The A.D. Aleksandrov problem and optimal mass transport on S^n

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Optimization and stochastic methods
for spatially distributed information
St. Petersburg, Russia
May 11-15, 2010

J. Moser, 1965

Given a C^∞ compact connected manifold M^n, $\partial M^n = \emptyset$, and α, β two n-volume forms on M^n s.t.

$$\int_{M^n} \alpha = \int_{M^n} \beta.$$

Then $\exists \phi \in Diff^\infty(M^n, M^n)$ such that

$$\phi^*(\beta) = \alpha.$$

In local coordinates: If $\alpha = f dx$ and $\beta = g dx$ then the equation

$$g(\phi(x))|J(\phi(x))| = f(x),\ x \in M^n,$$

has a C^∞-solution.

The solution ϕ is NOT UNIQUE.

When $\partial M^n \neq \emptyset$, see B. Dacorogna-J. Moser, 1990.

One can think of at least two ways to restrict the class of possible solutions:

I. Search for ϕ in a special (= restricted) class of maps.

II. Associate a COST $C(\phi)$ with each ϕ and look for ϕ that optimize the cost.

It seems that in many problems in geometry and physics the approach **I** is usually taken, while in economics, mechanics, etc., the main approach is **II.**

It turns out that for several classes of problems in geometry and geometrical optics these two approaches lead to the same solution.

The **Aleksandrov's problem** is a good case supporting this observation.

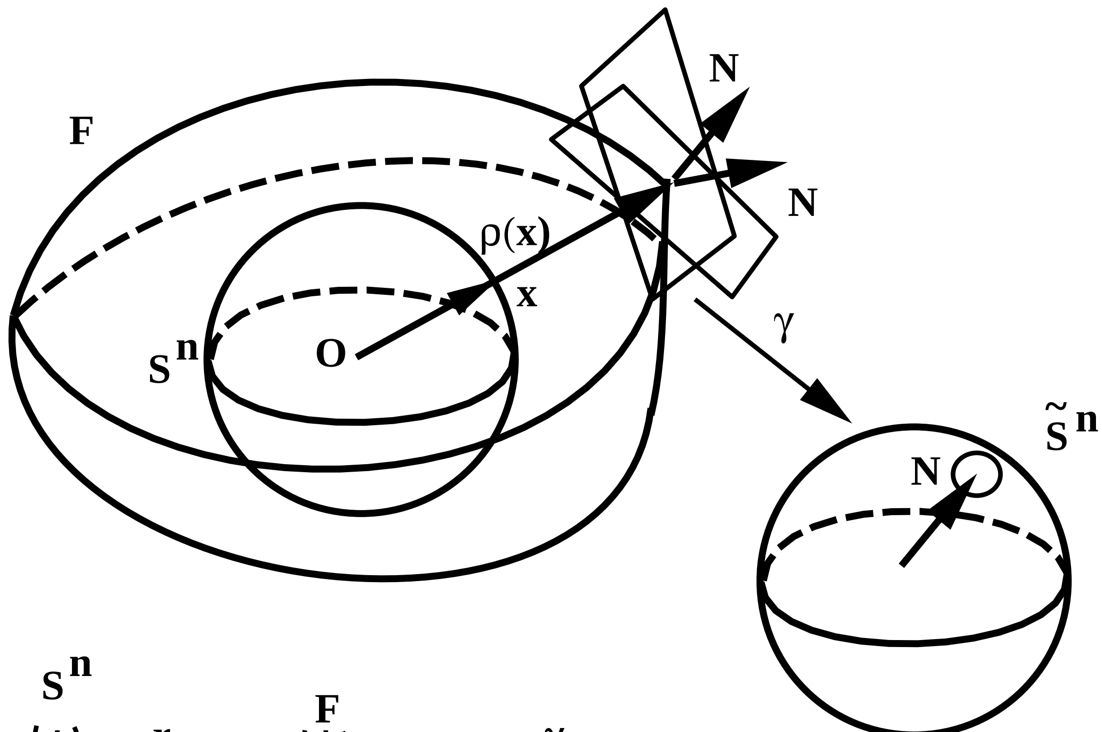
Aleksandrov's problem for Compact Convex Hypersurfaces

Statement of the problem

Notation.

$$\mathcal{F}^n \equiv \{\text{closed convex hypersurfaces in } \mathbf{R}^{n+1} \; *-\text{shaped with respect to the origin } \mathcal{O}\}, \; n \geq 1.$$

Generalized Gauss map



$$\begin{array}{ccccc} S^n & & F & & \\ \Cup & \xrightarrow{\ \mathbf{r}\ } & \Cup & \xrightarrow{\ \gamma\ } & \\ \mathbf{x} & & \mathbf{r}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{x} & & \{N(\mathbf{x})\} \subset \tilde{S}^n \end{array}$$

$$\alpha_F = \gamma \circ \mathbf{r}$$

The generalized Gauss map;

in general, multivalued

With each $F \in \mathcal{F}^n$ we associate two functions:

Radial function $\rho : \mathbb{S}^n \to (0, \infty)$, $\rho(x) =$ dist $(\mathcal{O}, F)$ in direction $x \in \mathbb{S}^n$

Support function $h : \tilde{\mathbb{S}}^n \to (0, \infty)$, $h(N) =$ distance from $\mathcal{O}$ to the supporting hyperplane to F with the outward unit normal $N \in \tilde{\mathbb{S}}^n$.

