Russian Math. Surveys **62**:3 409–451

Uspekhi Mat. Nauk **62**:3 5–46
DOI 10.1070/RM2007v062n03ABEH004410

Euler equations for incompressible ideal fluids

C. Bardos and E. S. Titi

Abstract. This article is a survey concerning the state-of-the-art mathematical theory of the Euler equations for an incompressible homogeneous ideal fluid. Emphasis is put on the different types of emerging instability, and how they may be related to the description of turbulence.

Contents

This work was supported in part by NSF grant no. DMS-0504619, BSF grant no. 2004271, and ISF grant no. 120/06.

AMS 2000 *Mathematics Subject Classification.* Primary 35Q05; Secondary 35Q30, 76Bxx, 76Dxx, 76Fxx.

1. Introduction

This contribution is mostly devoted to a time-dependent analysis of the $2D$ and $3D$ Euler equations

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \qquad \nabla \cdot u = 0 \tag{1}$$

of an incompressible homogeneous ideal fluid. We intend to connect several known (and maybe less known) points of view concerning this very classical problem. Furthermore, we will investigate the conditions under which one can consider the above problem as the limit of the incompressible Navier–Stokes equations

$$\partial_t u_\nu + \nabla \cdot (u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p_\nu = 0, \qquad \nabla \cdot u_\nu = 0, \tag{2}$$

when the viscosity ν goes to zero, that is, as the Reynolds number goes to infinity.

At the macroscopic level the Reynolds number Re corresponds to the ratio of *the strength of the non-linear effects* to *the strength of the linear viscous effects*. Therefore, with the introduction of a characteristic velocity U and a characteristic length scale L of the flow one has the dimensionless parameter

$$\mathrm{Re} = \frac{UL}{\nu}\,. \tag{3}$$

With the introduction of the characteristic time scale $T = L/U$ and the dimensionless variables,

$$x' = \frac{x}{L}, \qquad t' = \frac{t}{T}\,, \qquad \text{and} \qquad u' = \frac{u'}{U}\,,$$

the Navier–Stokes equations (2) take the non-dimensional form:

$$\partial_t u' + \nabla_{x'} \cdot (u' \otimes u') - \frac{1}{\mathrm{Re}} \Delta_{x'} u' + \nabla_{x'} p' = 0, \qquad \nabla \cdot u' = 0. \tag{4}$$

These are the equations to be considered below, omitting the prime ($'$) and returning to the notation ν for Re^{-1}.

In the presence of a physical boundary the problems (1) and (2) will be considered in an open domain $\Omega \subset \mathbb{R}^d$, $d = 2$, $d = 3$, with a piecewise smooth boundary $\partial\Omega$.

There are several good reasons to focus at present on the 'mathematical analysis' of the Euler equations rather than the Navier–Stokes equations.

1. Turbulence applications involving the Navier–Stokes equations (4) often correspond to very large Reynolds numbers; and a theorem which is valid for any finite, but very large, Reynolds number is expected to be compatible with results concerning infinite Reynolds number. In fact, this is the case when $\mathrm{Re} = \infty$, which drives other results, and we will give several examples of this fact.

2. Many non-trivial and sharp results for the incompressible Navier–Stokes equations rely on the smoothing effect of the Laplacian when the viscosity ν is > 0, and on the invariance of the set of solutions under the scaling

$$u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t). \tag{5}$$

However, simple examples with the same scalings but without an energy conservation law may exhibit very different behaviour concerning regularity and stability.

1. With ϕ a scalar function, the viscous Hamilton–Jacobi type or Burgers equation

$$\begin{aligned} &\partial_t\phi - \nu\Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \quad \text{in} \quad \Omega\times\mathbb{R}_t^+, \\ &\phi(x,t) = 0 \quad \text{for} \quad x\in\partial\Omega, \qquad \text{and} \quad \phi(\cdot,0) = \phi_0(\cdot)\in L^\infty(\Omega), \end{aligned} \tag{6}$$

has (because of the maximum principle) a global smooth solution for any $\nu > 0$. However, for $\nu = 0$ it is well known that certain solutions of the inviscid Burgers equation (6) will become singular (with shocks) in finite time.

2. Denote by $|\nabla|$ the square root of the operator $-\Delta$, defined in Ω and with Dirichlet homogeneous boundary conditions. Let us consider the solution $u(x,t)$ of the equation

$$\partial_t u - \nu\Delta u + \frac{1}{2}|\nabla|(u^2) = 0 \quad \text{in} \quad \Omega\times\mathbb{R}_t^+, \tag{7}$$

$$u(x,t) = 0 \quad \text{for} \quad x\in\partial\Omega, \qquad \text{and} \quad u(\cdot,0) = u_0(\cdot)\in L^\infty(\Omega). \tag{8}$$

Then one has the following proposition.

Proposition 1.1. *Assume that the initial data u_0 satisfies the relation*

$$\int_\Omega u_0(x)\phi_1(x)\,dx = -M < 0, \tag{9}$$

where $\phi_1(x) \geqslant 0$ denotes the first eigenfunction of the operator $-\Delta$ (with Dirichlet boundary condition), $-\Delta\phi_1 = \lambda_1\phi_1$. If M is sufficiently large, then the corresponding solution $u(x,t)$ of the system (7), (8) blows up in a finite time.

Proof. The L^2 scalar product of the equation (7) with $\phi_1(x)$ gives

$$\frac{d}{dt}\int_\Omega u(x,t)\phi_1(x)\,dx + \nu\lambda_1\int_\Omega u(x,t)\phi_1(x)\,dx = -\frac{\sqrt{\lambda_1}}{2}\int_\Omega u(x,t)^2\phi_1(x)\,dx.$$

Since $\phi_1(x) \geqslant 0$, the Cauchy–Schwarz inequality implies that

$$\left(\int u(x,t)\phi_1(x)\,dx\right)^2 \leqslant \left(\int_\Omega u(x,t)^2\phi_1(x)\,dx\right)\left(\int_\Omega \phi_1(x)\,dx\right).$$

As a result, the quantity $m(t) = -\int_\Omega u(x,t)\phi_1(x)\,dx$ satisfies the relation

$$\frac{dm}{dt} + \lambda_1 m \geqslant \frac{\sqrt{\lambda_1}\int_\Omega \phi_1(x)dx}{2}m^2 \quad \text{with} \quad m(0) = M,$$

and the conclusion of the proposition follows.

Remark 1.1. The above example was given with $\Omega = \mathbb{R}^3$ by Montgomery-Smith [1] (under the name 'cheap Navier–Stokes equations') with the purpose of revealing the role of the conservation of energy (which is not present in the above examples) in the Navier–Stokes dynamics. His proof shows that the same blowup property may appear in any space dimension for the solution of the 'cheap hyperviscosity equations'

$$\partial_t u + \nu(-\Delta)^m u + \frac{1}{2}|\nabla|(u^2) = 0.$$

On the other hand, one should observe that the above argument does not apply to the Kuramoto–Sivashinsky-like equations

$$\partial_t \phi + \nu(-\Delta)^m \phi + \alpha\Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \tag{10}$$

for $m \geqslant 2$. Without a maximum principle or without a control of some sort on the energy, the question of global existence of a smooth solution or finite-time blowup of some solution of the above equation is an open problem in $\Omega = \mathbb{R}^n$ for $n \geqslant 2$ and for $m \geqslant 2$. However, if in (10) the term $|\nabla\phi|^2$ is replaced by $|\nabla\phi|^{2+\gamma}$, $\gamma > 0$, then one can prove the blowup of some solutions (cf. [2] and references therein).

In conclusion, the above examples indicate that the conservation of some sort of energy, which is guaranteed by the structure of the equation, is essential in the analysis of the dynamics of the underlying problem. In particular, this very basic fact plays an essential role in the dynamics of the Euler equations.

Taking into account the above simple examples, the rest of the paper is organized as follows. In § 2 classical existence and regularity results for the time-dependent Euler equations are presented. § 3 provides more examples concerning the pathological behaviour of solutions of the Euler equations. The fact that the solutions of the Euler equations may exhibit oscillatory behaviour implies similar behaviour for the solutions of the Navier–Stokes equations as the viscosity tends to zero. The existence of (or lack thereof) strong convergence is analyzed in § 4 with the introduction of the Reynolds stress tensor and the notion of *dissipative solution*. A standard and very important problem for both theoretical study and applications is the vanishing-viscosity limit of solutions of the Navier–Stokes equations subject to the no-slip Dirichlet boundary condition in domains with physical boundaries. Very few mathematical results are available for this very unstable situation. One of the most striking results is a theorem of Kato [3], which is presented in § 5. § 6 is again devoted to the Reynolds stress tensor. We show that with the introduction of the Wigner measure the notion of Reynolds stress tensor, deduced from the defect in strong convergence as the viscosity tends to zero, plays the same role as the one originally introduced in the statistical theory of turbulence. When the zero viscosity limit of solutions of the Navier–Stokes equations is compared with the solution of the Euler equations, the main difference appears in a boundary layer which is described by the Prandtl equations. These equations are briefly described in § 7. There it is also recalled how the mathematical results are in agreement with the instability of the physical problem. The Kelvin–Helmholtz problem also exhibits some similar basic instabilities, but it is in some sense simpler. This is explained at the end of § 7, where it is also shown that some recent results of [4], [5], [6] on the regularity of the vortex sheet (interface) do contribute to an understanding of the instabilities of the original problem.

2. Classical existence and regularity results

2.1. Introduction. The Euler equations correspond formally to the limit case when the viscosity is 0 (or the Reynolds number is infinite):

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \qquad \nabla \cdot u = 0 \quad \text{in} \quad \Omega. \tag{11}$$

In the presence of physical boundaries, the above system is supplemented with the standard, no-normal flow, boundary condition:

$$u \cdot \vec{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{12}$$

where $\vec{n}$ denotes the outwards normal vector to the boundary $\partial\Omega$. It turns out that the vorticity $\omega = \nabla \wedge u$ is 'the basic quantity', from both the physical and mathematical analysis points of view. Therefore, equations (11) and (12), written in terms of the vorticity, are equivalent to the system

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u \quad \text{in} \quad \Omega, \tag{13}$$

$$\nabla \cdot u = 0, \quad \nabla \wedge u = \omega \quad \text{in} \quad \Omega, \qquad \text{and} \quad u \cdot \vec{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{14}$$

That is, the equations (14) completely determine u in terms of ω, which makes the whole system 'closed'. More precisely, the operator $K\colon \omega \mapsto u$ determined by (14) is a continuous linear map from $C^{\alpha}(\Omega)$ to $C^{\alpha+1}(\Omega)$ (with $\alpha > 0$), and also from $H^s(\Omega)$ to $H^{s+1}(\Omega)$.

Furthermore, for $2D$ flows the vorticity is perpendicular to the plane of motion, and therefore equation (13) reduces (this can also be checked directly) to the advection equation

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{15}$$

The structure of the quadratic non-linearity in (13) has a number of consequences, to be described below. We will be presenting only the essence of the essential arguments and not the full details of the proofs (see, for example, [7] or [8] for the details).

2.2. General results in 3*D*. The short-time existence of a smooth solution for the $3D$ incompressible Euler equations was obtained a long time ago, provided that the initial data are sufficiently smooth. To the best of our knowledge the original proof goes back to Lichtenstein [9]. This proof is based on a non-linear Gronwall estimate of the form

$$y' \leqslant C y^{\frac{3}{2}} \implies y(t) \leqslant \frac{y(0)}{\left(1 - 2tCy^{\frac{1}{2}}(0)\right)^2}. \tag{16}$$

Therefore, the value $y(t)$, which represents an adequate norm of the solution, is finite for a finite interval of time which depends on the size of the initial value $y(0)$, that is, the initial data of the solution of the Euler equations. These initial data have to be chosen from an appropriate space of sufficiently regular functions. In particular, if we consider the solution in the Sobolev space H^s, with $s > \frac{5}{2}$, then by taking the scalar product in H^s of the Euler equations with the solution u and by using the appropriate Sobolev inequalities we obtain:

$$\frac{1}{2}\frac{d\|u\|^2_{H^s}}{dt} = -\big(\nabla\cdot(u\otimes u), u\big)_{H^s} \leqslant C_s\|u\|^2_{H^s}\|\nabla u\|_{L^\infty} \leqslant C\|u\|^3_{H^s}. \tag{17}$$

As a result of (16) and (17) we obtain the local (in time) existence of a smooth solution.

As in many standard non-linear time-dependent problems, local regularity of smooth (strong) solutions implies local uniqueness and local stability (that is, continuous dependence on initial data). Furthermore, one may exhibit a threshold for this existence, uniqueness, and propagation of the regularity of the initial data (including analyticity; see Bardos and Benachour [10]). More precisely, one uses the following theorem.

Theorem 2.1 (Beale–Kato–Majda theorem [11]). *Let $u(t)$ be a solution of the $3D$ incompressible Euler equations which is regular for $0 \leqslant t < T$, that is,*

$$\textit{for all}\quad t\in[0,T],\quad u(t)\in H^s(\Omega)\quad \textit{for some}\quad s>\frac{5}{3}.$$

If

$$\int_0^T \big\|\nabla\wedge u(\cdot,t)\big\|_{L^\infty}\,dt<\infty, \tag{18}$$

then $u(t)$ can be uniquely extended up to a time $T+\delta$ ($\delta>0$) as a smooth solution of the Euler equations.

The main interest of this statement is that it shows that if one starts with smooth initial data, then instabilities can develop only if the size of the vorticity becomes arbitrarily large.

Remark 2.1. The Beale–Kato–Majda theorem was first proved in the whole space in [11]. Extension to a periodic 'box' is easy. For a bounded domain with the boundary condition $u\cdot\vec{n}=0$ it was established by Ferrari [12]. By combining arguments from [10] and [12] one can show, as in the Beale–Kato–Majda theorem, that the solution of the $3D$ Euler equations, with real-analytic initial data, remains real-analytic as long as (18) holds.

