A Benamou-Brenier approach to branched transport

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References

Some results of this talk are contained in

- L. B., G. Buttazzo, F. Santambrogio, *A Benamou-Brenier approach to branched transport*, submitted (`http://cvgmt.sns.it/people/brasco`)

Outline

Some notations

- $\Omega \subset \mathbb{R}^N$ compact and convex
- $\mathcal{P}(\Omega)$ = Borel probability measures over Ω
- $\mathcal{M}(\Omega; \mathbb{R}^N) = \mathbb{R}^N-$valued Radon measures over Ω
- $w_p = p-$Wasserstein distance

$$w_p(\rho_0, \rho_1) = \min \left\{ \left(\int_{\Omega \times \Omega} |x - y|^p \, d\gamma(x, y) \right)^{1/p} : \gamma \in \Pi(\rho_0, \rho_1) \right\}$$

- $\mathcal{W}_p(\Omega) = p-$Wasserstein space over Ω, i.e. $\mathcal{P}(\Omega)$ equipped with w_p
- $|\mu_t'|_{w_p} = \lim_{h \to 0} \dfrac{w_p(\mu_{t+h}, \mu_t)}{|h|}$ metric derivative
- α = exponent between 0 and 1

Branched transport: what's this?

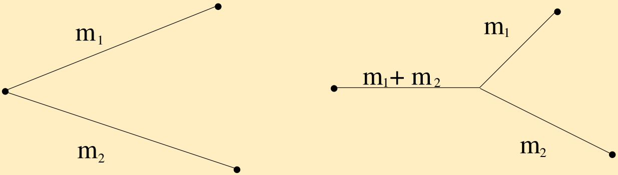
Transport problems where the cost has a **subadditive** dependence on the mass, i.e. moving a mass m for a distance ℓ costs

$$\varphi(m)\,\ell,$$

with $\varphi(m_1+m_2)<\varphi(m_1)+\varphi(m_2) \Longrightarrow$ **total cost** $=\sum \varphi(m)\,\ell$

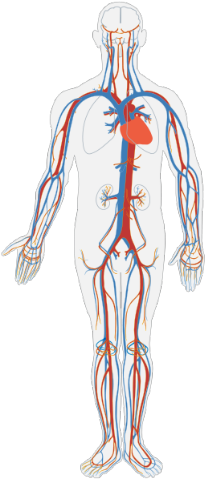
typical choice $\quad \varphi(t)=t^\alpha,\ \alpha\in[0,1]$

Due to concavity, **grouping the mass** during the transport could lower the total cost $\Longrightarrow$ typical optimal structures are **tree-shaped**



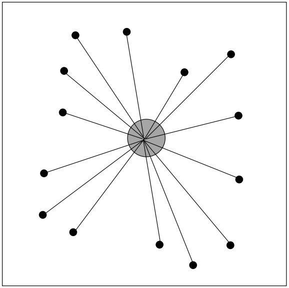
Remark

Many natural and artificial transportation systems satisfy this **cost saving requirement** (root systems in a tree, blood vessels...)

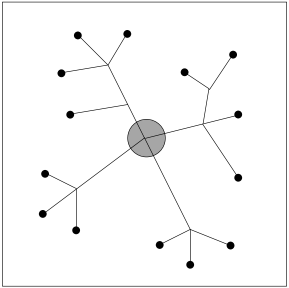


Example: a power supply station

- $\rho_0 = \delta_{x_0}$ power supply station
- $\rho_1 = \sum_{i=1}^{k} m_i \delta_{x_k}$ houses ($\sum_{i=1}^{k} m_i = 1$)



Monge–Kantorovich solution



Branched transport solution

Comment

it is better to construct an optimal network of wires (**right**) to save cost; this is not possible by looking at Monge-Kantorovich (**left**)

Some models: Gilbert's weighted oriented graphs

This is only suitable for **discrete measures**

$$\rho_0 = \sum_{i=1}^{k} a_i\, \delta_{x_i} \in \mathcal{P}(\Omega) \text{ and } \rho_1 = \sum_{j=1}^{m} b_j\, \delta_{y_j} \in \mathcal{P}(\Omega)$$