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Notation: $\sigma-$ standard Lebesgue measure on $\mathbb{S}^n$.

Fact. If $\omega \in \mathcal{B}(\mathbb{S}^n)$ then $\alpha_F(\omega) \subset \tilde{\mathbb{S}}^n$ is Lebesgue measurable.

Def-n. The **integral Gauss Curvature** is the “pull-back” of σ:

$$K_F(\omega) = \int_{\alpha_F(\omega)} d\sigma, \ \ \omega \in \mathcal{B}(\mathbb{S}^n).$$

QUESTION. Given a positive Borel measure μ on $\mathbb{S}^n$, under what conditions on μ there exists a $F \in \mathcal{F}^n$ such that

$$K_F(\omega) = \mu(\omega), \ \ \forall \omega \in \mathcal{B}(\mathbb{S}^n)?$$

Uniqueness - ?

Theorem 1. *(A.D. Aleksandrov '39) In order for a given function* μ *on* $\mathcal{B}(\mathbb{S}^n)$ *to be the integral Gauss curvature of* $F \in \mathcal{F}^n$ *it is necessary and sufficient that*

$$\mu \text{ is nonnegative and countably additive on Borel subsets of } \mathbb{S}^n, \tag{1}$$

$$\mu(\mathbb{S}^n) = \sigma(\mathbb{S}^n), \tag{2}$$

the inequality

$$\mu(\mathbb{S}^n \setminus \omega) > \sigma(\omega^*), \tag{3}$$

holds for any spherically convex $\omega \subset \mathbb{S}^n, \omega \neq \mathbb{S}^n$, *where (the dual)*

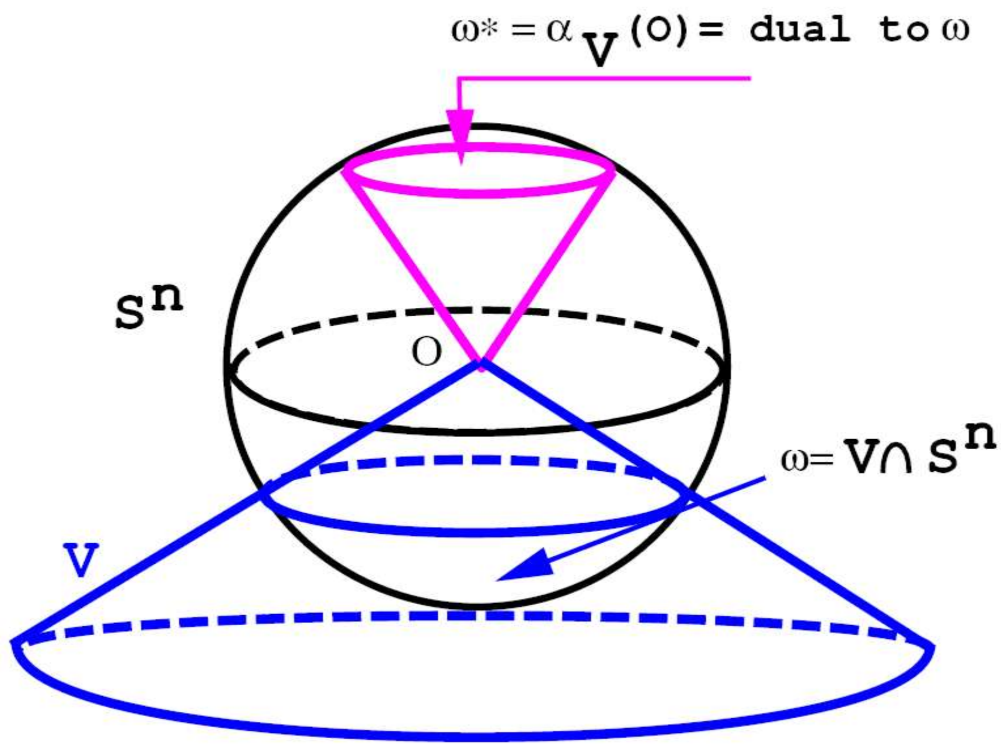
$$\omega^* = \{y \in \mathbb{S}^n \mid \langle x, y\rangle \leq 0 \; \forall x \in \omega.\}$$

Such F *is unique up to a homothety with respect to* $\mathcal{O}$*.*

Note. The measure μ may be a sum of point masses.