The Beale–Kato–Majda result has been slightly improved by Kozono [13], who proved that on the left-hand side of (18), the $\|\cdot\|_{L^\infty}$-norm can be replaced by the norm in the space BMO. This generalization is interesting because it employs harmonic analysis (or Fourier modes decomposition) techniques, which constitute an important tool for the study of 'turbulent' solutions; indeed, the space BMO, as the dual space of the Hardy space $\mathscr{H}^1$, is well defined in the frequency (Fourier) space. In fact (cf. [14]), BMO is the smallest space containing L^∞ which is also invariant under the action of zero-order pseudodifferential operators. The idea behind the Beale–Kato–Majda theorem, and its generalizations, is the fact that the solution u of the elliptic equations (14) satisfies for $1<p<\infty$ the estimate

$$\|\nabla u\|_{W^{s,p}} \leqslant C_{s,p}\big(\|u\|_{W^{s,p}}+\|\omega\|_{W^{s,p}}\big). \tag{19}$$

This relation could also be phrased in the context of the Hölder spaces $C^{k,\alpha}$, $\alpha > 0$. We stress, however, that the estimate (19) ceases to be true for $p = \infty$ (or $\alpha = 0$). This is due to the nature of the singularity, of the form $|x-y|^{2-d}$, in the kernel of the operator K, which leads (for $s > d/2+1$) to the estimate

$$\|\nabla u\|_{L^\infty} \leqslant C\big(\|\omega\|_{L^\infty} \log\big(1+\|u\|^2_{H^s}\big)\big), \quad \text{or, sharper,} \tag{20}$$

$$\|\nabla u\|_{L^\infty} \leqslant C\big(\|\omega\|_{\mathrm{BMO}} \log\big(1+\|u\|^2_{H^s}\big)\big). \tag{21}$$

If we let $z = 1 + \|u\|^2_{H^s}$, then by (21) the inequality (17) becomes

$$\frac{d}{dt} z \leqslant C\|\omega\|_{\mathrm{BMO}} z \log z.$$

This yields

$$\big(1+\|u(t)\|^2_{H^s}\big) \leqslant \big(1+\|u(0)\|^2_{H^s}\big)^{e^{C\int_0^t \|\omega(s)\|_{\mathrm{BMO}}\, ds}},$$

which proves the statement. The uniqueness of solutions can be proved along the same lines, as long as

$$\int_0^t \|\omega(s)\|_{\mathrm{BMO}}\, ds$$

remains finite.

Remark 2.2. The vorticity ω can be represented by the anti-symmetric part of the deformation tensor ∇u. However, in the estimates (20) or (21) this anti-symmetric part, that is, ω, can be replaced by the symmetric part of the deformation tensor

$$S(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^t\big). \tag{22}$$

Therefore, the theorems of Beale–Kato–Majda and Kozono can be rephrased in terms of this symmetric tensor, as was established in Ponce [15] and Kozono [13], respectively.

In fact, the above deformation tensor $S(u)$ (or $\widetilde{S}(\omega)$, when expressed in term of the vorticity), plays an important role in a complementary result of Constantin, Fefferman, and Majda [16] which shows that it is mostly the variations in the direction of the vorticity that produces singularities.

Proposition 2.1 [16]. *Let u, defined in $Q = \Omega \times (0,T)$, be a smooth solution of the Euler equations. Let $k_1(t)$ and $k_2(t)$ (which are well defined for $t < T$) be given by*

$$k_1(t) = \sup_{x\in\Omega} |u(x,t)|,$$

which measures the size of the velocity, and

$$k_2(t) = 4\pi \sup_{x,y\in\Omega, x\neq y} \frac{|\xi(x,t)-\xi(y,t)|}{|x-y|},$$

which measures the Lipschitz regularity of the direction

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$$

of the vorticity. Then under the assumptions

$$\int_0^T \big(k_1(t)+k_2(t)\big)\,dt<\infty \quad \text{and} \quad \int_0^T k_1(t)k_2(t)\,dt<\infty, \tag{23}$$

the solution $u(x,t)$ exists and is as smooth as the initial data up to a time $T+\delta$ for some $\delta>0$.

Proof. As before we present here only the basic ideas, and for simplicity we will focus on the case when $\Omega=\mathbb{R}^3$. First, since

$$S(u)=\frac{1}{2}\big(\nabla u+(\nabla u)^t\big)(x,t)=\widetilde{S}(\omega)(x,t), \tag{24}$$

we have

$$\frac{1}{2}\big(\partial_t|\omega|^2+u\cdot\nabla|\omega|^2\big)=(\omega\cdot\nabla u,\omega)=\big(\widetilde{S}(\omega)\omega,\omega\big), \tag{25}$$

which gives

$$\frac{d\|\omega\|_\infty}{dt}\leqslant\sup_x\big(|\widetilde{S}(\omega)|\big)\,\|\omega\|_\infty. \tag{26}$$

Next, we consider only the singular part of the operator $\omega\mapsto\widetilde{S}(\omega)$. The Biot–Savart law reproduces the velocity field from the vorticity according to the formula

$$u(x,t)=\frac{1}{4\pi}\int\frac{(x-y)\wedge\omega(y)}{|x-y|^3}\,dy. \tag{27}$$

For the essential part of this kernel, we introduce two smooth non-negative radial functions β_δ^1 and β_δ^2 with

$$\beta_\delta^1+\beta_\delta^2=1, \quad \beta_\delta^1=0 \quad \text{for} \quad |x|>2\delta, \quad \text{and} \quad \beta_\delta^2=0 \quad \text{for} \quad |x|<\delta. \tag{28}$$

Then we have

$$\begin{aligned}|\widetilde{S}(\omega)|\leqslant&\left|\int\Big(\frac{y}{|y|}\cdot\xi(x)\Big)\Big(\mathrm{Det}\Big(\frac{y}{|y|},\xi(x+y),\xi(x)\Big)\beta_\delta^1(|y|)\Big)|\omega(x+y)|\frac{dy}{|y|^3}\right|\\&+\left|\int\Big(\frac{y}{|y|}\cdot\xi(x)\Big)\Big(\mathrm{Det}\Big(\frac{y}{|y|},\xi(x+y),\xi(x)\Big)\beta_\delta^2(|y|)\Big)|\omega(x+y)|\frac{dy}{|y|^3}\right|.\end{aligned} \tag{29}$$

For the first term we use the estimate

$$\left|\mathrm{Det}\Big(\frac{y}{|y|},\xi(x+y),\xi(x)\Big)\beta_\delta^1(|y|)\right|\leqslant\frac{k_2(t)}{4\pi}|y| \tag{30}$$

to obtain

$$\left|\int\Big(\frac{y}{|y|}\cdot\xi(x)\Big)\Big(\mathrm{Det}\Big(\frac{y}{|y|},\xi(x+y),\xi(x)\Big)\beta_\delta^1(|y|)\Big)|\omega(x+y)|\frac{dy}{|y|^3}\right|\leqslant k_2(t)\delta\|\omega\|_\infty. \tag{31}$$

Next, we write the second term as

$$\int\Big(\frac{y}{|y|}\cdot\xi(x)\Big)\Big(\mathrm{Det}\Big(\frac{y}{|y|},\xi(x+y),\xi(x)\Big)\beta_\delta^2(|y|)\Big)\big(\xi(x+y)\cdot\big(\nabla_y\wedge u(x+y)\big)\big)\frac{dy}{|y|^3}$$

and integrate by parts with respect to y. By the Lipschitz regularity of ξ one has

$$\left|\nabla_y\left(\left(\frac{y}{|y|}\cdot\xi(x)\right)\left(\operatorname{Det}\left(\frac{y}{|y|},\xi(x+y),\xi(x)\right)\right)\right)\right|\leqslant Ck_2(t).$$

Therefore (observing that the terms coming from large values of $|y|$ and the terms coming from the derivatives of $\beta_\delta^2(|y|)$ give more regularity),

$$\begin{aligned}\left|\int\left(\frac{y}{|y|}\cdot\xi(x)\right)\left(\operatorname{Det}\left(\frac{y}{|y|},\xi(x+y),\xi(x)\right)\beta_\delta^2(|y|)\right)|\omega(x+y)|\frac{dy}{|y|^3}\right|&\\ \leqslant\int\left|\nabla_y\left(\frac{y}{|y|}\cdot\xi(x)\right)\left(\operatorname{Det}\left(\frac{y}{|y|},\xi(x+y),\xi(x)\right)\right)\right|\beta_\delta^2(|y|)\frac{dy}{|y|^3}\|u\|_\infty&\\ \leqslant Ck_2(t)|\log\delta|\,\|u\|_\infty\leqslant Ck_1(t)k_2(t)|\log\delta|.&\end{aligned}$$

Finally, inserting the above estimates in (26), we get that for $\|\omega\|_\infty>1$ and $\delta=\|\omega\|_\infty^{-1}$

$$\frac{d\|\omega\|_\infty}{dt}\leqslant Ck_2(t)\big(1+k_1(t)\big)\|\omega\|_\infty\log\|\omega\|_\infty,$$

and the conclusion follows as in the case of the Beale–Kato–Majda theorem.

The reader is referred, for instance, to the book of Majda and Bertozzi [7] and the recent survey of Constantin [17] for additional relevant material.

2.3. About the two-dimensional case. In the $2D$ case the vorticity $\omega=\nabla\wedge u$ obeys the equation

$$\partial_t(\nabla\wedge u)+(u\cdot\nabla)(\nabla\wedge u)=0. \tag{32}$$

This evolution equation preserves any L^p-norm ($1\leqslant p\leqslant\infty$) of the vorticity. Taking advantage of this observation in his remarkable paper [18], Yudovich proved the existence, uniqueness, and global regularity for all solutions with initial vorticity in L^∞. If the vorticity is in L^p, with $1<p\leqslant\infty$, then one can prove the existence of weak solutions. The same results hold also for $p=1$ and for vorticity as a finite measure with 'simple' changes of sign. The proof is more delicate in this limit case (cf. Delort [19] and § 7.2 below).

3. Pathological behaviour of solutions

Continuing with the comments of the previous section, we should recall some facts.

I. First, the following hold in the three-dimensional case.

i) There is no result on global (in time) existence of smooth solutions. More precisely, it is not known whether a solution of the Euler system defined on a finite time interval and with initial velocity in, say, H^s with $s>\frac{3}{2}+1$ can be extended as a regular or even a weak solution for all positive time.

ii) There is no result on the existence, even for a small time interval, of a weak solution for initial data less regular than in the above case.

iii) Due to the scaling property of the Euler equations in $\mathbb{R}^3$, the problem of global (in time) existence for small initial data is equivalent to the global existence for all initial data and for all $t\in\mathbb{R}$.

II. Second, in both the $2D$ ($d=2$) and the $3D$ ($d=3$) cases the fact that a function $u \in L^2\big([0,T];L^2(\mathbb{R}^d)\big)$ is a weak solution, that is, satisfies the equations

$$\partial_t u + \nabla\cdot(u\otimes u) + \nabla p = 0, \qquad \nabla\cdot u = 0, \qquad u(x,0) = u_0(x) \tag{33}$$

in the distribution sense, is not sufficient to determine it uniquely from the initial data u_0 (except in $2D$ with the additional regularity assumption that $\nabla\wedge u_0 \in L^\infty$). More precisely, in both $2D$ and $3D$ one can construct, following Scheffer [20] and Shnirelman [21], non-trivial solutions $u \in L^2\big(\mathbb{R}_t;L^2(\mathbb{R}^d)\big)$ of (33) that are of compact support in space and time.

The following examples may contribute to the understanding of the underlying difficulties. First, one can exhibit (cf. Constantin [17], Gibbon and Ohkitani [23], and references therein) blowup for smooth solutions, with infinite energy, of the $3D$ Euler equations. Such solutions can be constructed as follows. The solution u is (x_1,x_2)-periodic on the lattice $(\mathbb{R}/L\mathbb{Z})^2$ and is defined for all $x_3\in\mathbb{R}$ according to the formula

$$u = \big(u_1(x_1,x_2,t), u_2(x_1,x_2,t), x_3\gamma(x_1,x_2,t)\big) = (\tilde u, x_3\gamma),$$

where $\tilde u$ and γ are determined by the following equations.

To maintain the divergence-free condition, it is required that

$$\nabla\cdot\tilde u + \gamma = 0,$$

and to enforce the Euler dynamics, it is required that

$$\partial_t(\nabla\wedge\tilde u) + (\tilde u\cdot\nabla)(\nabla\wedge\tilde u) = \gamma\tilde u,$$
$$\partial_t\gamma + (\tilde u\cdot\nabla)\gamma = -\gamma^2 + I(t),$$

and finally to enforce the (x_1,x_2)-periodicity it is required that

$$I(t) = -\frac{2}{L^2}\int_{[0,L]^2}\big(\gamma(x_1,x_2,t)\big)^2\,dx_1\,dx_2.$$

Therefore, the scalar function γ satisfies an integro-differential Riccati equation of the form

$$\partial_t\gamma + \tilde u\nabla\gamma = -\gamma^2 - \frac{2}{L^2}\int_{[0,L]^2}\big(\gamma(x_1,x_2,t)\big)^2\,dx_1\,dx_2,$$

from which the proof of the blowup including the explicit nature of this blowup, follows.

The above example can be regarded as non-physical, because the initial energy

$$\int_{(\mathbb{R}^2/L)^2\times\mathbb{R}}|u(x_1,x_2,x_3,0)|^2\,dx_1\,dx_2\,dx_3$$

is infinite. On the other hand, it is instructive because it shows that the conservation of energy in the Euler equations may play a crucial role in the absence of singularities. Furthermore, an approximation of the above solution by a family of finite-energy solutions would probably be possible, but to the best of our knowledge this has not yet been done. Such an approximation procedure might lead to the idea that no uniform bound can be obtained for the stability or regularity of the $3D$ Euler equations. Along these lines, one has the following proposition.