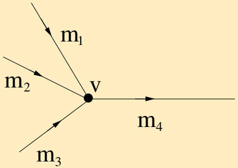
Transport path between ρ_0 and ρ_1

$\mathfrak{g}$ **weigthed oriented graph** consisting of:

- $\{v_s\}_{s\in V}$ vertices (comprising x_i **sources** and y_j **sinks**)
- $\{e_h\}_{h\in H}$ edges
- $\{\overrightarrow{\tau_h}\}_{h\in H}$ orientations of the edges
- $\{m_h\}_{h\in H}$ weigths (i.e. transiting mass on the edge e_h)

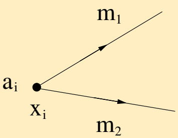
$+$ **Kirchhoff's Law** for circuits

Interior vertices

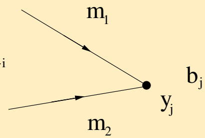


$$m_1 + m_2 + m_3 = m_4$$

“Boundary” vertices



$$m_1 + m_2 = a_i$$



$$m_1 + m_2 = b_j$$

Total cost

$$M_\alpha(\mathfrak{g}) = \sum_{h \in H} m_h^\alpha \, \mathcal{H}^1(e_h)$$ **(Gilbert-Steiner energy)**

Some models: Xia's transport path model I

Idea: for the discrete case...

- $\mathfrak{g} \rightsquigarrow \phi_{\mathfrak{g}}$ **vector measure** $\langle \phi_{\mathfrak{g}}, \overrightarrow{\varphi} \rangle = \sum_{h \in H} m_h \int_{e_h} \overrightarrow{\varphi} \cdot \overrightarrow{\tau_h} \, d\mathcal{H}^1$
- Kirchhoff's Law $\rightsquigarrow \mathrm{div}\phi_{\mathfrak{g}} = \rho_0 - \rho_1$

...for the general case

ϕ **transport path** between ρ_0 and ρ_1 if $\exists \{\mathfrak{g}_{\mathfrak{n}}, \rho_0^n, \rho_1^n\}_{n \in \mathbb{N}}$ s.t.

$$\phi_{\mathfrak{g}_n} \rightharpoonup \phi, \;\; \rho_i^n \rightharpoonup \rho_i, i = 0, 1$$

Total cost

$$M_\alpha^* := \text{relaxation of } M_\alpha$$

$$M_\alpha^*(\phi) = \begin{cases} \int_\Sigma m(x)^\alpha \, d\mathcal{H}^1(x), & \text{if } \phi = m \overrightarrow{\tau} \mathcal{H}^1 \llcorner \Sigma, \\ +\infty, & \text{otherwise} \end{cases}$$

Some models: Xia's transport path model II

Theorem (Xia, Morel-Santambrogio)

Let $\alpha \in (1 - 1/N, 1]$ *and* $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$, *then*

$$d_\alpha(\rho_0, \rho_1) := \min\{M_\alpha^*(\phi) \, : \, \mathrm{div}\, \phi = \rho_0 - \rho_1\} < +\infty$$

Moreover d_α *defines a* **distance** *on* $\mathcal{P}(\Omega)$, **equivalent** *to* w_1 *(and thus to any* w_p, *with* $1 \leq p < \infty$*)*

$$w_1(\rho_0, \rho_1) \leq d_\alpha(\rho_0, \rho_1) \leq C\, w_1(\rho_0, \rho_1)^{N(\alpha-1)+1}$$

Remark

- the exponent $N(\alpha - 1) + 1$ can not be improved
- the lower bound is not optimal, actually we have $w_{1/\alpha} \leq d_\alpha$ (Devillanova-Solimini)

Some models: a Lagrangian approach I

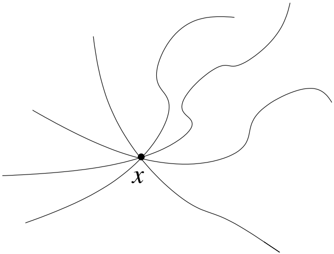
Transportation is described through Q **probability measures on Lipschitz paths** (parametrized on $[0,1]$, let us say)