Illustration of condition

$$\mu(\mathbb{S}^n \setminus \omega) > \sigma(\omega^*),$$



If $F \in C^2$ then

$$K_F(\omega) = \int_{\alpha_F(\omega) \subset \tilde{\mathbb{S}}^n} d\sigma(N) = \int_{r(\omega)} \bar{K} dF = \int_{\omega \subset \mathbb{S}^n} \bar{K} g d\sigma = \mu(\omega), \ \forall \omega \in \mathcal{B}(\mathbb{S}^n),$$

where $\bar{K}$ is the Gauss-Kronecker curvature and $gd\sigma$ is the volume element of F.

The solution of Aleksandrov is a weak solution.

Outline of Aleksandrov's proof

Step 1. First solve the problem in the class of convex polytopes $\mathcal{P}_k^n \subset \mathcal{F}^n$ with vertices only on some fixed set of rays $x_1, ..., x_k$ emanating from $\mathcal{O}$ and not pointing in one hemisphere. The equation of the problem is

$$(K_P(x_i) \equiv) \ \ \sigma(\alpha_P(x_i)) = \mu_i, \ i = 1, 2, ..., k, \ P \in \mathcal{P}_k^n$$

where $\mu = (\mu_1, ..., \mu_k)$ is a given atomic measure satisfying conditions of Aleksandrov's theorem.

Step 2. Given a general μ, approximate it by $\sum_{i=1}^k \mu_i \delta(x_i)$. Do Step 1 for the $\mu_k = \sum_i \mu_i \delta(x_i)$ to obtain a sequence $\{P_k\}$.

Step 3. Show that the set $\{P_k\}$ is compact in $C(\mathbb{S}^n)$. Extract a converging subsequence (making sure that $F = \lim_{k\to\infty} P_k \in \mathcal{F}^n$) and use weak continuity of the $K_{P_k}(\equiv \sigma(\alpha_{P_k}))$ to conclude that $K_{P_k}(\omega) \longrightarrow K_F(\omega)$ (weakly).

To do Step I, Aleksandrov used his **Mapping Lemma**, which is a variant of the domain invariance theorem. However, to apply Aleksandrov's mapping lemma one needs to establish first the uniqueness of a solution.

Aleksandrov's theorem stimulated further research on this and many related problems.

Various generalizations (including noncompact case) were investigated by A.D. Aleksandrov, A.V. Pogorelov, I. Bakel'man, A. Kagan, V. Oliker, L. Caffarelli-L. Nirenberg-J. Spruck, A. Treibergs, P. Delanoe, J. Urbas, L. Barbosa-H. Lira-V. Oliker, Y.Y. Li-V. Oliker, Q. Jin-Y.Y.Li, R. McCann, V. Bogachev-A. Kolesnikov,...

This list is by no means complete!!

In his classical book on Convex Polyhedra, ’50, A.D. Aleksandrov asked for variational proofs of geometric existence problems for convex polytopes and said that this is a difficult problem.

Next, I will describe such a variational formulation and a proof of Aleksandrov’s theorem. In fact, the proposed approach is really a generic procedure applicable in many other existence problems in geometry (and optics). In addition, this approach provides significantly more information about the solution (including a way to compute it!) than the original approach of Aleksandrov.

The Theorems 2-4 below were established by V.O., Adv. in Math., 213 (2007).

Theorem 2. *Let* $\mu : \mathbb{S}^n \to [0, \infty)$ *satisfy* (1)-(3) *in Theorem 1. Put:*

$$c(x, N) = \begin{cases} \log\langle x, N\rangle & \textit{when } \langle x, N\rangle > 0 \\ -\infty & \textit{otherwise} \end{cases} \quad (x, N) \in \mathbb{S}^n \times \tilde{\mathbb{S}}^n,$$

$$\mathcal{A} = \{(h, \rho) \in C(\tilde{\mathbb{S}}^n) \times C(\mathbb{S}^n) \mid h > 0,\ \rho > 0, \qquad (4)$$
$$\log h(N) - \log \rho(x) \geq c(x, N),\ (x, N) \in \mathbb{S}^n \times \tilde{\mathbb{S}}^n\},$$

$$\mathcal{Q}[h, \rho] = \int_{\tilde{\mathbb{S}}^n} \log h(N) d\sigma(N) - \int_{\mathbb{S}^n} \log \rho(x) d\mu(x). \qquad (5)$$

