpha-\beta)\,\mathcal{G}_\varepsilon(s)\;+\;\beta\,\mathcal{G}_\varepsilon(\sigma)\;+\;\xi_\varepsilon\int_\Omega (s^2+\xi_\varepsilon)|\nabla u|^\gamma d\mathbf{x}$$

with $\kappa_\varepsilon, \xi_\varepsilon, \zeta_\varepsilon$ suitable infinitesimal weights and

$$\mathcal{G}_\varepsilon(s) = \int_\Omega \left(\varepsilon\,|\nabla s|^2 + \frac{(s-1)^2}{4\varepsilon}\right)\,d\mathbf{x}$$

- [M.Carriero, I.Farina, A.Sgura] implemented a finite difference approach via Euler-Lagrange equations.



Theorem - Asymptotic expansion of loc.min. triplets with crack-tip

Assume $(\Gamma, \emptyset, u)$ is a locally minimizing triplet of E in $\mathbb{R}^2$,
where $\Gamma =$ denotes the closed negative real axis.
Then there are constants A, B with $(A, B) \neq (0, 0)$ and A_h, B_h s.t.

$$
\begin{aligned}
u(r,\theta) &= \\
&= r^{3/2}\Big(A\left(\sin\left(\tfrac{\theta}{2}\right) - \tfrac{5}{3}\sin\left(\tfrac{3}{2}\theta\right)\right) + B\left(\cos\left(\tfrac{\theta}{2}\right) - \tfrac{7}{3}\cos\left(\tfrac{3}{2}\theta\right)\right)\Big) + \\
&\quad + \sum_{h=1}^{+\infty} r^{h+\frac{3}{2}}\Big(A_h\cos\left(\left(h+\tfrac{3}{2}\right)\theta\right) + B_h\sin\left(\left(h+\tfrac{3}{2}\right)\theta\right) + \\
&\quad - \tfrac{2h+3}{2h+7}A_h\cos\left(\left(h-\tfrac{1}{2}\right)\theta\right) - \tfrac{2h+3}{2h-5}B_h\sin\left(\left(h-\tfrac{1}{2}\right)\theta\right)\Big)
\end{aligned}
$$

where u is expressed by polar coordinates in $\mathbb{R}^2$
with $\theta \in (-\pi, \pi)$ and $r \in (0, +\infty)$.

This expansion is strongly convergent in $H^2(B_\varrho \setminus \Gamma)$, moreover ...

... the lower order term ($h = 0$) in the expansion must have the following form

$$\begin{cases} W_0 = (A\,\omega(\vartheta) + B\,w(\vartheta))\, r^{3/2} \quad \text{in } B_\varrho \setminus \Gamma\,, \ \textit{referring to } \textbf{modes}: \\ \textit{Mode 1 (Jump)} \qquad \omega(\vartheta) = \left(\sin\left(\frac{\vartheta}{2}\right) - \frac{5}{3}\sin\left(\frac{3}{2}\vartheta\right)\right) \\ \textit{Mode 2 (Crease)} \qquad w(\vartheta) = \left(\cos\left(\frac{\vartheta}{2}\right) - \frac{7}{3}\cos\left(\frac{3}{2}\vartheta\right)\right) \end{cases}$$

where $\vartheta \in (-\pi, \pi)$ and constants A, B verify

$$35\,A^2 + 37\,B^2 = \frac{4\,\alpha}{\pi}\,, \qquad A \neq 0\,.$$