Constraints

$(e_0)_\sharp Q = \rho_0$, $(e_1)_\sharp Q = \rho_1$
(where $e_t(\sigma) = \sigma(t)$ *evaluation at* t)

Multiplicity (i.e. "*transiting mass*")

$[x]_Q = Q(\{\widetilde{\sigma} \,:\, x \in \widetilde{\sigma}([0,1])\}) \leq 1$



Energy (Bernot-Caselles-Morel)

$$E_\alpha(Q) = \int_{Lip([0,1];\Omega)} \int_0^1 [\sigma(t)]_Q^{\alpha-1} \, |\sigma'(t)| \, dt \, dQ(\sigma)$$

Some models: a Lagrangian approach II

If $E_\alpha(Q) < +\infty$ and Q gives full mass to injective curves...

Gilbert-Steiner energy, again!

$$E_\alpha(Q) = \int_\Omega [x]_Q^\alpha \, d\mathcal{H}^1(x)$$

Theorem (Bernot-Caselles-Morel)

For every ρ_0, ρ_1, this Lagrangian model is **equivalent** *to Xia's one (i.e. same optimal structures, different description of the same energy)*

There exist other Lagrangian models (Maddalena-Morel-Solimini[a], Bernot-Figalli) that we are neglecting, differing for the definition of the multiplicity: the one chosen here is **not local in time**

[a] This was actually the first!

Aim of the talk

We want to present a model for branched transport of the type

Energy

$$\mathcal{G}(\mu, v) = \int_0^1 G_\alpha(\mu_t, v_t)\, dt \quad \textit{with} \quad \begin{array}{c} t \mapsto \mu_t \text{ curve in } \mathcal{P}(\Omega) \\ t \mapsto v_t \text{ velocity field} \end{array}$$

Constraints: the continuity equation

$$\begin{cases} \partial_t \mu_t + \mathrm{div}_x(v_t \mu_t) = 0 & \text{in } \Omega, \\ \mu_0 = \rho_0, \quad \mu_1 = \rho_1 \end{cases}$$

Remark

This is **Eulerian** and **dynamical**, i.e. an optimal μ provides the evolution in time of the branched transport with its **velocity field** v, not just the optimal ramified structure

The Benamou-Brenier formula I

First of all, recall the dynamical formulation for w_p $(p > 1)$

Benamou-Brenier [*Numer. Math.* **84** (2000)]

$$w_p(\rho_0, \rho_1) = \min \left\{ \int_0^1 \int_\Omega |v_t(x)|^p d\mu_t(x)\, dt : \begin{array}{c} \partial_t \mu_t + \operatorname{div}_x(v_t\, \mu_t) = 0 \\ \mu_0 = \rho_0, \;\; \mu_1 = \rho_1 \end{array} \right\}$$

Important

It can be reformulated as a **convex optimization** + **linear constraints**, introducing

$$\phi_t := v_t \cdot \mu_t \;(momentum) \implies |v_t|^p \mu_t = |\phi_t|^p\, \mu_t^{1-p} \;\; convex$$

Thanks to the Disintegration Theorem...

(μ, ϕ) can be thought as measures on $[0,1] \times \Omega$ disintegrating as

$$\mu = \int \mu_t\, dt \;\; \text{and} \;\; \phi = \int \phi_t\, dt$$

The Benamou-Brenier formula II

$$f_p(x,y) = \begin{cases} |y|^p\, x^{1-p}, & \text{if } x>0,\ y\in\mathbb{R}^N, \\ 0, & \text{if } x=0,\ y=0, \\ +\infty, & \text{otherwise} \end{cases}$$

is **jointly convex** and $1-$**homogeneous**

The functional can be rewritten as follows

Benamou-Brenier functional

$$\mathcal{F}_p(\mu,\phi) = \int_{[0,1]\times\Omega} f_p\left(\frac{d\mu}{dm}, \frac{d\phi}{dm}\right)\, dm$$