Then $\exists$! *(up to a homothety with respect to* $\mathcal{O}$*) closed convex hypersurface* $\tilde{F} \in \mathcal{F}^n$ *with support function* $\tilde{h}$ *and radial function* $\tilde{\rho}$ *such that*

$$\mathcal{Q}[\tilde{h}, \tilde{\rho}] = \inf_{\mathcal{A}} \mathcal{Q}[h, \rho]. \qquad (6)$$

The hypersurface $\tilde{F}$ *is the unique (up to a homothety w. r. to* $\mathcal{O}$*) solution of the Aleksandrov problem, that is,*

$$K_{\tilde{F}}(\omega) = \mu(\omega), \quad \forall \omega \in \mathcal{B}(\mathbb{S}^n). \qquad (7)$$

- **Connection with the Monge problem on $\mathbb{S}^n$**

Let μ be as before and let $\Theta = \{\theta\}$:

(a) each θ is measurable, possibly multivalued, map of $\mathbb{S}^n$ onto $\tilde{\mathbb{S}}^n$,

(b) $\forall$ Borel set $\omega \subset \mathbb{S}^n$ the image $\theta(\omega)$ is Lebesgue measurable and $\sigma\{N \in \tilde{\mathbb{S}}^n \mid \theta^{-1}(N)$ contains more than one point$\} = 0$,

(c) each θ is measure preserving, that is,

$$\int_{\tilde{\mathbb{S}}^n} f(\theta^{-1}(N))d\sigma(N) = \int_{\mathbb{S}^n} f(x)d\mu(x) \;\; \forall f \in C(\mathbb{S}^n).$$

Observe that $\Theta \neq \emptyset$. For example, the generalized Gauss map $\alpha_{\tilde{F}} \in \Theta$, where $\tilde{F} \sim (\tilde{h}, \tilde{\rho})$ is the minimizer of $\mathcal{Q}$, but there are also other maps in Θ.

Here is a construction for an atomic measure μ. Let $x_1, ..., x_k \in \mathbb{S}^n$ and $\mu_1, ..., \mu_k \geq 0$, $\sum \mu_i = \sigma(\mathbb{S}^n)$. Subdivide $\tilde{\mathbb{S}}^n = \bigcup_{i=1}^k \bar{E}_i$ where

$$|\partial E_i| = 0,\ E_i \bigcap E_j = \emptyset \text{ when } i \neq j, \text{ and } \int_{E_i} d\sigma = \mu_i,\ i = 1, ..., k.$$

Consider a multivalued map $\theta : \mathbb{S}^n \to \tilde{\mathbb{S}}^n$ s.t. $\theta(x_i) = \bar{E}_i$ and $\forall x \neq x_i\ \ \theta(x) \in \bigcup_{i=1}^k \partial E_i$. Then $\theta \in \Theta$.

For $(x, N) \in \mathbb{S}^n \times \tilde{\mathbb{S}}^n$ define the "cost density" and the cost

$$c(x,N) = \begin{cases} \log\langle x, N\rangle & \text{when } \langle x, N\rangle > 0 \\ -\infty & \text{otherwise} \end{cases} \quad (x, N) \in \mathbb{S}^n \times \tilde{\mathbb{S}}^n \tag{8}$$

$$T[\theta] := \int_{\tilde{\mathbb{S}}^n} c(\theta^{-1}(N), N) d\sigma(N), \ \theta \in \Theta, \tag{9}$$

Problem of Monge's type (MP) on $\mathbb{S}^n \times \tilde{\mathbb{S}}^n$: Find $\tilde{\theta} \in \Theta$ such that

$$T[\tilde{\theta}] = sup_{\Theta} T[\theta].$$

Theorem 3. *Let $\tilde{F} \sim (\tilde{h}, \tilde{\rho})$ be the minimizer of $\mathcal{Q}$ in Theorem 2. The generalized Gauss map $\alpha_{\tilde{F}}$ is a solution of MP and any other solution $\tilde{\theta}$ of MP satisfies $\tilde{\theta} = \alpha_{\tilde{F}} \ \mu -$ a.e. Furthermore,*

$$\mathcal{Q}[\tilde{h}, \tilde{\rho}] = T[\tilde{\theta}].$$

In addition, $\tilde{\theta}^{-1} = \alpha_{\tilde{F}}^{-1} \ \sigma -$ a.e.

Geometrically, for $F \in \mathcal{F}^n$ the function $c(x, N_F(x)) \equiv \log\langle x, N_F(x)\rangle$ gives a scale invariant quantitative measure of "asphericity" of a hypersurface F with respect to $\mathcal{O}$. For example, for a sphere centered at $\mathcal{O}$ it is identically zero, while for a sufficiently elongated ellipsoid of revolution centered at $\mathcal{O}$ it has large negative values at points where the radial direction is nearly orthogonal to the normal.