Proposition 3.1. *For $1 < p < \infty$ there is no continuous function $\tau \mapsto \phi(\tau)$ such that any smooth solution of the Euler equations satisfies the estimate*

$$\big\|u(\cdot,t)\big\|_{W^{1,p}(\Omega)} \leqslant \phi\big(\big\|u(\cdot,0)\big\|_{W^{1,p}(\Omega)}\big).$$

Observe that the above statement is not in contradiction to the local stability results which produce local control of the higher norms at time t in terms of these norms at time 0 in the form of the inequality

$$\big\|u(t)\big\|_{H^s(\Omega)} \leqslant \frac{\|u(0)\|_{H^s(\Omega)}}{1 - Ct\|u(0)\|_{H^s(\Omega)}} \qquad \text{for } s > \frac{5}{2},$$

which follows from (17).

Proof. The proof is by inspection of a pressureless solution which is defined on a period box $(\mathbb{R}/\mathbb{Z})^3$ according to the formula

$$u(x,t) = \big(u_1(x_2), 0, u_3\big(x_1 - tu_1(x_2), x_2\big)\big),$$

and which satisfies the equation

$$\nabla \cdot u = 0, \qquad \partial_t u + u \cdot \nabla u = 0.$$

Therefore, the initial data satisfies the relation

$$\begin{aligned}\big\|u(\cdot,0)\big\|^p_{W^{1,p}(\Omega)} \simeq &\int_0^1 |\partial_{x_2} u_1(x_2)|^p \, dx_1 \\ &+ \int_0^1\int_0^1 \big(|\partial_{x_1} u_3(x_1,x_2)|^p + |\partial_{x_2} u_3(x_1,x_2)|^p\big)\, dx_1\, dx_2. \end{aligned} \tag{34}$$

And for $t > 0$,

$$\begin{aligned}\big\|u(\cdot,t)\big\|^p_{W^{1,p}(\Omega)} \simeq &\int |\partial_{x_2} u_1(x_2)|^p \, dx_1\, dx_2\, dx_3 \\ &+ \int_0^1\int_0^1 \big(|\partial_{x_1} u_3(x_1,x_2)|^p + |\partial_{x_2} u_3(x_1,x_2)|^p\big)\, dx_1\, dx_2 \\ &+ t^p \int_0^1\int_0^1 |\partial_{x_2} u_1(x_2)|^p |\partial_{x_1} u_3(x_1,x_2)|^p \, dx_1\, dx_2. \end{aligned} \tag{35}$$

Then a suitable choice of u_1 and u_3 makes the left-hand side of (34) bounded, while the term

$$t^p \int_0^1\int_0^1 |\partial_{x_2} u_1(x_2)|^p |\partial_{x_1} u_3(x_1,x_2)|^p \, dx_1\, dx_2$$

on the right-hand side of (35) may be infinite. The proof is then completed by a regularization argument.

Remark 3.1. With smooth initial data, the above construction gives an example of a global (in time) smooth solution with vorticity growing (here only linearly) as $t \to \infty$.

As in the case of the Riccati differential inequality $y' \leqslant Cy^2$, one can obtain sufficient conditions for the existence of a smooth solution during a finite interval of time (say $0 \leqslant t < T$). On the other hand, this gives no indication of the possible appearance of blowup after such time. Complicated phenomena that take place in the fluid due to strong non-linearities may later interact in such a way that they balance each other and bring the fluid back to a smooth regime. Such a phenomenon is called *singularity depletion*.

An example which seems to illustrate such cancellation has been constructed by Hou and Li [24]. It involves axisymmetric solutions of the $3D$ Euler equations of the form $rf(z)$, which obviously possess infinite energy.

Specifically, let us start with the following system of integro-differential equations with solutions that are defined for $(z,t) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^+$:

$$u_t + 2\psi u_z = -2vu, \qquad v_t + 2\psi v_z = u^2 - v^2 + c(t), \tag{36}$$

$$\psi_z = v, \qquad \int_0^1 v(z,t)\,dz = 0. \tag{37}$$

In (36) the z-independent function $c(t)$ is chosen to enforce the second relation of (37), which in turn makes the function $\psi(z,t)$ 1-periodic in the z direction. As a result one has the following.

Lemma 3.1. *For any initial data* $\big(u(z,0), v(z,0)\big) \in C^m(\mathbb{R}/\mathbb{Z})$ *with* $m \geqslant 1$, *the system* (36), (37) *has a unique global* (*in time*) *smooth solution.*

Proof. This proof relies on a global *a priori* estimate. Taking the derivative with respect to z gives (using the notation $(u_z, v_z) = (u', v')$)

$$\begin{aligned} u'_t + 2\psi u'_z - 2\psi_z u' &= -2v'u - 2vu', \\ v'_t + 2\psi v'_z - 2\psi_z v' &= 2uu' - 2vv'. \end{aligned}$$

Next, one uses the relation $\psi_z = -v$, multiplies the first equation by u and the second by v, and adds them to obtain

$$\frac{1}{2}(u_z^2 + v_z^2)_t + \psi(u_z^2 + v_z^2)_z = 0. \tag{38}$$

The relation (38) provides a uniform L^∞ bound on the z-derivatives of u and v. A uniform L^∞ bound for v follows from the Poincaré inequality, and finally one uses for u the Gronwall inequality

$$\big\|u(z,t)\big\|_{L^\infty} \leqslant \big\|u(z,0)\big\|_{L^\infty} e^{t\|(u(z,0))^2 + (v(z,0))^2\|_{L^\infty}}. \tag{39}$$

Remark 3.2. The global existence of solutions of the system (36), (37) with no restriction on the size of the initial data is the result of a delicate balance/cancelation which depends on the coefficients of the system. Any modification of these coefficients may lead to a finite-time blowup of the solutions of the modified system. On the other hand, the solutions of the system (36), (37) can grow exponentially in time. Numerical simulations in [24] indicate that the exponential growth rate in (39) may get saturated.

The special structure of the system (36), (37) is related to *the* $3D$ *axisymmetric Euler equations with swirl* as follows. We introduce the orthogonal basis

$$e_r = \left(\frac{x}{r}, \frac{y}{r}, 0\right), \qquad e_\theta = \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \qquad e_z = (0, 0, 1),$$

and for a solution of the system (36), (37) we construct solutions of the $3D$ $(2+1/2)$ Euler equation according to the following proposition.

Proposition 3.2. *If* $u(z,t)$ *and* $\psi(z,t)$ *are solutions of the system* (36), (37), *then the function*

$$U(z,t) = -r\frac{\partial\psi(z,t)}{\partial z}e_r + ru(z,t)e_\theta + 2r\psi(z,t)e_z$$

is a smooth solution of the $3D$ *Euler system, but with infinite energy. Moreover, this solution is defined for all time and without any smallness assumption on the size of the initial data.*

4. Weak limit of solutions of the Navier–Stokes dynamics

As we have already remarked in the Introduction, for practical problems as well as for mathematical analysis one can regard Euler dynamics as the limit of Navier–Stokes dynamics as the viscosity tends to zero. Therefore, this section is devoted to the study of the *weak limit* as $\nu \to 0$ of Leray–Hopf type solutions of the Navier–Stokes equations in $2D$ and $3D$. We will consider only convergence on finite intervals of time $0 < t < T < \infty$. Let us recall once again that ν denotes the dimensionless viscosity, that is, the inverse of the Reynolds number.

4.1. Reynolds stress tensor. As above, we denote by Ω an open set in $\mathbb{R}^d$. For any initial data $u_\nu(x,0) = u_0(x) \in L^2(\Omega)$ and any given viscosity $\nu > 0$, the pioneering works of Leray [25]–[27] and Hopf [28]–[30] (see also the detailed survey of Ladyzhenskaya [31] and the later generalizations by Scheffer [20] and by Caffarelli, Kohn, and Nirenberg [32]) showed the existence of functions u_ν and p_ν with the property

$$u_\nu \in L^\infty\big((0,T); L^2(\Omega)\big) \cap L^2\big((0,T); H_0^1(\Omega)\big) \quad \text{for every } T \in (0,\infty) \tag{40}$$

that satisfy the Navier–Stokes equations

$$\partial_t u_\nu + \nabla\cdot(u_\nu \otimes u_\nu) - \nu\Delta u_\nu + \nabla p_\nu = 0, \tag{41}$$

$$\nabla\cdot u_\nu = 0, \qquad u_\nu = 0 \quad \text{on} \quad \partial\Omega \tag{42}$$

in the sense of distributions. Moreover, such solutions satisfy the 'pointwise' energy inequality

$$\frac{1}{2}\partial_t|u_\nu(x,t)|^2 + \nu|\nabla u_\nu(x,t)|^2 + \nabla\cdot\left((u_\nu\otimes u_\nu)(x,t) - \nu\nabla\frac{|u_\nu(x,t)|^2}{2}\right) + \nabla\cdot\big(p_\nu(x,t)u_\nu(x,t)\big) \leqslant 0 \tag{43}$$

or, in integrated form,

$$\frac{1}{2}\partial_t \int_\Omega |u_\nu(x,t)|^2\,dx + \nu\int_\Omega |\nabla u_\nu(x,t)|^2\,dx \leqslant 0. \tag{44}$$

A pair $\{u_\nu, p_\nu\}$ which satisfies (40), (42), (43) is called a suitable weak solution of the Navier–Stokes equations in the Caffarelli–Kohn–Nirenberg sense. If, however, it satisfies the integrated version (44) of the energy inequality instead of the pointwise energy inequality (43), then it will be called a Leray–Hopf weak solution of the Navier–Stokes equations.

In dimension two (or in an arbitrary dimension but with stronger smallness hypothesis on the size of the initial data with respect to the viscosity), these solutions are shown to be smooth, unique, and continuously dependent on the initial data. Furthermore, in this case equality holds in (43) and (44).

Therefore, as a result of the above, and in particular the energy inequality (44), one concludes that, modulo the extraction of a subsequence, the sequence $\{u_\nu\}$ converges in the weak-$*$ topology of $L^\infty\big(\mathbb{R}_t^+, L^2(\Omega)\big)$ to a limit $\bar u$, the sequence $\{\nabla p_\nu\}$ converges to a distribution $\nabla \bar p$ as $\nu \to 0$, and the following hold:

$$\bar u \in L^\infty\big(\mathbb{R}_t^+, L^2(\Omega)\big), \quad \nabla\cdot\bar u = 0 \quad \text{in} \quad \Omega, \quad \bar u\cdot\vec n = 0 \quad \text{on} \quad \partial\Omega,$$

$$\int_\Omega |\bar u(x,t)|^2\,dx + 2\nu\int_0^t\int_\Omega |\nabla u|^2\,dx\,dt \leqslant \int_\Omega |\bar u_0(x)|^2\,dx,$$

$$\lim_{\nu\to 0}(u_\nu\otimes u_\nu) = \bar u\otimes\bar u + \lim_{\nu\to 0}\big((u_\nu-\bar u)\otimes(u_\nu-\bar u)\big), \tag{45}$$

$$\partial_t\bar u + \nabla\cdot(\bar u\otimes\bar u) + \lim_{\nu\to 0}\nabla\cdot\big((u_\nu-\bar u)\otimes(u_\nu-\bar u)\big) + \nabla\bar p = 0. \tag{46}$$

Observe that the term

$$\mathrm{RT}(x,t) = \lim_{\nu\to 0}\big(u_\nu(x,t)-\bar u(x,t)\big)\otimes\big(u_\nu(x,t)-\bar u(x,t)\big) \tag{47}$$

is a positive, symmetric, measure-valued tensor. In analogy with (see below) the statistical theory of turbulence, this tensor may be called the Reynolds stress tensor or turbulence tensor. In particular, certain turbulent regions will correspond to the support of this tensor.

This approach leads to the following questions.

1. What are the basic properties (if any) of the tensor $\mathrm{RT}(x,t)$?

2. When is the tensor $\mathrm{RT}(x,t)$ identically equal to zero? Or, what is equivalent, when does the limit pair $\{\bar u, \bar p\}$ satisfies the Euler equations?

3. When does the energy dissipation

$$\nu\int_0^T\int_\Omega |\nabla u_\nu(x,t)|^2\,dx\,dt$$

tend to zero as $\nu \to 0$?

4. Assuming that the pair $\{\bar u, \bar p\}$ is a solution of the Euler equations, is such a solution sufficiently regular to imply the conservation of energy?

Hereafter, we will use the following notation for the L^2-norm:

$$|\Phi| = \left(\int_\Omega |\Phi(x)|^2\,dx\right)^{1/2}.$$

Remark 4.1. The tensor $\mathrm{RT}(x,t)$ is generated by the high frequency spatial oscillations of the weak solution. This feature will be explained in more detail in § 6.1. Therefore, such behaviour should be intrinsic and, in particular, independent of orthogonal changes of coordinates (rotations). For instance, in the $2D$ case if the function $\bar{u}$ is regular, then invariance under rotation implies the relation

$$\mathrm{RT}(x,t) = \alpha(x,t)\,\mathrm{Id} + \frac{1}{2}\beta(x,t)\big(\nabla\bar{u} + (\nabla\bar{u})^T\big),$$

where $\alpha(x,t)$ and $\beta(x,t)$ are some (unknown) scalar-valued functions. Thus, the equation (46) becomes

$$\partial_t\bar{u} + \nabla\cdot(\bar{u}\otimes\bar{u}) + \nabla\cdot\left(\beta(x,t)\frac{1}{2}\big(\nabla\bar{u} + (\nabla\bar{u})^T\big)\right) + \nabla\big(\bar{p} + \alpha(x,t)\big) = 0. \tag{48}$$

Of course, this 'soft information' does not indicate whether $\beta(x,t)$ is zero or not. It also does not indicate whether this coefficient is positive, nor how to compute it. But this turns out to be the turbulent eddy diffusion coefficient that is present in classical engineering turbulence models like the Smagorinsky model or the $k\varepsilon$-models (see, for example, [33]–[36]).