Comment

$\mathcal{F}_p$ **l.s.c.** and does **not** depend on the choice of m

$$w_p(\rho_0,\rho_1) = \min\left\{\mathcal{F}_p(\mu,\phi) \ :\ \partial_t\mu + \mathrm{div}_x\phi = \delta_0\otimes\rho_0 - \delta_1\otimes\rho_1\ \right\}$$

The Benamou-Brenier formula III

Remark

By its very definition

$$\mathcal{F}_p(\mu, \phi) < +\infty \Longrightarrow \phi \ll \mu$$

and in this case

$$\mathcal{F}_p(\mu, \phi) = \int_{[0,1]\times\Omega} \left| \frac{d\phi}{d\mu} \right|^p \, d\mu$$

If moreover $\mu = \int \mu_t \, dt$, then $\phi = \int \phi_t \, dt$ with $\phi_t = v_t \cdot \mu_t$ and

$$\mathcal{F}_p(\mu, \phi) = \int_0^1 \int_\Omega \left| \frac{d\phi_t}{d\mu_t} \right|^p \, d\mu_t \, dt = \int_0^1 \int_\Omega |v_t|^p d\mu_t \, dt$$

A possible variant for branched transport: heuristics

We consider the **local** and **l.s.c. functional on measures**

$$g_\alpha(\lambda) = \begin{cases} \int_\Omega |\lambda(\{x\})|^\alpha \, d\#(x), & \text{if } \lambda \text{ is atomic} \\ +\infty, & \text{otherwise} \end{cases}$$

Energy?

For $\mu = \int \mu_t \, dt$ and $\phi = \int \phi_t \, dt$ with $\phi_t \ll \mu_t$

$$\mathcal{G}_\alpha(\mu, \phi) = \int_0^1 g_\alpha\left(\left| \frac{d\phi_t}{d\mu_t} \right|^{1/\alpha} \mu_t \right) dt = \int_0^1 g_\alpha(|v_t|^{1/\alpha} \mu_t) \, dt$$

This is a Gilbert-Steiner energy!

$$\mathcal{G}_\alpha(\mu, \phi) = \int_0^1 \sum_{k \in \mathbb{N}} |v_t(x_{k,t})| \, \mu_t(\{x_{k,t}\})^\alpha \, dt$$

A possible variant for branched transport: setting

$\mathfrak{D}$ = admissible pairs (μ, ϕ)

$\mu \in C([0,1]; \mathcal{P}(\Omega))$ $\quad \partial_t \mu_t + \mathrm{div}_x \phi_t = 0$ in Ω

$\phi \in L^1([0,1]; \mathcal{M}(\Omega; \mathbb{R}^N))$

Dynamical branched energy

$$G_\alpha(\mu_t, \phi_t) = \begin{cases} \int_\Omega |v_t(x)| \mu_t(\{x\})^\alpha \, d\#(x) & \text{if } \phi_t = v_t \cdot \mu_t, \\ +\infty & \text{if } \phi_t \not\ll \mu_t \end{cases}$$

$$\mathcal{G}_\alpha(\mu, \phi) = \int_0^1 G_\alpha(\mu_t, \phi_t)\, dt, \;\; (\mu, \phi) \in \mathfrak{D}$$

Important remark

$\mathcal{G}_\alpha(\mu, \phi) < +\infty \quad \not\Longrightarrow \quad \mu_t$ atomic $\forall t$

$\Longrightarrow \quad \phi \ll \mu$ and μ_t atomic on $\{|v_t(x)| > 0\}$

Main result

Theorem (B.-Buttazzo-Santambrogio)

For every $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$, the minimization problem

$$\mathfrak{B}_\alpha(\rho_0, \rho_1) = \min_{(\mu,\phi)\in\mathfrak{D}} \{\mathcal{G}_\alpha(\mu, \phi) \,:\, \mu_0 = \rho_0, \ \mu_1 = \rho_1\}$$

admits a solution

Remark 1

The proof uses Direct Methods...l.s.c.? coercivity? As always, it is a matter of choosing the right **topology**

Remark 2

Observe that the problem is **not** convex, but rather **concave**

Choice of the topology

Proposal: pointwise convergence

What about "$\mu_t^n \rightharpoonup \mu_t$ *for every* t *and* $\phi_t^n \rightharpoonup \phi_t$ *for a.e.* t"?