The above result shows that the most efficient way (with respect to the cost $c(x, N)$) to transfer to σ an abstractly given measure μ on $\mathbb{S}^n$ is to move it by the generalized Gauss map $\alpha_{\tilde{F}}$ of the convex hypersurface $\tilde{F}$ solving the Aleksandrov problem and the variational problem (4)-(6).

The conditions on μ and Θ are in fact intrinsic on $\mathbb{S}^n$. But the solution solves an extrinsic embedding problem!

Connection with the Kantorovich problem on $\mathbb{S}^n$.
In the framework of the mass transport theory on $\mathbb{S}^n \times \tilde{\mathbb{S}}^n$ the problem of minimizing $\mathcal{Q}[h, \rho]$ over $\mathcal{A}$ is the dual of the following **Kantorovich-type** (primal) maximization problem.

Let $\Gamma(\mu, \sigma)$ be a set of joint finite Borel measures on $\mathbb{S}^n \times \tilde{\mathbb{S}}^n$ with marginals μ and σ as in Theorem 1; that is, for each $\gamma \in \Gamma(\mu, \sigma)$:

$$\gamma[\omega, \tilde{\mathbb{S}}^n] = \mu(\omega) \ \ \forall \omega \in \mathcal{B}(\mathbb{S}^n)$$

and

$$\gamma[\mathbb{S}^n, \tilde{\omega}] = \sigma(\tilde{\omega}) \ \ \forall \tilde{\omega} \in \mathcal{B}(\tilde{\mathbb{S}}^n).$$

Put

$$\mathcal{C}[\gamma] = \int_{\mathbb{S}^n} \int_{\tilde{\mathbb{S}}^n} c(x, N) d\gamma(x, N), \ \ \gamma \in \Gamma(\mu, \sigma).$$

Each γ is an admissible transport "plan" with marginals μ and σ.

A Kantorovich type problem: Find $\tilde{\gamma} \in \Gamma(\mu, \sigma)$ s.t.

$$\mathcal{C}[\tilde{\gamma}] = \sup_{\Gamma(\mu,\sigma)} \mathcal{C}[\gamma].$$

Theorem 4. *Such optimal measure $\tilde{\gamma}$ exists and it is generated by the map $\alpha_{\tilde{F}}$ by setting*

$$\tilde{\gamma}[U, V] = \sigma[\alpha_{\tilde{F}}(U) \cap V]$$

for any Borel subsets U and V on $\mathbb{S}^n \times \tilde{\mathbb{S}}^n$.

In addition, the duality relation

$$\mathcal{Q}[\tilde{h}, \tilde{\rho}] = \mathcal{C}[\tilde{\gamma}]$$

holds.

Main steps of the variational proof of A.D.'s theorem.

1. **Availability of two representations of $F \in \mathcal{F}^n$:**

   (a) via the radial function $\rho : \mathbb{S}^n \to (0, \infty)$

   (b) via the support function $h : \tilde{\mathbb{S}}^n \to (0, \infty)$

   (c) ρ and h are related by a "Legendre"-like transform:

$$h(N) = \sup_{x \in \mathbb{S}^n} \rho(x) \langle x, N \rangle, \ N \in \tilde{\mathbb{S}}^n, \tag{10}$$

$$\frac{1}{\rho(x)} = \sup_{N \in \tilde{\mathbb{S}}^n} \frac{\langle x, N \rangle}{h(N)}, \ x \in \mathbb{S}^n. \tag{11}$$

(d) Conversely, a pair $(h,\rho) \in C(\tilde{\mathbb{S}}^n) \times C(\mathbb{S}^n)$, $h,\rho > 0$, satisfying (10),(11) defines a unique $F \in \mathcal{F}^n$ with support function h and radial function ρ.

(e) The generalized Gauss map $\alpha_F : \mathbb{S}^n \to \tilde{\mathbb{S}}^n$ and its inverse are

$$\alpha_F(x) = \left\{N \in \tilde{\mathbb{S}}^n | h(N) = \rho(x)\langle x, N\rangle\right\},\ x \in \mathbb{S}^n,$$

$$\alpha_F^{-1}(N) = \{x \in \mathbb{S}^n | h(N) = \rho(x)\langle x, N\rangle\},\ N \in \tilde{\mathbb{S}}^n.$$

2. **Splitting:** (10), (11) imply that for any $F \in \mathcal{F}^n$ the pair (h, ρ) satisfies

$$\log h(N) - \log \rho(x) \geq c(x, N) \ \ \forall x \in \mathbb{S}^n, \ N \in \tilde{\mathbb{S}}^n, \tag{12}$$

and for each x equality is attained for some N and for each N equality is attained for some x.