Remark 4.2. Assume that the limit $\{\bar{u},\bar{p}\}$ is a solution of the Euler equations which is sufficiently regular to ensure conservation of energy, that is, $|\bar{u}(t)|^2 = |u_0|^2$. Then by virtue of the energy relation (44) we have

$$\frac{1}{2}|u_\nu(t)|^2 + \nu\int_0^t |\nabla u_\nu(s)|^2\,ds \leqslant \frac{1}{2}|u_0|^2,$$

and by the weak limit relation

$$\liminf_{\nu\to 0}\frac{1}{2}|u_\nu(t)|^2 \geqslant \frac{1}{2}|\bar{u}(t)|^2$$

we get strong convergence and the equality

$$\liminf_{\nu\to 0}\nu\int_0^t |\nabla u_\nu(s)|^2\,ds = 0.$$

The following question was then raised by Onsager in [37]: "What is the minimal regularity needed to be satisfied by the solutions of the $2D$ or $3D$ Euler equations that would imply conservation of energy?" The question was pursued by several authors up to the contribution of Eyink [38] and of Constantin, E, and Titi [39]. Basically, in $3D$ it was shown that if u is bounded in $L^\infty\big(\mathbb{R}_t^+, H^\beta(\Omega)\big)$ with $\beta > 1/3$, then the energy

$$\frac{1}{2}\int_\Omega |u(x,t)|^2\,dx$$

is independent of the time. On the other hand, arguments borrowed from the statistical theory of turbulence (cf. § 6.2) show that the sequence u_ν will in general be bounded in $L^\infty\big(\mathbb{R}_t^+, H^{\frac{1}{3}}(\Omega)\big)$, and one should observe that such a statement does not contradict the possibility of decay of the energy in the limit as $\nu\to 0$.

4.2. Dissipative solutions of the Euler equations. To study the weak limit of Leray–Hopf solutions of the Navier–Stokes dynamics as $\nu \to 0$, DiPerna and Lions (see [40], p. 153) introduced the notion of *dissipative solution* of the Euler equations. To motivate this notion, let $w(x,t)$ be a divergence-free test function which satisfies $w\cdot\vec{n}=0$ on the boundary $\partial\Omega$ and let

$$E(w)=\partial_t w+P(w\cdot\nabla w), \tag{49}$$

where P is the Leray–Helmholtz projector (see, for example, [41]). Then for any smooth divergence-free solution $u(x,t)$ of the Euler equations in Ω with the boundary condition $u\cdot\vec{n}=0$ on $\partial\Omega$, one has

$$\begin{gathered}\partial_t u+\nabla\cdot(u\otimes u)+\nabla p=0,\\ \partial_t w+\nabla\cdot(w\otimes w)+\nabla q=E(w),\\ \frac{d|u-w|^2}{dt}+2\big(S(w)(u-w),(u-w)\big)=2\big(E(w),u-w\big),\end{gathered}$$

where $S(w)$ denotes, as before, the symmetric tensor

$$S(w)=\frac{1}{2}\big(\nabla w+(\nabla w)^T\big).$$

By integration with respect to time this gives

$$\begin{aligned}|u(t)-w(t)|^2 &\leqslant e^{\int_0^t 2\|S(w)(s)\|_\infty\,ds}|u(0)-w(0)|^2\\ &\quad+2\int_0^t e^{\int_s^t 2\|S(w)(\tau)\|_\infty\,d\tau}\big(E(w)(s),(u-w)(s)\big)\,ds.\end{aligned} \tag{50}$$

The above observation leads to the following definition.

Definition 4.1. A divergence-free vector field

$$u\in C_w\big(\mathbb{R}_t;\big(L^2(\Omega)\big)^d\big)$$

which satisfies the boundary condition $u\cdot\vec{n}=0$ on $\partial\Omega$ is called a *dissipative solution* of the Euler equations (11) if (50) holds for any smooth divergence-free vector field w with $w\cdot\vec{n}=0$ on $\partial\Omega$.

The following statement is easy to verify, but we mention it here for the sake of clarity.

Theorem 4.1. i) *Any classical solution u of the Euler equations* (11) *is a dissipative solution.*

ii) *Every dissipative solution satisfies the energy inequality*

$$|u(t)|^2\leqslant|u(0)|^2. \tag{51}$$

iii) *The dissipative solutions are 'stable with respect to classical solutions'. More precisely, if w is a classical solution and u is a dissipative solution of the Euler equations, then*

$$|u(t)-w(t)|^2\leqslant e^{\int_0^t 2\|S(w)(s)\|_\infty\,ds}|u(0)-w(0)|^2.$$

In particular, if there exists a classical solution for specific initial data, then any dissipative solution with the same initial data coincides with it.

iv) *In the absence of physical boundaries, that is, in the case of periodic boundary conditions or in the whole space $\mathbb{R}^d$, $d = 2, 3$, any weak limit as $\nu \to 0$ of Leray–Hopf solutions of the Navier–Stokes equations is a dissipative solution of the Euler equations.*

Proof and remarks. The point i) is a direct consequence of the construction. To prove ii) we regard $w \equiv 0$ as a classical solution. As a result, one obtains for any dissipative solution the inequality (51), which justifies the term 'dissipative'. Furthermore, it shows that the pathological examples constructed by Scheffer [20] and Shnirelman [21] are not dissipative solutions of the Euler equations.

For the point iii) we use in (50) the fact that w, being a classical solution, must satisfy $E(w) \equiv 0$. We also observe that this statement is in the spirit of the 'weak with respect to strong' stability result of Dafermos (cf. [72; p. 66, Theorem 5.2.1]) for hyperbolic systems.

Next, we prove iv) in the absence of physical boundaries. Let u_ν be a Leray–Hopf solution of the Navier–Stokes system which satisfies the energy inequality (44), and let w be a classical solution of the Euler equations. By subtracting the two equations

$$\begin{aligned}
\partial_t u_\nu + \nabla \cdot (u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p_\nu = 0,\\
\partial_t w + \nabla \cdot (w \otimes w) - \nu \Delta w + \nabla p = -\nu \Delta w
\end{aligned}$$

one from the other and taking the L^2 inner product of the difference with $(u_\nu - w)$, one obtains

$$\begin{aligned}
\frac{d|u_\nu - w|^2}{dt} &+ 2\big(S(w)(u_\nu - w), (u_\nu - w)\big) - 2\nu\big(\Delta(u_\nu - w), (u_\nu - w)\big)\\
&\leqslant 2\big(E(w), u_\nu - w\big) - \big(\nu \Delta w, (u_\nu - w)\big). \qquad (52)
\end{aligned}$$

We stress that the above step is formal, and only through rigorous arguments can one see the reason for obtaining the inequality in (52) instead of equality. However, this should not be a surprise, because we are dealing with Leray–Hopf solutions u_ν of the Navier–Stokes system which satisfy the energy inequality (44) instead of equality.

To conclude our proof we now observe that in the absence of physical boundaries we can use the equality

$$-\nu \int \Delta(w - u_\nu)(x,t) \cdot (w - u_\nu)(x,t)\, dx = \nu \int |\nabla(w - u_\nu)(x,t)|^2\, dx, \qquad (53)$$

and the result follows by letting ν tend to zero.

Remark 4.3. The above theorem states, in particular, that, in the absence of physical boundaries, as long as a smooth solution of the Euler equations does exist, it is the limit as $\nu \to 0$ of any sequence of Leray–Hopf solutions of the Navier–Stokes equations with the same initial data. In a series of papers starting with Bardos, Golse, and Levermore [43], connections were established between the notion of Leray–Hopf solutions for the Navier–Stokes equations and *renormalized solutions* *of the Boltzmann equations*, as defined by DiPerna and Lions [44]. In particular, it was ultimately shown by Golse and Saint-Raymond [45] that, modulo the extraction of a subsequence and for a suitable choice of space-time scaling, any sequence of such renormalized solutions of the Boltzmann equations converges (in some weak sense) to a Leray–Hopf solution of the Navier–Stokes system. On the other hand, it was shown by Saint-Raymond [46] that, under a scaling which reinforces the non-linear effect (corresponding at the macroscopic level to the Reynolds number going to infinity), any sequence (modulo extraction of a subsequence) of renormalized solutions of the Boltzmann equations converges to a dissipative solution of the Euler equations. Therefore, such a sequence of normalized solutions of the Boltzmann equations converges to the classical solution of the Euler equations, as long as such a solution exists. In this situation one should observe that, with the notion of dissipative solutions of the Euler equations, classical solutions of the Euler equations play a role similar to the 'Leray–Hopf limit' and the 'Boltzmann limit'.

Remark 4.4. There are at least two situations where the notion of dissipative solution of the Euler equations is not helpful.

The first situation is concerned with the $2D$ Euler equations. Let $u_\varepsilon(x,t)$ be the sequence of solutions of the $2D$ Euler equations corresponding to a sequence of smooth initial data $u_\varepsilon(x,0)$. Suppose that the sequence of initial data $u_\varepsilon(x,0)$ converges weakly but not strongly in $L^2(\Omega)$ to initial data $\bar u(x,0)$ as $\varepsilon \to 0$. Then thanks to (50), for any smooth divergence-free vector field w one has the inequality

$$\begin{aligned} |u_\varepsilon(t)-w(t)|^2 &\leqslant e^{\int_0^t 2\|S(w)(s)\|_\infty\,ds}|u_\varepsilon(0)-w(0)|^2 \\ &\quad + 2\int_0^t e^{\int_s^t 2\|S(w)(\tau)\|_\infty\,d\tau}\big(E(w),u_\varepsilon-w\big)(s)\,ds. \end{aligned} \tag{54}$$

However, with weak convergence as $\varepsilon \to 0$ one has only

$$|\bar u(0)-w(0)|^2 \leqslant \liminf_{\varepsilon\to 0}|u_\varepsilon(0)-w(0)|^2,$$

and (54) might not hold at the limit as $\varepsilon \to 0$. To illustrate this situation, we consider a sequence of oscillating solutions of the $2D$ Euler equations of the form

$$u_\varepsilon(x,t) = U\left(x,t,\frac{\phi(x,t)}{\varepsilon}\right) + O(\varepsilon),$$

where the map $\theta \mapsto U(x,t,\theta)$ is a non-trivial 1-periodic function. In Cheverry [47] a specific example was constructed such that

$$\bar u = w - \lim_{\varepsilon\to 0} u_\varepsilon = \int_0^1 U(x,t,\theta)\,d\theta$$

is no longer a solution of the Euler equations. The obvious reason for this (in comparison to the notion of dissipative solution) is the fact that

$$U\left(x,0,\frac{\phi(x,0)}{\varepsilon}\right)$$

does not converge strongly in $L^2(\Omega)$.

The second situation, which will be discussed at length below, corresponds to a weak limit of solutions of the Navier–Stokes equations in a domain with physical boundary, subject to the no-slip Dirichlet boundary condition.

As we have already indicated in Theorem 4.1, one of the most important features of the above definition of dissipative solution of the Euler equations is that it coincides with the classical solution of the Euler equations when the latter exists. This can be seen by replacing w in (50) with this classical solution of the Euler equations. Therefore, any procedure for approximating dissipative solutions of the Euler equations must in the limit lead to the inequality (50). Indeed, in the absence of physical boundaries we showed almost at once in Theorem 4.1 that the Leray–Hopf weak solutions of the Navier–Stokes equations converge to dissipative solutions of the Euler equations. On the other hand, in the presence of physical boundaries the proof does not go as smoothly, because of boundary effects. Specifically, in the case of domains with physical boundaries, the inequality (52) leads to the inequality

$$\begin{aligned}\frac{1}{2}\frac{d|u_\nu - w|^2}{dt} + \big(S(w)(u_\nu - w), (u_\nu - w)\big) + \nu \int |\nabla(w - u_\nu)|^2\,dx \\ \leqslant \big(E(w), u_\nu - w\big) - \nu\big(\Delta w, (u_\nu - w)\big) + \nu \int_{\partial\Omega} \partial_n u_\nu \cdot w\,d\sigma. \qquad (55)\end{aligned}$$

The very last term in (55) represents the boundary effect. We will discuss below the subtleties in handling this term.

5. No-slip Dirichlet boundary conditions for the Navier–Stokes dynamics

This section is devoted to the very few available results concerning the limit as $\nu \to 0$ of solutions of the Navier–Stokes equations in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with the homogeneous (no-slip) Dirichlet boundary condition $u_\nu = 0$ on $\partial\Omega$. This boundary condition is not the easiest to deal with, as far as the zero viscosity limit is concerned. For instance, the solutions of the of $2D$ Navier–Stokes equations, subject to the boundary conditions $u_\nu \cdot n = 0$ and $\nabla \wedge u_\nu = 0$, are much better understood and much easier to analyze mathematically (see, for example, [48]) as $\nu \to 0$. However, the no-slip boundary condition corresponds better to the physical picture for the following reasons.

i) It can be deduced in the smooth (laminar) regime from the Boltzmann kinetic equations when the interaction with the boundary is described by a scattering kernel.

ii) It generates the pathology that is observed in physical experiments, like the von Kármán vortex streets. Moreover, one should keep in mind that almost all high Reynolds number turbulence experiments involve a physical boundary (very often turbulence is generated by a pressure driven flow through a grid).

The problem emerges first from the boundary layer. This is because for the Navier–Stokes dynamics, the whole velocity field equals zero on the boundary, that is, $u_\nu = 0$ on $\partial\Omega$, while for the Euler dynamics only the normal component of velocity field is equal to zero on the boundary, that is, $u \cdot \vec{n} = 0$ on $\partial\Omega$. Therefore, in the limit as $\nu \to 0$ the tangential component u_ν of the velocity field of the

Navier–Stokes dynamics generates a boundary layer by its 'jump'. Then, unlike the situation with linear singular perturbation problems, the non-linear advection term of the Navier–Stokes equations may propagate this instability inside the domain.