Answer: NO

Good for l.s.c. (you simply apply Fatou Lemma, because G_α is l.s.c) **but** not so good for coercivity (how can we infer compactness from $\mathcal{G}_\alpha \leq C$?)

Choice: weak topology

$(\mu^n, \phi^n) \rightharpoonup (\mu, \phi)$ (as measures on $[0,1] \times \Omega$)

The basic inequalities

If $(\mu, \phi) \in \mathfrak{D}$ such that $\phi \ll \mu$ and $\phi_t = v_t \cdot \mu_t$

$(B.I.)_1$

$$G_\alpha(\mu_t, \phi_t) = \sum_i \mu_t(\{x_i\})^\alpha |v_t(x_i)| = \sum_i \left(\mu_t(\{x_i\}) |v_t(x_i)|^{1/\alpha}\right)^\alpha$$

$$\geq \left(\sum_i \mu_t(\{x_i\}) |v_t(x_i)|^{1/\alpha}\right)^\alpha = \|v_t\|_{L^{1/\alpha}(\mu_t)} \geq |\mu_t'|_{W_{1/\alpha}}$$

$(B.I.)_2$

$$\mathcal{G}_\alpha(\mu, \phi) = \int_0^1 G_\alpha(\mu_t, \phi_t)\, dt \geq \int_0^1 |\phi_t|(\Omega)\, dt = |\phi|([0,1] \times \Omega)$$

Remark

$\sup_t G_\alpha \leq C \implies |\phi|([0,1] \times \Omega) \leq C$ and μ_t Lipschitz in $\mathcal{W}_{1/\alpha}$

Proof of the main result I

Stage 1 – Exctraction of a subsequence

- $\{(\mu^n, \phi^n)\} \subset \mathfrak{D}$ minimizing sequence
- we can assume $\mathcal{G}_\alpha(\mu^n, \phi^n) \le C$ for every n
- $\mathcal{G}_\alpha$ 1−homogeneous w.r.t. v_t (i.e. reparametrization invariant)
- $(\mu^n, \phi^n) \rightsquigarrow (\tilde{\mu}^n, \tilde{\phi}^n)$, with $\tilde{\mu}^n_s = \mu^n_{\mathfrak{t}(s)}$ and $\tilde{\phi}^n_s = \mathfrak{t}'(s) \cdot \phi^n_{\mathfrak{t}(s)}$
- choose $\mathfrak{t}$ s.t. $G_\alpha(\tilde{\mu}^n_s, \tilde{\phi}^n_s) \equiv \mathcal{G}_\alpha(\tilde{\mu}^n, \tilde{\phi}^n) = \mathcal{G}_\alpha(\mu^n, \phi^n) \le C$
- $\Longrightarrow \tilde{\mu}^n \rightharpoonup \mu$ and $\tilde{\phi}^n \rightharpoonup \phi$ (thanks to $(B.I.)_1$ and $(B.I.)_2$)

Proof of the main result II

Stage 2 – Admissibility of the limit

- clearly $\mu = \int \mu_t \, dt$ (uniform limit of continuous curves)
- to show that $\phi = \int \phi_t \, dt$, we use l.s.c. of Benamou-Brenier functional

$$\mathcal{F}_{1/\alpha}(\mu, \phi) \leq \liminf_{n\to\infty} \mathcal{F}_{1/\alpha}(\tilde{\mu}^n, \tilde{\phi}^n) \overset{(B.I.)_1}{\leq} C$$