3. **Define the functional**

$$\mathcal{Q}[h, \rho] := \int_{\tilde{\mathbb{S}}^n} \log h(N) d\sigma(N) - \int_{\mathbb{S}^n} \log \rho(x) d\mu(x)$$

on the set

$$\mathcal{A} := \{(h, \rho) \in C(\tilde{\mathbb{S}}^n) \times C(\mathbb{S}^n), \ h > 0, \rho > 0 \ \text{ and satisfy (12)}\}.$$

and consider the problem:

$$Q[h, \rho] \longmapsto \text{min over} \mathcal{A}.$$

4. **Notes on the proof:**

(i) The problem is first solved for convex polytopes $P_k \in \mathcal{P}_k^n \subset \mathcal{F}^n$ with vertices on a fixed set of k rays $x_1, ..., x_k$ originating at $\mathcal{O}$ and atomic

$$\mu = \sum_{i=1}^{k} \mu_i \delta_{x_i}.$$

(ii) For polytopes one needs to minimize

$$\mathcal{Q}[h, \rho] := \int_{\tilde{\mathbb{S}}^n} \log h(N) d\sigma(N) - \sum_{i=1}^{k} \log \rho(x_i) \mu_i$$

on a (suitably modified) set $\mathcal{A}$.

(iii) Because $\mu(\mathbb{S}^n) = \sigma(\tilde{\mathbb{S}}^n)$, the functional $\mathcal{Q}$ is scale invariant,

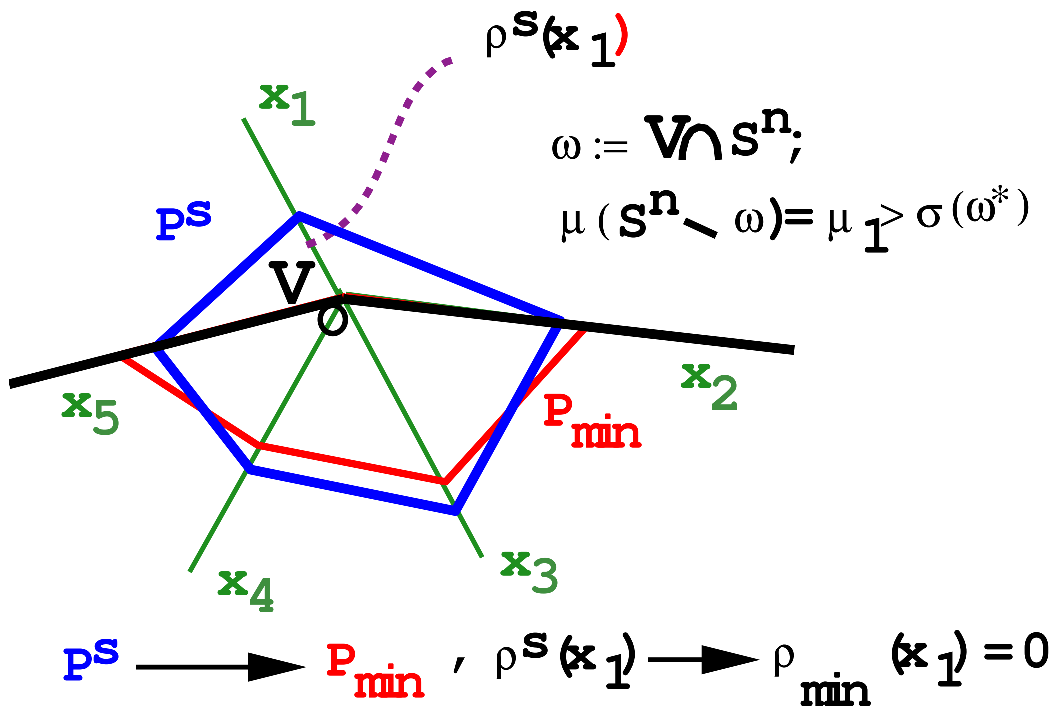
$$\mathcal{Q}[\lambda h, \lambda \rho] = \mathcal{Q}[\log h, \log \rho] \;\; \forall \lambda > 0.$$

Then the set $\mathcal{A}$ may be normalized to (h, ρ) s.t. $\rho(x), h(N) \leq 1$ and $\rho(\tilde{x}) = 1$ for some $\tilde{x}_\rho \in \mathbb{S}^n$.

(iv) It suffices to minimize $\mathcal{Q}$ on $(h, \rho) \in \mathcal{A}$ on which equality in (12) holds for some $x \in \mathbb{S}^n$ and for some $N \in \tilde{\mathbb{S}}^n$. By step 1(d) these are convex polytopes in $\mathcal{P}_k^n$. By Blaschke's theorem the set of such polytopes is compact in $C(\mathbb{S}^n)$. Hence the $\min \mathcal{Q}$ is attained on some convex P.