As we have already pointed out, the very last term in (55), that is, the boundary integral term in the case of the no-slip boundary condition,

$$\nu \int_{\partial\Omega} \frac{\partial u_\nu}{\partial n} \cdot w \, d\sigma = \nu \int_{\partial\Omega} \left(\frac{\partial u_\nu}{\partial n}\right)^\tau \cdot w^\tau \, d\sigma = \nu \int_{\partial\Omega} (\nabla \wedge u_\nu) \cdot (\vec{n} \wedge w) \, d\sigma, \tag{56}$$

is possibly responsible for the loss of regularity in the limit as $\nu \to 0$. This is stated more precisely in the following.

Proposition 5.1. *Let $u(x,t)$ be a solution of the incompressible Euler equations in $\Omega \times (0,T]$, with the following regularity assumptions*:

$$S(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big) \in L^1\big((0,T); L^\infty(\Omega)\big),$$
$$u \in L^2\big((0,T); H^s(\Omega)\big) \quad \textit{for} \quad s > \frac{1}{2}\,.$$

Further, suppose that the sequence u_ν of Leray–Hopf solutions of the Navier–Stokes dynamics (*with no-slip boundary condition*) *with the initial data $u_\nu(x,0) = u(x,0)$, satisfies the relation*

$$\lim_{\nu\to 0} \nu \big\| P_{\partial\Omega}(\nabla \wedge u_\nu) \big\|_{L^2((0,T); H^{-s+\frac{1}{2}}(\partial\Omega))} = 0,$$

where $P_{\partial\Omega}$ denotes the projection on the tangent plane to $\partial\Omega$ according to the formula

$$P_{\partial\Omega}(\nabla \wedge u_\nu) = \nabla \wedge u_\nu - \big((\nabla \wedge u_\nu)\cdot \vec{n}\big)\vec{n}.$$

Then the sequence u_ν converges to u in $C\big((0,T); L^2(\Omega)\big)$.

The proof is a direct consequence of (55) and (56) with w replaced by u. This proposition can be improved with the following simple and beautiful theorem of Kato which takes into account the vorticity production in the boundary layer $\big\{x \in \Omega \mid d(x,\partial\Omega) < \nu\big\}$, where $d(x,y)$ denotes the Euclidean distance between the points x and y.

Theorem 5.1. *Let $u(x,t) \in W^{1,\infty}\big((0,T)\times\Omega\big)$ be a solution of the Euler dynamics, and let u_ν be a sequence of Leray–Hopf solutions of the Navier–Stokes dynamics with no-slip boundary condition*

$$\partial_t u_\nu - \nu\Delta u_\nu + \nabla\cdot(u_\nu \otimes u_\nu) + \nabla p_\nu = 0, \qquad u_\nu(x,t) = 0 \quad \textit{on} \quad \partial\Omega \tag{57}$$

with initial data $u_\nu(x,0) = u(x,0)$. *Then the following are equivalent*:

$$\text{(i)} \quad \lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} (\nabla \wedge u_\nu)\cdot(\vec{n}\wedge u)\, d\sigma\, dt = 0, \tag{58}$$

$$\text{(ii)} \quad u_\nu(t) \to u(t) \quad \textit{in } L^2(\Omega) \textit{ uniformly with respect to } t\in[0,T], \tag{59}$$

$$\text{(iii)} \quad u_\nu(t) \to u(t) \quad \textit{weakly in } L^2(\Omega) \textit{ for each } t\in[0,T], \tag{60}$$

$$\text{(iv)} \quad \lim_{\nu\to 0} \nu \int_0^T \int_{\Omega} |\nabla u_\nu(x,t)|^2\, dx\, dt = 0, \tag{61}$$

$$\text{(v)} \quad \lim_{\nu\to 0} \nu \int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2\, dx\, dt = 0. \tag{62}$$

Sketch of the proof. The statement (59) is deduced from (58) by replacing w by u in (55) and (56). No proof is needed to deduce (60) from (59) and (62) from (61).

Next, we recall the energy inequality (44) satisfied by the Leray–Hopf solutions of the Navier–Stokes dynamics:

$$\frac{1}{2}\int_\Omega |u_\nu(x,T)|^2\, dx + \nu\int_0^T\int_\Omega |\nabla u_\nu(x,t)|^2\, dx\, dt \leqslant \frac{1}{2}\int_\Omega |u(x,0)|^2\, dx. \tag{63}$$

By the weak convergence stated in (60) and the fact that u is a smooth solution of the Euler dynamics,

$$\begin{aligned}\lim_{\nu\to 0}\frac{1}{2}\int_\Omega |u_\nu(x,T)|^2\, dx &\geqslant \frac{1}{2}\int_\Omega \Big|\lim_{\nu\to 0} u_\nu(x,T)\Big|^2\, dx \\ &= \frac{1}{2}\int_\Omega |u(x,T)|^2\, dx = \frac{1}{2}\int_\Omega |u(x,0)|^2\, dx.\end{aligned} \tag{64}$$

Together with (63) this shows that (60) implies (61).

The most subtle part in the proof of this theorem is the fact that (62) implies (58). The first step is the construction of a divergence-free function $v_\nu(x,t)$ with support in the region $\{x\in\Omega \mid d(x,\partial\Omega)\leqslant\nu\}\times[0,T)$, coinciding with u on $\partial\Omega\times[0,T]$, and satisfying the estimates (with K a constant independent of ν)

$$\|v_\nu\|_{L^\infty(\Omega\times(0,T))} + \big\|d(x,\partial\Omega)\nabla v_\nu\big\|_{L^\infty(\Omega\times(0,T))} \leqslant K, \tag{65}$$

$$\big\|\big(d(x,\partial\Omega)\big)^2\nabla v_\nu\big\|_{L^\infty(\Omega\times(0,T))} \leqslant K\nu, \tag{66}$$

$$\|v_\nu\|_{L^\infty((0,T);L^2(\Omega))} + \|\partial_t v_\nu\|_{L^\infty((0,T);L^2(\Omega))} \leqslant K\nu^{\frac{1}{2}}, \tag{67}$$

$$\|\nabla v_\nu\|_{L^\infty((0,T);L^2(\Omega))} \leqslant K\nu^{-\frac{1}{2}}, \tag{68}$$

$$\|\nabla v_\nu\|_{L^\infty(\Omega\times(0,T))} \leqslant K\nu^{-1}. \tag{69}$$

Then we multiply the Navier–Stokes equations by v_ν and integrate to obtain

$$
\begin{aligned}
-\nu\int_0^T\int_{\partial\Omega}(\nabla\wedge u_\nu)\cdot(\vec n\wedge u)\,d\sigma\,dt &= -\nu\int_0^T\int_{\partial\Omega}\frac{\partial u_\nu}{\partial n}\cdot v_\nu\,d\sigma\,dt\\
&= -\nu\int_0^T\int_\Omega\Delta u_\nu\cdot v_\nu\,dx\,dt-\nu\int_0^T\int_\Omega(\nabla u_\nu:\nabla v_\nu)\,dx\,dt\\
&= -\int_0^T(\partial_t u_\nu,v_\nu)\,dt-\int_0^T\big(\nabla\cdot(u_\nu\otimes u_\nu),v_\nu\big)\,dt-\nu\int_0^T(\nabla u_\nu,\nabla v_\nu)\,dt. \quad (70)
\end{aligned}
$$

By using the above estimates and (62), one can show finally that

$$
\lim_{\nu\to 0}\left(\int_0^T\big((\partial_t u_\nu,v_\nu)+\big(\nabla\cdot(u_\nu\otimes u_\nu),v_\nu\big)+\nu(\nabla u_\nu,\nabla v_\nu)\big)\,dt\right)=0, \quad (71)
$$

which completes the proof.

Remark 5.1. In [49] Constantin and Wu study the rate of convergence of solutions of the $2D$ Navier–Stokes equations to solutions of the Euler equations in the absence of physical boundaries and for finite intervals of time. Their main observation is that while the rate of convergence in the L^2-norm for smooth initial data is of order $\mathscr{O}(\nu)$, it is of order $\mathscr{O}(\sqrt{\nu})$ for less smooth initial data. For instance, the order of convergence $\mathscr{O}(\sqrt{\nu})$ is attained for vortex patch data with a smooth boundary. In this case the fluid develops an internal boundary layer which is responsible for this reduction in the order of convergence.

In the $2D$ case and for initial data of finite $W^{1,p}$-norm with $p>1$ (and also for initial data with the vorticity a finite measure with a 'simple' change of sign [19], [50]), one can prove the existence of 'weak solutions of the Euler dynamics'. In the absence of physical boundaries, such solutions are limit points of a family of (uniquely determined) Leray–Hopf solutions of the Navier–Stokes dynamics. However, these weak solutions of the Euler equations are not uniquely determined, and the issue of the conservation of energy for them is, to the best of our knowledge, completely open.

On the other hand, in a domain with physical boundary and with smooth initial data, Theorem 5.1 shows a clear-cut difference between the following two situations (the same remark is valid locally in time for the $3D$ problem).

i) The mean rate of dissipation of energy

$$
\varepsilon=\frac{\nu}{T}\int_0^T\int|\nabla u_\nu(x,t)|^2\,dx\,dt
$$

goes to zero as $\nu\to 0$, and the sequence u_ν of Leray–Hopf solutions converges strongly to a regular solution $\bar u$ of the Euler dynamics.

ii) The mean rate of dissipation of energy does not go to zero as $\nu\to 0$ (modulo the extraction of a subsequence), so the corresponding weak limit u of the sequence u_ν does not conserve energy, that is,

$$
\frac{1}{2}|u(t)|^2<\frac{1}{2}|u(0)|^2 \quad \text{for some } t\in(0,T),
$$

and one of the following two scenarios may occur in the limit.

a) In the limit one obtains a weak solution (not a strong solution) of the Euler dynamics that exhibits energy decay. Such a scenario is compatible with a uniform estimate for the Fourier spectrum

$$E_\nu(k,t) = |\hat{u}_\nu(k,t)|^2 |k|^{d-1}, \qquad \hat{u}_\nu(k,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ikx} u_\nu(x,t)\,dx,$$

which can satisfy an estimate, uniform in ν, of the type

$$E_\nu(k,t) \leqslant C|k|^{-\beta} \tag{72}$$

with $\beta < 5/3$. Otherwise, this would be in contradiction to results of Onsager [37], Eyink [38], and Constantin, E, and Titi [39].

b) No estimate of the type (72) is true uniformly with respect to the viscosity, and the limit is not even a solution of the Euler dynamics, rather a solution of a modified system of equations with a term connected with turbulence modeling — an 'eddy-viscosity' term.

6. Deterministic and statistical spectrum of turbulence

6.1. Deterministic spectrum and Wigner transform. The purpose of this subsection is the introduction of Wigner measures for the analysis (in dimensions $d = 2$ and 3) of the Reynolds stress tensor

$$\mathrm{RT}(u_\nu)(x,t) = \lim_{\nu \to 0} \big((u_\nu - \bar{u}) \otimes (u_\nu - \bar{u}) \big), \tag{73}$$

which appears in the weak limit process of solutions u_ν of the Navier–Stokes equations as $\nu \to 0$ (cf. (45)). (Note that $\mathrm{RT}(u_\nu)(x,t)$ is independent of the viscosity ν, but it depends on the sequence $\{u_\nu\}$.) This point of view will be compared below (cf. § 6.2) to ideas emerging from statistical theory of turbulence.

Let $\{u_\nu, p_\nu\}$ be a sequence of Leray–Hopf solutions of the Navier–Stokes equations subject to the no-slip Dirichlet boundary condition in a domain Ω (with a physical boundary). By the energy inequality (44) (possibly an equality in some cases),

$$\frac{1}{2}|u_\nu(\cdot,t)|^2 + \nu \int_0^t |\nabla u_\nu(\cdot,t)|^2\,dt \leqslant \frac{1}{2}|u(\cdot,0)|^2, \tag{74}$$

the sequence $\{u_\nu\}$ converges (modulo the extraction of a subsequence) as $\nu \to 0$ in the weak-$*$ topology of the Banach space $L^\infty\big((0,T); L^2(\Omega)\big)$, to a divergence-free vector field $\bar{u}$, and the sequence of distributions $\{\nabla p_\nu\}$ converges to a distribution $\nabla \bar{p}$. Moreover, the pair $\{\bar{u}, \bar{p}\}$ satisfies the system of equations

$$\nabla \cdot \bar{u} = 0 \quad \text{in} \quad \Omega, \qquad \bar{u} \cdot \vec{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{75}$$

$$\partial_t \bar{u} + \nabla \cdot (\bar{u} \otimes \bar{u}) + \nabla \cdot \mathrm{RT}(u_\nu) + \nabla \bar{p} = 0 \quad \text{in} \quad \Omega \tag{76}$$

(cf. (46)). Being concerned with the behaviour of the solution inside the domain, we consider an arbitrary open subset Ω' whose closure is compact in Ω, that is, $\overline{\Omega'} \Subset \Omega$. Assuming that the weak-$*$ limit function belongs to $L^2\big((0,T); H^1(\Omega)\big)$, we introduce the function

$$v_\nu = a(x)(u_\nu - \bar{u}),$$

where $a(x) \in \mathscr{D}(\Omega)$ and $a(x) \equiv 1$ for all $x \in \Omega'$. As a result of (74), and the above assumptions, the sequence v_ν satisfies the uniform estimate

$$\nu \int_0^\infty \int_\Omega |\nabla v_\nu|^2 \, dx \leqslant C. \tag{77}$$

Consequently, the sequence v_ν is (in the sense of Gérard, Markowich, Mauser, and Poupaud [51]) $\sqrt{\nu}$-oscillating. Accordingly, we introduce the deterministic correlation spectrum, or *Wigner transform*, at the scale $\sqrt{\nu}$:

$$\widehat{\mathrm{RT}(v_\nu)}(x,t,k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d_y} e^{ik\cdot y} \left(v_\nu \left(x - \frac{\sqrt{\nu}}{2} y \right) \otimes v_\nu \left(x + \frac{\sqrt{\nu}}{2} y \right) \right) dy.$$