$\Longrightarrow \phi \ll \mu$ and $\phi = \int \phi_t \, dt$

- (μ, ϕ) still solves the continuity equation $\Longrightarrow (\mu, \phi) \in \mathfrak{D}$
- $\mu_0 = \rho_0$ and $\mu_1 = \rho_1$

Proof of the main result III (conclusion)

Stage 3 – l.s.c. along a minimizing sequence

- remember that $\tilde{\phi}^n = \tilde{v}^n \cdot \tilde{\mu}^n$ and $\mathcal{G}_\alpha(\tilde{\mu}^n, \tilde{\phi}^n) \leq C$
- define $\mathfrak{m}^n = \int \sum_i |\tilde{v}_t^n(x_{i,t})| \, \tilde{\mu}_t^n(\{x_{i,t}\})^\alpha \, \delta_{x_{i,t}} \, dt \in \mathcal{M}([0,1] \times \Omega)$
- $\mathfrak{m}^n([0,1] \times \Omega) = \mathcal{G}_\alpha(\tilde{\mu}^n, \tilde{\phi}^n) \leq C$
- $\Longrightarrow \mathfrak{m}^n \rightharpoonup \mathfrak{m}$ and $\mathfrak{m} = \int \mathfrak{m}_t \, dt$
- $\mathfrak{m}^n([0,1] \times \Omega) \to \mathfrak{m}([0,1] \times \Omega) \Longrightarrow \mathcal{G}_\alpha(\tilde{\mu}^n, \tilde{\phi}^n) \to \int_0^1 \mathfrak{m}_t(\Omega) \, dt$
- show $\mathfrak{m}_t(\Omega) \geq G_\alpha(\mu_t, \phi_t)$ (a little bit delicate)
- $\Longrightarrow \mathcal{G}_\alpha(\mu, \phi) \leq \liminf_{n \to \infty} \mathcal{G}_\alpha(\tilde{\mu}^n, \tilde{\phi}^n) = \inf \mathcal{G}_\alpha$

Equivalences with other models

Theorem (B.-Buttazzo-Santambrogio)

$$\mathfrak{B}_\alpha(\rho_0,\rho_1) = \min\{E_\alpha(Q) \,:\, (e_i)_\sharp Q = \rho_i\} = d_\alpha(\rho_0,\rho_1)$$

As always, we have **equivalence of the problems**, *not just equality of the minima*

Recall that

$$E_\alpha(Q) = \int_{Lip([0,1];\Omega)} \int_0^1 [\sigma(t)]_Q^{\alpha-1} |\sigma'(t)|\, dt\, dQ(\sigma)$$

Remark

In order to compare the two models, we need to switch from **curves of measures** to **measures on curves** (and back!)

Some preliminary comments

Alert!

- Transiting mass in our model $\Longrightarrow \mu_t(\{x\})$ (local in space/time)
- Transiting mass in E_α model $\Longrightarrow [x]_Q$ (not local in time)

We will need the following

Theorem (Superposition principle (AGS, Theorem 8.2.1))

Let (μ, v) solve the continuity equation, with $\|v_t\|^p_{L^p(\mu_t)}$ integrable in time. Then $\mu_t = (e_t)_\sharp Q$ with Q concentrated on solutions of the ODE $\sigma'(t) = v_t(\sigma(t))$

Comment

This is a probabilistic version of the *method of characteristics*

Sketch of the proof: $\mathfrak{B}_\alpha(\rho_0, \rho_1) \geq d_\alpha(\rho_0, \rho_1)$

Step 1

(μ, ϕ) optimal $\overset{(B.I.)_1}{\Longrightarrow}$ $\phi = v \cdot \mu$ and $\int_0^1 \|v_t\|_{L^{1/\alpha}(\mu_t)}\, dt \leq \mathfrak{B}_\alpha(\rho_0, \rho_1)$

Step 2 - superposition principle

$\exists Q$ s.t. $\mu_t = (e_t)_\# Q$ and $\sigma'(t) = v_t(\sigma(t))$ for $Q-$a.e. σ