HOWEVER, if $\mathcal{O} \in P$ then $P \notin \mathcal{F}^n$!!!. But this is impossible!

Proof. Let $\{P^s\}$ with radial functions $\{\rho^s\}$ be a minimizing sequence,



By (iii) not all $\rho^s(x_i) \to 0$. Assume, that it is only $\rho^s(x_1) \longrightarrow 0$ and $\rho^s(x_i) \geq \varepsilon > 0$ for $i \neq 1$. When $P^s \longrightarrow P_{\text{min}} \Rightarrow$

$K_{P_s}(x_1) := \sigma(\alpha_{P^s}(x_1)) \to \sigma(\alpha_{P_{\text{min}}}(x_1)) = \sigma(\alpha_V(\mathcal{O})) = \sigma(\omega^*) < \mu_1$, where $\omega = V \cap \mathbb{S}^n$. Then

$$\mathcal{Q}[h^s, \rho^s] = \sum_{i=1}^{k} \left\{ \int_{\alpha_{P^s}(x_i)} \log h^s(N) d\sigma(N) - \log \rho^s(x_i)\mu_i \right\}$$

$$= \sum_{i=1}^{k} \left\{ \int_{\alpha_{P^s}(x_i)} [\log \rho^s(x_i) + \log\langle x_i, N\rangle]\, d\sigma(N) - \log \rho^s(x_i)\mu_i \right\}$$

$$= \log \rho^s(x_1)\, [\sigma(\alpha_{P^s}(x_1) - \mu_1]$$

$$+ \sum_{i>1} \log \rho^s(x_i)\, [\sigma(\alpha_{P^s}(x_i)) - \mu_i] + \sum_{i=1}^{k} \int_{\alpha_{P^s}(x_i)} \log\langle x_i, N\rangle d\sigma(N).$$

Since $\sigma(\alpha_{P^s}(x_1)) \to \sigma(\omega^*)$ and $\sigma(\omega^*) - \mu_1 < 0$, we get $\log \rho^s(x_1)\, [\sigma(\alpha_{P^s}(x_1)) - \mu_1] \to +\infty$. The remaining terms are bounded.

(v) A geometric perturbation argument is used to show that

$$K_{P_{\text{min}}}(x_i) = \mu_i,\ i = 1, ..., k.$$

(vi) The general variational problem for a measure μ satisfying (1)-(3) in Theorem 1 is handled by approximation by polyhedra and uses the fact that the integral Gauss curvature is weakly continuous.

(vii) Uniqueness [using (12), show uniqueness μ-a.e. of the generalized Gauss map of the minimizer; then show uniqueness of the minimizing pair $(\rho_{\text{min}}, h_{\text{min}})$].

(viii) **Note.** The typical in optimal transport requirement $\mu(\mathbb{S}^n) = \sigma(\tilde{\mathbb{S}}^n)$ is sufficient for compactness in $C(\mathbb{S}^n)$ but not sufficient to stay in $\mathcal{F}^n$.

Now one can construct polytopes with prescribed curvatures by using linear programming (at least in principle!).

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Regularity

If μ has density m and e is the usual metric on $\mathbb{S}^n$ then the equation is

$$\frac{\rho^{1-n} det[-\rho Hess(\rho) + 2\nabla\rho \otimes \nabla\rho + \rho^2 e]}{(\rho^2 + |\nabla\rho|^2)^{(n+1)/2} det(e)} = m \text{ on } \mathbb{S}^n,$$

The main **result** - Pogorelov '69, $n = 2$, Oliker '83, $n \geq 2$ (using polarity, extending C^2 estimates by Pogorelov, plus C^3 estimates by Calabi):

If $m > 0,\ m \in C^k(\mathbb{S}^n),\ k \geq 3$, then $F \in C^{k+1,\beta}(\mathbb{S}^n), \beta \in (0,1)$. If $m \in C^a(\mathbb{S}^n)$ then $F \in C^a(\mathbb{S}^n)$.

The case when $m \geq 0$ was studied by P. Guan and Y.Y. Li ('97) under various additional assumptions.

Proof of Theorem 3. Let $\theta \in \Theta$. Then for any pair $(h, \rho) \in \mathcal{A}$

$$\log h(N) - \log \rho(\theta^{-1}(N)) \geq c(\theta^{-1}(N), N).$$

It suffices to consider θ such that $\langle \theta^{-1}(N), N \rangle \leq 0$ only on sets of measure zero. Multiply by $d\sigma$, integrate and use the change of variable on the second term in the left (using (c) in def-n of Θ). Then

$$\mathcal{Q}[h, \rho] = \int_{\tilde{\mathbb{S}}^n} \log h(N) d\sigma(N) - \int_{\mathbb{S}^n} \log \rho(x) d\mu(x) \geq T[\theta].$$

Using the minimizing pair $(h_{\text{min}}, \rho_{\text{min}})$, we obtain

$$\mathcal{Q}[h_{\text{min}}, \rho_{\text{min}}] = T[\alpha_{\tilde{F}}] \geq T[\theta],$$

that is, $T[\alpha_{\tilde{F}}] = T[\tilde{\theta}]$.