By means of the inverse Fourier transform one has

$$v_\nu(x,t) \otimes v_\nu(x,t) = \int_{\mathbb{R}^d_k} \widehat{\mathrm{RT}(v_\nu)}(x,t,k) \, dk. \tag{78}$$

The tensor $\widehat{\mathrm{RT}(v_\nu)}(x,t,k)$ is the main object in § 1 of [51]. Modulo the extraction of a subsequence, the tensor $\widehat{\mathrm{RT}(v_\nu)}(x,t,k)$ converges weakly as $\nu \to 0$ to a non-negative symmetric matrix-valued measure $\widehat{\mathrm{RT}}(x,t,dk)$, which is called a *Wigner measure* or *Wigner spectrum*. Moreover, interior to the open subset Ω' the weak limit $\bar{u}$ is a solution of the equation

$$\partial_t \bar{u} + \nabla \cdot (\bar{u} \otimes \bar{u}) + \nabla \cdot \int_{\mathbb{R}^d_k} \widehat{\mathrm{RT}}(x,t,dk) + \nabla \bar{p} = 0.$$

The Wigner spectrum has the following properties.

i) It is defined by a two-point correlation formula.

ii) It is an (x,t)-locally dependent object. Specifically, for any $\phi \in \mathscr{D}(\Omega)$

$$\begin{aligned} &\lim_{\nu \to 0} \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d_y} e^{ik\cdot y} \left((\phi v_\nu) \left(x - \frac{\sqrt{\nu}}{2} y \right) \otimes (\phi v_\nu) \left(x + \frac{\sqrt{\nu}}{2} y \right) \right) dy \right) \\ &\qquad = |\phi(\cdot)|^2 \widehat{\mathrm{RT}}(x,t,dk). \end{aligned} \tag{79}$$

Therefore, the construction of $\widehat{\mathrm{RT}}(x,t,dk)$ is independent of the choice of the function $a(x)$ and the open subset Ω'.

iii) It is a *criteria for turbulence*: points (x,t) around which the sequence u_ν remains smooth and converges locally strongly to $\bar{u}$ as $\nu \to 0$ are characterized by the relation

$$\mathrm{Trace}\big(\widehat{\mathrm{RT}}(x,t,dk)\big) = 0. \tag{80}$$

iv) It is a *microlocal object*. In fact, it depends only on the behaviour of the Fourier spectrum of the sequence $\phi(x)v_\nu$ (or in fact $\phi(x)u_\nu$) in the *frequency band*

$$A \leqslant |k| \leqslant \frac{B}{\sqrt{\nu}}.$$

Proposition 6.1. *For any pair (A, B) of strictly positive constants and any test functions $\big(\psi(k), \phi(x), \theta(t)\big) \in C_0^\infty(\mathbb{R}_k^d) \times C_0^\infty(\mathbb{R}_x^d) \times C_0^\infty(\mathbb{R}_t^+)$,*

$$\int_0^\infty \int \psi(k)|\phi(x)|^2\theta(t)\,\mathrm{Trace}\big(\widehat{\mathrm{RT}(u_\nu)}\big)(x,t,dk)\,dx\,dt \\ = \lim_{\nu\to 0}\int_0^\infty \theta(t)\int_{A\leqslant|k|\leqslant\frac{B}{\sqrt{\nu}}}\big(\psi\big(\sqrt{\nu}\,k\big)\widehat{(\phi v_\nu)}, \widehat{(\phi v_\nu)}\big)\,dk\,dt. \tag{81}$$

The only difference between this representation and that in [51] is that the weak limit $\bar{u}$ has been subtracted from the sequence u_ν. Otherwise, the formula (78) together with the energy estimate is the first statement in Proposition 1.7 of [51], while (81) is deduced from (1.32) in [51] by observing that the weak convergence of $(u_\nu - \bar{u})$ to 0 implies that

$$\lim_{\nu\to 0}\left(\int_0^\infty \theta(t)\int_{|k|\leqslant A}\big(\psi\big(\sqrt{\nu}\,k\big)\widehat{(\phi v_\nu)}, \widehat{(\phi v_\nu)}\big)\,dk\,dt\right) = 0.$$

6.2. The energy spectrum in the statistical theory of turbulence. The Wigner spectrum studied in the previous subsection turns out to be the deterministic version of the 'turbulence spectrum', which is a classical concept in the statistical theory of turbulence. The two points of view can be correlated by introducing homogeneous random variables. Let $(\mathfrak{M}, \mathfrak{F}, dm)$ be an underlying probability space. A random variable $u(x,\mu)$ is said to be homogeneous if for any function F the expectation of $F\big(u(x,\mu)\big)$, namely,

$$\big\langle F\big(u(x,\cdot)\big)\big\rangle = \int_{\mathfrak{M}} F\big(u(x,\mu)\big)\,dm(\mu),$$

is independent of x, that is,

$$\nabla_x\big(\big\langle F\big(u(x,\cdot)\big)\big\rangle\big) = 0.$$

In particular, if $u(x,\mu)$ is a homogeneous random vector-valued function, then

$$\big\langle u(x+r,\cdot)\otimes u(x,\cdot)\big\rangle = \left\langle u\left(x+\frac{r}{2},\cdot\right)\otimes u\left(x-\frac{r}{2},\cdot\right)\right\rangle, \tag{82}$$

which leads to the following proposition.

Proposition 6.2. *Let $u(x,\mu)$ be a homogeneous random variable and denote by $\hat{u}(k,\mu)$ its Fourier transform. Then*

$$\big\langle \hat{u}(k,\cdot)\otimes\overline{\hat{u}(k,\cdot)}\,\big\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ik\cdot r}\left\langle u\left(x+\frac{r}{2},\cdot\right)\otimes u\left(x-\frac{r}{2},\cdot\right)\right\rangle dr. \tag{83}$$

Proof. The proof will be given for a homogeneous random variable which is periodic with respect to the variable x, with basic periodic box of size $2\pi L$. The formula (83) is then deduced by letting L go to infinity. From the Fourier series expansion in $(\mathbb{R}/2\pi L\mathbb{Z})^d$,

$$\begin{aligned}\hat{u}(k,\mu)&\otimes\overline{\hat{u}(k,\mu)}\\&=\frac{1}{(2\pi L)^{2d}}\int_{(\mathbb{R}/2\pi L\mathbb{Z})^d}\int_{(\mathbb{R}/2\pi L\mathbb{Z})^d}\big(u(y,\mu)e^{-i\frac{k\cdot y}{L}}\otimes u(y',\mu)e^{i\frac{k\cdot y'}{L}}\big)\,dy'\,dy\\&=\frac{1}{(2\pi L)^{2d}}\int_{(\mathbb{R}/2\pi L\mathbb{Z})^d}e^{-i\frac{k\cdot r}{L}}\int_{(\mathbb{R}/2\pi L\mathbb{Z})^d}\big(u(y,\mu)\otimes u(y+r,\mu)\big)\,dy\,dr.\end{aligned}$$

Averaging with respect to the probability measure dm, using the homogeneity of the random variable $u(x,\mu)$, and integrating with respect to dy, we have

$$\Big\langle\hat{u}(k,\cdot)\otimes\overline{\hat{u}(k,\cdot)}\Big\rangle=\frac{1}{(2\pi L)^d}\int_{(\mathbb{R}/2\pi L\mathbb{Z})^d}e^{-i\frac{k\cdot r}{L}}\Big\langle u_\nu\Big(y+\frac{r}{2},\cdot\Big)\otimes u_\nu\Big(y-\frac{r}{2},\cdot\Big)\Big\rangle\,dr. \tag{84}$$

This concludes our proof of Proposition 6.2.

Assuming now that in addition to homogeneity, the expectation of the two-point correlation tensor $\langle u(x+r,\cdot)\otimes u(x,\cdot)\rangle$ is *isotropic* (that is, it does not depend on the direction of the vector r, but only on its length), we obtain the formula

$$\begin{aligned}\Big\langle\hat{u}(k,\cdot)\otimes\overline{\hat{u}(k,\cdot)}\Big\rangle&=\frac{1}{(2\pi L)^d}\int_{(\mathbb{R}/2\pi L\mathbb{Z})^d}e^{-i\frac{k\cdot r}{L}}\Big\langle u_\nu\Big(y+\frac{r}{2},\cdot\Big)\otimes u_\nu\Big(y-\frac{r}{2},\cdot\Big)\Big\rangle\,dr\\&=\frac{E(|k|)}{S_{d-1}|k|^{d-1}}\Big(I-\frac{k\otimes k}{|k|^2}\Big),\end{aligned}$$

where $S_1=2\pi$ and $S_2=4\pi$, and this defines the turbulence spectrum $E(|k|)$.

Homogeneity implies that solutions of the Navier–Stokes equations satisfy a local version of energy balance often called the Kármán–Howarth relation (cf. (85) below). Specifically, let $\{u_\nu,p_\nu\}$ be a solution of the forced Navier–Stokes equations in Ω (subject to either no-slip Dirichlet boundary condition in the presence of a physical boundary, otherwise Ω is the whole space or a periodic box)

$$\partial_t u_\nu+\nabla\cdot(u_\nu\otimes u_\nu)-\nu\Delta u_\nu+\nabla p_\nu=f.$$

Here u_ν and p_ν are random variables which depend on (x,t). Below we will drop the explicit dependence on μ when this does not cause any confusion.

We multiply the Navier–Stokes equations by $u_\nu(x,t,\mu)$ and *assume* the equality

$$\begin{aligned}\frac{1}{2}\partial_t|u_\nu(x,t,\mu)|^2-\nabla_x\cdot\big((\nu\nabla_x u_\nu-p_\nu I)u_\nu\big)(x,t,\mu)&\\+\nu|\nabla_x u_\nu(x,t,\mu)|^2=f(x,t)\cdot u_\nu(x,t,\mu),&\end{aligned}$$

which in the $2D$ case is a proven fact. However, in the $3D$ case the suitable solutions of the Navier–Stokes equations in the Caffarelli–Kohn–Nirenberg sense are only known to satisfy a weaker form of the above relation, involving an inequality instead of equality (cf. (43)).

By the homogeneity assumption,

$$\Big\langle \big((\nu\nabla_x u_\nu - p_\nu I)u_\nu\big)(x,t,\cdot)\Big\rangle$$

does not depend on x, and therefore,

$$\Big\langle \nabla_x \cdot \big((\nu\nabla_x u_\nu - p_\nu I)u_\nu\big)(x,t,\cdot)\Big\rangle = \nabla_x \cdot \Big\langle \big((\nu\nabla_x u_\nu - p_\nu I)u_\nu\big)(x,t,\cdot)\Big\rangle = 0.$$

Thus, the averaged pointwise energy relation

$$\frac{1}{2}\partial_t\big\langle |u_\nu(x,t,\cdot)|^2\big\rangle + \nu\big\langle |\nabla u_\nu(x,t,\cdot)|^2\big\rangle = \big\langle f(x,t)\cdot u_\nu(x,t,\cdot)\big\rangle \tag{85}$$

is obtained. This equality is often called the Kármán–Howarth relation; it implies that the quantities

$$\mathfrak{e} = \big\langle |u_\nu(x,t,\cdot)|^2\big\rangle \qquad \text{and} \qquad \varepsilon(\nu) = \frac{1}{t}\int_0^t \nu\big\langle |\nabla u_\nu(x,s,\cdot)|^2\big\rangle\, ds$$

are uniformly bounded in time, under reasonable assumptions on the forcing term.

Finally, assume also that the random process is stationary in time. Then the equation (85) gives an *a priori* estimate for the mean rate of dissipation of energy

$$\varepsilon(\nu) = \nu\big\langle |\nabla u_\nu(x,t,\cdot)|^2\big\rangle$$

which is independent of (x,t). With the forcing term f acting only on low Fourier modes, one may now assume the existence of a region called the inertial range where the turbulence spectrum $E(|k|)$ behaves according to a universal power law. *The term 'inertial' refers to the fact that in this range of wave numbers the energy cascades from low modes to high modes with no leakage of energy. That is, there are no viscous effects in this range and only the inertial term* $(u\cdot\nabla)u$ *is active.* Combining the above assumptions of homogeneity, isotropy, and stationarity of the random process, together with the existence of an inertial range of size

$$A \leqslant |k| \leqslant B\nu^{-\frac{1}{2}}$$

where the spectrum behaves according to a power law, we finally get the 'Kolmogorov law'

$$E(|k|) \simeq \varepsilon(\nu)^{\frac{2}{3}}|k|^{-\frac{5}{3}}, \tag{86}$$

using dimensional arguments in the three-dimensional case. It is important to keep in mind the fact that this derivation is based on the analysis of a random family of solutions. Therefore, the formula (86) combined with the formula (84) implies that *in the average* the solutions have a spectrum which behaves in the turbulent regime according to the prescription

$$\begin{aligned}
&\big\langle \hat{u}_\nu(k,t,\cdot)\otimes\overline{\hat{u}_\nu(k,t,\cdot)}\,\big\rangle \\
&\quad= \frac{1}{(2\pi L)^d}\int_{(\mathbb{R}/L\mathbb{Z})^d} e^{-i\frac{k\cdot r}{L}}\Big\langle u_\nu\Big(y+\frac{r}{2},t,\cdot\Big)\otimes u_\nu\Big(y-\frac{r}{2},t,\cdot\Big)\Big\rangle\, dr \\
&\quad\simeq \frac{\varepsilon(\nu)^{\frac{2}{3}}|k|^{-\frac{5}{3}}}{4\pi|k|^{d-1}}\Big(I - \frac{k\otimes k}{|k|^2}\Big).
\end{aligned}$$

Remark 6.1. The main difficulty in the full justification of the above derivation is the construction of a probability measure dm on the ensemble of solutions of the Navier–Stokes equations that would satisfy the conditions of homogeneity, isotropy, and stationarity. In particular, the construction of such a measure should be uniformly valid when the viscosity ν tends to 0; see, for instance, the books of Vishik and Fursikov [52] and of Foias et al. [53] and their references for further study regarding this challenging problem.