Step 3 - comparison of the multiplicities

$\mu_t = (e_t)_\sharp Q \Longrightarrow [x]_Q \geq Q(\{\tilde{\sigma} : \tilde{\sigma}(t) = x\}) = \mu_t(\{x\})$

$$\int [\sigma(t)]_Q^{\alpha-1} |\sigma'(t)|\, dQ(\sigma) \overset{Step2}{=} \int [x]_Q^{\alpha-1} |v_t(x)|\, d\mu_t(x)$$
$$\overset{Step3}{\leq} \int \mu_t(\{x\})^{\alpha-1} |v_t(x)|\, d\mu_t(x)$$

Sketch of the proof: $\mathfrak{B}_\alpha(\rho_0, \rho_1) \le d_\alpha(\rho_0, \rho_1)$

Step 0 – approximation

Approximate (ρ_0, ρ_1) with (ρ_0^n, ρ_1^n) (finite sums of Dirac masses) s.t. $d_\alpha(\rho_0^n, \rho_1^n) \to d_\alpha(\rho_0, \rho_1)$

Remark: why approximation?

$\exists Q$ optimal s.t.

$$[\sigma(t)]_Q = Q(\{\widetilde{\sigma}(t) = \sigma(t)\})$$ the mass is **synchronized**

(this is true **if** ρ_0 **is finitely atomic**)

Step 1 – curve in $\mathcal{P}(\Omega)$

$\mu_t := (e_t)_\sharp Q$ and disintegrate $Q = \int Q_x^t \, d\mu_t(x)$ (i.e. Q_x^t is concentrated on $\{\sigma \,:\, \sigma(t) = x\}$)

Sketch of the proof: $\mathfrak{B}_\alpha(\rho_0, \rho_1) \leq d_\alpha(\rho_0, \rho_1)$

Step 2 – velocity field

$v_t(x) := \int_{\{\sigma\,:\,\sigma(t)=x\}} \sigma'(t)\, dQ_x^t(\sigma)$ (average velocity)

Step 3

$(\mu, v\cdot\mu) \in \mathfrak{D}$ and $\mathcal{G}_\alpha(\mu, v\cdot\mu) \leq E_\alpha(Q) = d_\alpha(\rho_0^n, \rho_1^n)$, with $\mu_0 = \rho_0^n$ and $\mu_1 = \rho_1^n$

Step 4

Putting all together, we have

$$\mathfrak{B}_\alpha(\rho_0, \rho_1) \leq \liminf_{n\to\infty} \mathfrak{B}_\alpha(\rho_0^n, \rho_1^n) \leq \lim_{n\to\infty} d_\alpha(\rho_0^n, \rho_1^n) = d_\alpha(\rho_0, \rho_1)$$

A final remark: comparison of d_α and $w_{1/\alpha}$

Taking (μ, ϕ) optimal for $\mathfrak{B}_\alpha(\rho_0, \rho_1)$

$$\int_0^1 |\mu_t'|_{w_{1/\alpha}}\, dt \overset{(B.I.)_1}{\leq} \mathfrak{B}_\alpha(\rho_0, \rho_1) \overset{equivalence}{=} d_\alpha(\rho_0, \rho_1)$$

i.e. we have another proof of

$$w_{1/\alpha}(\rho_0, \rho_1) \leq d_\alpha(\rho_0, \rho_1)$$

Remark

d_α and $w_{1/\alpha}$ have exactly the same scaling

$$d_\alpha = \sum m^\alpha\, \ell \quad w_{1/\alpha} = \left(\sum m\, \ell^{1/\alpha}\right)^\alpha$$

Further readings

- Standard reference on branched transport

- M. Bernot, V. Caselles, J.-M. Morel *Optimal transportation networks – Models and theory*, Springer Lecture Notes (2009)

- Other models employing curves in Wasserstein spaces (but avoiding the use of the continuity equation) have been studied

- A. Brancolini, G. Buttazzo, F. Santambrogio, *Path functionals over Wasserstein spaces*, JEMS (2006)
- L. B., F. Santambrogio, *An equivalent path functional formulation of branched transportation problems*, accepted