Note. The uniqueness is obtained using the uniqueness of $\alpha_{\tilde{F}}$.

Outline of the proof of Theorem 4. Observe that any map $\theta \in \Theta$ gives a measure in $\Gamma(\mu, \sigma)$ by setting:

$$\gamma_\theta[U, V] = \sigma[\theta(U) \cap V] \;\forall \text{ Borel } U \in \mathbb{S}^n,\; V \in \tilde{\mathbb{S}}^n.$$

Then for $\gamma_0 := \alpha_{\tilde{F}}$ we have (after switching to θ^{-1})

$$\begin{aligned}\sup_{\Gamma(\mu,\sigma)} \mathcal{C}[\gamma] &\geq \sup_{\theta\in\Theta} \int_{\tilde{\mathbb{S}}^n} c(\theta^{-1}(N), N) d\sigma(N) \\ &= \int_{\tilde{\mathbb{S}}^n} c(\alpha_{\tilde{F}}^{-1}(N), N) d\sigma(N) = \mathcal{Q}[\tilde{h}, \tilde{\rho}].\end{aligned}$$

We prove now the reverse inequality.
Since $c(x, N) \leq 0 \;\forall (x, N) \in \mathbb{S}^n \times \tilde{\mathbb{S}}^n$, it is clear that

$$\sup_{\Gamma(\mu,\sigma)} \mathcal{C}[\gamma] = \sup_{\Gamma_+(\mu,\sigma)} \mathcal{C}[\gamma], \text{ where}$$

$$\Gamma_+(\mu, \sigma) := \{\gamma \in \Gamma(\mu, \sigma) \mid spt\gamma \subset \{(x, N) \in \mathbb{S}^n \times \tilde{\mathbb{S}}^n \mid \langle x, N\rangle \geq 0\}.$$

On the other hand, any pair (h, ρ) in the set $\mathcal{A}$ of admissible functions satisfies

$$\log h(N) - \log \rho(x) \geq c(x, N), \; (x, N) \in \mathbb{S}^n \times \tilde{\mathbb{S}}^n,$$

and by integrating against any $\gamma \in \Gamma_+(\mu, \sigma)$ we obtain:

$$\mathcal{C}[\gamma] \leq \int_{\mathbb{S}^n} \int_{\tilde{\mathbb{S}}^n} [\log h(N) - \log \rho(x)] \gamma(dx, dN)$$

$$= \int_{\tilde{\mathbb{S}}^n} \log h(N) \gamma(\mathbb{S}^n, dN) - \int_{\tilde{\mathbb{S}}^n} \log \rho(x) \gamma(dx, \tilde{\mathbb{S}}^n) = \mathcal{Q}[h, \rho].$$

Thus, the $\sup_{\Gamma(\mu,\sigma)} \mathcal{C}[\gamma]$ is attained on $\tilde{\gamma}$ corresponding to $(\tilde{h}, \tilde{\rho})$, that is, $\tilde{\gamma}$ is defined by $\alpha_{\tilde{F}}$ and the equality $\mathcal{Q}[\tilde{h}, \tilde{\rho}] = \mathcal{C}[\tilde{\gamma}]$ holds.

Entropy and monotonicity

(a) For $F \in \mathcal{F}^n$ put $u_F := \langle \alpha_F^{-1}(N), N \rangle$ and consider the "**entropy**"

$$\mathcal{E}(F) := \int_{\tilde{\mathbb{S}}^n} u_F(N) \log u_F(N) d\sigma.$$

Then $\mathcal{E}(F) \leq 0$. Also, it increases when passing from F to a parallel hypersurface. Its maximum ($= 0$) is attained when

$$F = \mathbb{S}^n.$$

(b) To prove this in a more general case one might consider a flow with rescaling ($diam(F_t) = 1$ to factor out homotheties):

$$\frac{\partial F_t}{\partial t} = K_{F_t} N, \text{ on } \tilde{\mathbb{S}}^n \times (0, \infty), \; F_0 = F_{init} \in \mathcal{F}^n.$$

The claim is that under this flow the limiting state is $\mathbb{S}^n$ and $\mathcal{E}(F) \to 0$. This should follow from the evolution equation for the entropy density $u_{F_t} \log u_{F_t}$.