The next difficulty (which is a controversial subject) is the justification for the spectrum of an inertial range with a power law. In Foias et al. [53] (see also Foias [54]) it was established, for example, that there is an inertial range of wave numbers where one has a forward energy cascade. However, we are unaware of a rigorous justification for a power law in this inertial range.

Nevertheless, the construction of a power law of the spectrum is often used as a benchmark for validation of numerical computations and experiments. Since in general one would have only one run of an experiment or one run of a simulation, Birkhoff's theorem corresponding to assuming ergodicity is then used. This allows the replacement of the ensemble average by the time average. For instance, in the presence of a forcing term one may assume in addition to stationarity that *for almost all solutions* (that is, for almost every μ)

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u_\nu\Big(y+\frac{r}{2},t,\mu\Big)\otimes u_\nu\Big(y-\frac{r}{2},t,\mu\Big)\,dt=\Big\langle u_\nu\Big(y+\frac{r}{2},t,\cdot\Big)\otimes u_\nu\Big(y-\frac{r}{2},t,\cdot\Big)\Big\rangle,$$

which would give the following relation for almost every solution u_ν:

$$\begin{aligned}\lim_{T\to\infty}&\frac{1}{T}\int_0^T u_\nu(k,t,\mu)\otimes\overline{\hat u_\nu(k,t,\mu)}\,dt\\&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-ik\cdot r}\Big\langle u_\nu\Big(y+\frac{r}{2},t,\cdot\Big)\otimes u_\nu\Big(y-\frac{r}{2},t,\cdot\Big)\Big\rangle\,dr\\&=\big\langle\hat u(k,t,\cdot)\otimes\overline{\hat u(k,t,\cdot)}\,\big\rangle\simeq\frac{\varepsilon(\nu)^{\frac23}|k|^{-\frac53}}{4\pi|k|^{d-1}}\Big(I-\frac{k\otimes k}{|k|^2}\Big).\end{aligned}$$

6.3. Comparison between deterministic and statistical spectra. The deterministic point of view considers families of solutions u_ν of the Navier–Stokes dynamics with viscosity $\nu>0$, and interprets the notion of turbulence in terms of the weak limit behaviour (the asymptotic behaviour of such sequences as $\nu\to0$) with the Wigner spectrum:

$$\begin{aligned}&\widehat{\mathrm{RT}}(x,t,dk)\\&\quad=\lim_{\nu\to0}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_y}e^{ik\cdot y}\Big((u_\nu-\bar u)\Big(x-\frac{\sqrt{\nu}}{2}y,t\Big)\otimes(u_\nu-\bar u)\Big(x+\frac{\sqrt{\nu}}{2}y,t\Big)\Big)\,dy.\end{aligned}$$

As already observed earlier, this is a local object (it takes into account the dependence on (x,t)). Moreover, one could define the support of turbulence for such a family of solutions as the support of the measure $\mathrm{Trace}\,\widehat{\mathrm{RT}}(x,t,dk)$. Of course, determining such a support is extremely hard and is a configuration-dependent problem. This is perfectly described in often-quoted words of Leonardo da Vinci on this subject, in particular, in [55] (p. 112):

doue la turbolenza dellacqua sigenera
doue la turbolenza dellacqua simantiene plugho
doue la turbolenza dellacqua siposa.[1]

Up to this point nothing much can be said without extra hypotheses, except that the formula (81) indicates the existence of an essential, if not an 'inertial', range

$$A \leqslant |k| \leqslant \frac{B}{\sqrt{\nu}}. \tag{87}$$

On the other hand, the statistical theory of turbulence starts from hypotheses (seemingly difficult to formulate in a rigorous mathematical setting) about the existence of statistics (a probability measure) with respect to which the two-point correlations for any family of solutions u_ν of the Navier–Stokes equations are homogeneous and isotropic. Under these assumptions one proves properties on the decay of the turbulence spectrum. Moreover, from all these assumptions one obtains by simple dimensional arguments that *for averages of solutions with respect to the probability measure $dm(\mu)$*, the following formula holds in the inertial range:

$$\left\langle |\hat{u}_\nu(k,t,\cdot)|^2 \right\rangle = \frac{1}{(2\pi)^4} \frac{E(|k|)}{|k|^2} \simeq \big(\varepsilon(\nu)\big)^{\frac{2}{3}} |k|^{-\frac{11}{3}}. \tag{88}$$

Finally, by assuming and using the stationarity (in time) of these two-point correlations, and by applying sometimes the Birkhoff ergodic theorem, one should be able to obtain for almost every solution the formula

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\hat{u}_\nu(k,t,\mu) \otimes \overline{\hat{u}_\nu(k,t,\mu)} \,\big)\, dt \\ &\qquad = \left\langle \hat{u}_\nu(k,t,\cdot) \otimes \overline{\hat{u}_\nu(k,t,\cdot)} \right\rangle \simeq \frac{\big(\varepsilon(\nu)\big)^{\frac{2}{3}} |k|^{-\frac{5}{3}}}{4\pi |k|^{d-1}} \left(I - \frac{k \otimes k}{|k|^2} \right). \end{aligned}$$

Remark 6.2. Some further connections between these two aspects of spectra may be considered.

i) Assuming that near a point (x_0, t_0) the Wigner spectrum is isotropic, define the local mean rate of energy dissipation as

$$\varepsilon_{(x_0,t_0)}(\nu) = \nu \int_0^\infty \int_\Omega |\phi(x,t)|^2 |\nabla u_\nu(x,t)|^2 \, dx\, dt, \tag{89}$$

[1] *Editor's note*: The English translation given in [55] is

where the turbulence of water is generated,
where the turbulence of water maintains for long,
where the turbulence of water comes to rest.

where ϕ is a localized function near (x_0, t_0). Then one can prove that for $|k|$ in the range (87),

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_y} e^{ik\cdot y}\left((u_\nu-\bar{u})\left(x-\frac{\sqrt{\nu}}{2}y,t\right)\otimes(u_\nu-\bar{u})\left(x+\frac{\sqrt{\nu}}{2}y,t\right)\right)dy$$
$$\simeq \varepsilon_{(x_0,t_0)}(\nu)|k|^{-\frac{11}{3}}\left(I-\frac{k\otimes k}{|k|^2}\right)$$

as ν tends to zero.

ii) Since large-scale forcing and boundary effects tend to break isotropy, isotropy can hold only at small spatial scales, and hence away from these effects. Since the Wigner spectrum is localized and involves only a range of high wave numbers (that is, small spatial scales), it is reasonable to ask for sufficient conditions which guarantee the isotropy of the Wigner spectrum.

iii) Another approach for establishing the existence of an inertial range for a forward energy cascade in $3D$, and a forward enstrophy cascade in $2D$, is presented in [53] (see also references therein). This approach is based on statistically stationary solutions of the Navier–Stokes equations. These are time-independent probability measures which are invariant under the solution operator of the Navier–Stokes equations. Furthermore, in Foias [54] some semirigorous arguments are presented to justify the Kolmogorov power law for the energy spectrum.

7. Prandtl and Kelvin–Helmholtz problems

In this section we assume that the sequence $\{u_\nu\}$ of solutions of the Navier–Stokes equations with no-slip Dirichlet boundary condition (in the presence of a physical boundary) converges to a solution of the Euler equations. According to Theorem 5.1 of Kato, in this situation one has

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\{x\in\Omega: d(x,\partial\Omega)<\nu\}}|\nabla u_\nu(x,t)|^2\,dx\,dt=0. \tag{90}$$

However, since the tangential velocity of the solution of the Euler equations is not zero on the boundary as $\nu\to 0$, a boundary layer is going to appear. On the one hand, the scaling of the boundary layer has to be compatible with (90), and on the other hand, the equations that model the behaviour in this boundary layer have to reflect the fact that the problem is very unstable. This is because the instabilities (and possible singularities) that occur near the boundary may not remain confined near the boundary, and will in fact propagate inside the domain due to the non-linear advection term in the Navier–Stokes equations. These considerations explain why the Prandtl equations (PE) of the boundary layer are complicated.

There are good reasons to justify comparing the Prandtl equations with the Kelvin–Helmholtz problem (KH).

1. Even though some essential issues remain unsolved for KH, it is much better understood from the mathematical point of view than the PE problem. However, the two problems share similar properties such as instabilities and appearance of singularities.

2. At the level of modeling (in particular, for the problem of the wake behind an airplane and the vortices generated by the tip of the wings) it is not clear whether turbulence should be described by singularities in KH or PE (or both)!

For simplicity these problems are considered in the $2D$ case, and for PE in the half-space $x_2 > 0$.

7.1. The Prandtl boundary layer. We start with the $2D$ Navier–Stokes equations in the half-plane $x_2 > 0$ with the no-slip boundary condition $u^\nu(x_1, 0, t) \equiv 0$,

$$\partial_t u_1^\nu - \nu\Delta u_1^\nu + u_1^\nu \partial_{x_1} u_1^\nu + u_2^\nu \partial_{x_2} u_1^\nu + \partial_{x_1} p^\nu = 0, \tag{91}$$

$$\partial_t u_2^\nu - \nu\Delta u_2^\nu + u_1^\nu \partial_{x_1} u_2^\nu + u_2^\nu \partial_{x_2} u_2^\nu + \partial_{x_2} p^\nu = 0, \tag{92}$$

$$\partial_{x_1} u_1^\nu + \partial_{x_2} u_2^\nu = 0, \tag{93}$$

$$u_1^\nu(x_1, 0, t) = u_2^\nu(x_1, 0, t) = 0 \quad \text{for} \quad x_1 \in \mathbb{R}, \tag{94}$$

and assume that inside the domain (away from the boundary) the vector field $u^\nu(x_1, x_2, t)$ converges to the solution $u_{\text{int}}(x_1, x_2, t)$ of the Euler equations with the same initial data. The tangential component of this solution on the boundary $x_2 = 0$ and the pressure are denoted by

$$U(x_1, t) = u_1^{\text{int}}(x_1, 0, t), \qquad \widetilde{P}(x_1, t) = p(x_1, 0, t).$$

We now introduce the scale $\varepsilon = \sqrt{\nu}$. Taking into account that the normal component of the velocity remains 0 on the boundary, we use the following ansatz, which corresponds to a boundary layer in a parabolic PDE problem:

$$\begin{pmatrix} u_1^\nu(x_1, x_2) \\ u_2^\nu(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \tilde{u}_1^\nu(x_1, x_2/\varepsilon) \\ \varepsilon \tilde{u}_2^\nu(x_1, x_2/\varepsilon) \end{pmatrix} + u^{\text{int}}(x_1, x_2). \tag{95}$$

Inserting the right-hand side of (95) into the Navier–Stokes equations, returning to the notation (x_1, x_2) for the variables

$$X_1 = x_1, \qquad X_2 = \frac{x_2}{\varepsilon},$$

and letting ε go to zero, we obtain formally the equations

$$\tilde{u}_1(x_1, 0, t) + U_1(x_1, 0, t) = 0, \tag{96}$$

$$\partial_{x_2} \tilde{p}(x_1, x_2) = 0 \implies \tilde{p}(x_1, x_2, t) = \widetilde{P}(x_1, t), \tag{97}$$

$$\partial_t \tilde{u}_1 - \partial_{x_2}^2 \tilde{u}_1 + \tilde{u}_1 \partial_{x_1} \tilde{u}_1 + \tilde{u}_2 \partial_{x_2} \tilde{u}_1 = \partial_{x_1} \widetilde{P}(x_1, t), \tag{98}$$

$$\partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 = 0, \qquad \tilde{u}_1(x_1, 0) = \tilde{u}_2(x_1, 0) = 0 \quad \text{for} \quad x_1 \in \mathbb{R}, \tag{99}$$

$$\lim_{x_2 \to \infty} \tilde{u}_1(x_1, x_2) = \lim_{x_2 \to \infty} \tilde{u}_2(x_1, x_2) = 0. \tag{100}$$

Remark 7.1. As an indication of the validity of the Prandtl equations we observe that (95) is consistent with Theorem 5.1 of Kato. Specifically, thanks to (95) one has

$$\nu \int_0^T \int_{\Omega \cap \{d(x, \partial\Omega) \leqslant c\nu\}} |\nabla u^\nu(x, s)|^2 \, dx \, ds \leqslant C\sqrt{\nu}\,.$$

Remark 7.2. The following example, constructed by Grenier [56], shows that the Prandtl expansion cannot always be valid. In the case when the solutions are considered in the domain

$$(\mathbb{R}_{x_1}/\mathbb{Z}) \times \mathbb{R}^+_{x_2},$$

Grenier starts with a solutions u^ν_{ref} of the pressureless Navier–Stokes equations given by

$$u^\nu_{\mathrm{ref}} = \big(u_{\mathrm{ref}}\big(t, y/\sqrt{\nu}\,\big), 0\big),$$
$$\partial_t u_{\mathrm{ref}} - \partial_{YY} u_{\mathrm{ref}} = 0,$$

where $Y = y/\sqrt{\nu}$. Using a convenient and explicit choice of the function u_{ref}, together with some sharp results on instabilities, he constructs a solution of the Euler equations of the form

$$\tilde{u} = u_{\mathrm{ref}} + \delta v + O(\delta^2 e^{2\lambda t}) \quad \text{for} \quad 0 < t < \frac{1}{\lambda \log \delta}\,.$$

He then shows that the vorticity generated by the boundary for the solution of the Navier–Stokes equations (with the same initial data) is too strong to allow convergence of the Prandtl expansion. One should observe, however, that once again this is an example which involves solutions with infinite energy. It would be interesting to see if such an example could be modified to belong to the class of finite-energy solutions; and then to analyze how the modified finite-energy solution might violate the Kato criterion in Theorem 5.1.

It is important to observe that in their mathematical properties the PE exhibit the pathology of the situation that they are trying to model. First, one can prove the following proposition.

Proposition 7.1. *Let $T > 0$ be a finite positive time, and let $\big(U(x,t), P(x,t)\big) \in C^{2+\alpha}\big(\mathbb{R}_t \times (\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2})\big)$ be a smooth solution of the 2D Euler equations satisfying at time $t = 0$ the compatibility condition $U_1(x_1, 0, t) = U_2(x_1, 0, t) = 0$ (note that only the boundary condition $U_2(x_1, 0, t) = 0$ is preserved by the Euler dynamics). Then the following statements are equivalent:*

i) *with initial data $\tilde{u}(x, 0) = 0$, the boundary condition $\tilde{u}_1(x_1, 0, t) = U_1(x_1, 0, t)$ in (96), and right-hand side in (97) given by $\widetilde{P}(x_1, t) = P(x_1, 0, t)$, the Prandtl equations have a smooth solution $\tilde{u}(x, t)$, for $0 < t < T$;*

ii) *the solution $u_\nu(x, t)$ of the Navier–Stokes equations with initial data $u_\nu(x, 0) = U(x, 0)$ and with the no-slip boundary condition at the boundary $x_1 = 0$ converges in $C^{2+\alpha}$ to the solution of the Euler equations as $\nu \to 0$.*

The fact that i) and ii) can fail for some t is related to the appearance of a detachment zone and the generation of turbulence. This is well illustrated in the analysis of the Prandtl equations written in the simplified form

$$\partial_t \tilde{u}_1 - \partial^2_{x_2} \tilde{u}_1 + \tilde{u}_1 \partial_{x_1} \tilde{u}_1 + \tilde{u}_2 \partial_{x_2} \tilde{u}_1 = 0, \qquad \partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 = 0, \tag{101}$$

$$\tilde{u}_1(x_1, 0) = \tilde{u}_2(x_1, 0) = 0 \quad \text{for} \quad x_1 \in \mathbb{R}, \tag{102}$$

$$\lim_{x_2 \to \infty} \tilde{u}_1(x_1, x_2) = 0, \tag{103}$$

$$\tilde{u}_1(x_1, x_2, 0) = \tilde{u}_0(x_1, x_2). \tag{104}$$

Regularity in the absence of detachment corresponds to a theorem of Oleinik [57]. She proved that global smooth solutions of the above system do exist provided that the initial profile is monotonic, that is, for any initial profile satisfying

$$\tilde{u}(x,0) = \big(\tilde{u}_1(x_1,x_2),0\big), \qquad \partial_{x_2}\tilde{u}_1(x_1,x_2) \neq 0.$$

On the other hand, initial conditions with 'recirculation properties' leading to a finite-time blowup have been constructed by E and Engquist [58] and E [59]. An interesting aspect of these examples is that the blowup generally does not occur on the boundary, but rather inside the domain.

The above pathology appears in the fact that the PE are highly unstable. This comes from the determination of $\tilde{u}_2$ in terms of $\tilde{u}_1$ in the equation

$$\partial_{x_1}\tilde{u}_1 + \partial_{x_2}\tilde{u}_2 = 0.$$

Therefore, it is only with analytic initial data (in fact, analytic with respect to the tangential variable is sufficient) that one can obtain (using an abstract version of the Cauchy–Kovalevskaya theorem) the existence of a smooth solution of the Prandtl equations on a finite time interval and the convergence to the solution of the Euler equations on the same time interval (Asano [60], Caflisch and Sammartino [61], and Cannone, Lombardo, and Sammartino[62]).

7.2. The Kelvin–Helmholtz problem. The Kelvin–Helmholtz (KH) problem concerns the evolution of a solution of the $2D$ Euler equations

$$\partial_t u + \nabla\cdot(u\otimes u) + \nabla p = 0, \qquad \nabla\cdot u = 0 \tag{105}$$

with initial vorticity $\omega(x,0)$ a measure concentrated on a curve $\Gamma(0)$.

This is already simpler than the PE because the pathology, if any, should in principle be concentrated on a curve. Furthermore, the dynamics in this case inherits the general properties of the $2D$ dynamics. In particular, it will obey the equation

$$\partial_t(\nabla\wedge u) + (u\cdot\nabla)(\nabla\wedge u) = 0,$$

for the conservation (for smooth solutions) of the density of the vorticity. Therefore, one can guarantee the existence of a weak solution when the initial vorticity $\omega(x,0)$ is a Radon measure. This was first done by Delort [19], assuming that the initial measure has definite sign. Then the result was generalized to situations where the change of sign was sufficiently simple (see [50]). However, this remarkable positive result is impaired by the non-uniqueness result of Shnirelman [21].

For smooth solutions of the KH problem, that is, solutions with vorticity ω a bounded Radon measure with support on the curve $\Sigma_t = \{r(\lambda,t),\ \lambda\in\mathbb{R}\}$, the velocity field is given for $x\notin\Sigma_t$ by the so-called Biot–Savart law:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}R_{\frac{\pi}{2}}\int \frac{x-r'}{|x-r'|^2}\omega(r',t)\,ds' \\ &:= \frac{1}{2\pi}R_{\frac{\pi}{2}}\int \frac{x-r(\lambda',t)}{|x-r(\lambda',t)|^2}\omega\big(r(\lambda',t),t\big)\frac{\partial s(\lambda',t)}{\partial\lambda'}\,d\lambda'. \end{aligned} \tag{106}$$

Here $r(\lambda,t)$ with $\lambda\in\mathbb{R}$ is a parametrization of the curve Σ_t, and $s(\lambda,t)=|r(\lambda,t)|$ is the corresponding arc length; $R_{\frac{\pi}{2}}$ denotes the $\frac{\pi}{2}$-counterclockwise rotation matrix

$$R_{\frac{\pi}{2}}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Furthermore, the velocity field u has two-sided limits $u_\pm$ as x approaches the curve Σ_t. By virtue of the incompressibility condition one has the continuity condition for the normal component of the velocity field, that is,

$$u_-\cdot\vec{n}=u_+\cdot\vec{n},$$

where hereafter $\vec{\tau}$ and $\vec{n}$ will denote the unit tangent and unit normal vectors to the curve Σ_t, respectively. In addition, the average

$$\langle u\rangle=\frac{u_++u_-}{2}$$

is given by the principal value of the singular integral appearing in (106):

$$v=\langle u\rangle=\frac{1}{2\pi}R_{\frac{\pi}{2}}\,\mathrm{p.\,v.}\int\frac{x-r'}{|x-r'|^2}\omega(r',t)\,ds'. \tag{107}$$

Using the calculus of distributions, one can show that as long as the curve Σ_t is smooth, the velocity field u defined above, being a weak solution of the Euler equations

$$\partial_t u+\nabla\cdot(u\otimes u)+\nabla p=0,\qquad \nabla\cdot u=0,$$

is equivalent to the density of the vorticity ω, and the curve Σ_t satisfies the coupled system of equations

$$\omega_t-\partial_s\big(\omega(\partial_t r-v)\cdot\vec{\tau}\big)=0, \tag{108}$$
$$(r_t-v)\cdot\vec{n}=0, \tag{109}$$
$$v(r,t)=\frac{1}{2\pi}R_{\frac{\pi}{2}}\,\mathrm{p.\,v.}\int\frac{r-r'}{|r-r'|^2}\omega(r',t)\,ds'. \tag{110}$$

The equations (108), (109), (110) do not completely determine $r(\lambda,t)$. This is due to the freedom in the choice of the parametrization of the curve Σ_t. Assuming that $\omega\neq 0$, one can introduce a new parametrization $\lambda(t,s)$ which reduces the problem to the equation

$$\partial_t r(\lambda,t)=\frac{1}{2\pi}R_{\frac{\pi}{2}}\,\mathrm{p.\,v.}\int\frac{r(\lambda,t)-r(\lambda',t)}{|r(\lambda,t)-r(\lambda',t)|^2}\,d\lambda', \tag{111}$$

or, introducing the complex variable $z=r_1+ir_2$ with $r=(r_1,r_2)$, one obtains the Birkhoff–Rott equation

$$\partial_t\bar{z}(\lambda,t)=\frac{1}{2\pi i}\,\mathrm{p.\,v.}\int\frac{d\lambda'}{z(\lambda,t)-z(\lambda',t)}\,. \tag{112}$$

Remark 7.3. The following are certain mathematical similarities between the KH problem and the PE.

1. As in the PE, one has for the evolution equation (112) a local (in time) existence and uniqueness result in the class of analytic initial data. This is done by implementing a version of the Cauchy–Kovalevskaya theorem (C. Sulem, P.-L. Sulem, Bardos, and Frisch [63]).

2. As also for the PE, one can construct solutions that blow up in finite time.

3. One observes that the singular behaviour in experiments and numerical simulations with the KH problem is very similar to the phenomena that can be observed in the problem with the no-slip boundary condition as the viscosity approaches zero.

The best way to understand the structure of the KH problem is to use the fact that the Euler equations are invariant under both space and time translations, and under space rotations, and to consider a weak solution of the $2D$ Euler dynamics either in the whole plane $\mathbb{R}^2$, satisfying

$$u \in C\big((-T,T); L^2(\mathbb{R}^2)\big), \qquad T>0,$$

or subject to periodic boundary conditions satisfying

$$u \in C\big((-T,T); L^2\big((\mathbb{R}/\mathbb{Z})^2\big)\big).$$

Assuming that in a small neighbourhood $\mathscr{U}$ of the point $(t=0, z=0)$ the vorticity is concentrated on a smooth curve in the complex plane which takes the form

$$z(\lambda,t) = \big(\alpha t + \beta\big(\lambda + \varepsilon f(\lambda,t)\big), \qquad f(0,0) = \nabla f(0,0) = 0. \tag{113}$$

Then using the relations $\nabla \cdot u = 0$, $\nabla \wedge u = \omega$ and the Biot–Savart law, one obtains

$$\varepsilon|\beta|^2 \partial_t \bar{f}(\lambda,t) = \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\{z(t,\lambda')\in\mathscr{U}\}} \frac{d\lambda'}{(\lambda-\lambda')\big(1-\varepsilon\frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'}\big)} + E\big(z(\lambda,t)\big) \tag{114}$$

where, here and below, $E(z)$ denotes the 'remainder', which is analytic with respect to z. Next, we use the expansion

$$\begin{aligned}\frac{1}{2\pi}\,\mathrm{p.\,v.}\int &\frac{d\lambda'}{(\lambda-\lambda')\big(1-\varepsilon\frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'}\big)}\,d\lambda' \\ &= \frac{\varepsilon}{2\pi}\,\mathrm{p.\,v.}\int \frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}\,d\lambda' + \sum_{n\geqslant 2}\frac{\varepsilon^n}{2\pi}\,\mathrm{p.\,v.}\int \frac{\big(f(\lambda,t)-f(\lambda',t)\big)^n}{(\lambda-\lambda')^{(n+1)}}\,d\lambda'\end{aligned} \tag{115}$$

and employ for the Hilbert transform the formulae

$$\frac{1}{2\pi}\,\mathrm{p.\,v.}\int \frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'}\,d\lambda' = -\frac{i}{2}\,\mathrm{sgn}(D_\lambda)f, \tag{116}$$

$$\frac{1}{2\pi}\,\mathrm{p.\,v.}\int \frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}\,d\lambda' = |D_\lambda|f, \tag{117}$$

to deduce from (114) and (115) that the real and imaginary parts of $f(\lambda,t) = X(\lambda,t)+iY(\lambda,t)$ are local solutions of the system

$$\partial_t X = \frac{1}{2|\beta|^2}|D_\lambda|Y + \varepsilon R_1(X,Y) + E_1(X,Y), \tag{118}$$

$$\partial_t Y = \frac{1}{2|\beta|^2}|D_\lambda|X + \varepsilon R_2(X,Y) + E_2(X,Y), \tag{119}$$

or

$$\begin{aligned}\Big(\partial_t^2 + \frac{1}{4|\beta|^4}\partial_\lambda^2\Big)X &= \varepsilon\Big(\partial_t R_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_2(X,Y)\Big)\\ &\qquad + \partial_t E_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_2(X,Y),\end{aligned} \tag{120}$$

$$\begin{aligned}\Big(\partial_t^2 + \frac{1}{4|\beta|^4}\partial_\lambda^2\Big)Y &= \varepsilon\Big(\partial_t R_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_1(X,Y)\Big)\\ &\qquad + \partial_t E_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_1(X,Y).\end{aligned} \tag{121}$$

In (120) and (121) the terms

$$\partial_t E_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_2(X,Y) \quad \text{and} \quad \partial_t E_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_1(X,Y)$$

are the first-order derivatives of analytic functions with respect to (X,Y) while the terms

$$\partial_t R_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_2(X,Y) \quad \text{and} \quad \partial_t R_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_1(X,Y)$$

are the second-order derivatives of analytic functions with a small factor ε. Therefore, one observes that, up to a small perturbation, the KH problem behaves like a second-order constant-coefficient elliptic equation. This fact has several important consequences.

1. It explains why the evolution equation is well-posed only on a short time interval, and only with initial data belonging to the class of analytic functions. It is like solving an elliptic equation simultaneously with both the Dirichlet and Neumann boundary conditions.

2. It is a tool for the construction of solutions that blow up in finite time.

3. It explains, by an indirect regularity argument, the very singular behaviour of the solution after the first breakdown of its regularity.

These three points are discussed in further detail below.

7.2.1. *Local solution.* When the curve Σ_t is a graph of a function, say $y = y(x,t)$, the equations (108)–(110) become

$$y_t + y_x v_1 = v_2, \qquad \partial_t\omega + \partial_x(v_1\omega) = 0, \tag{122}$$

$$v_1(t,x) = -\frac{1}{2\pi}\,\mathrm{p.\,v.}\int_{\mathbb{R}} \frac{y(x,t)-y(x',t)\omega(x',t)}{(x-x')^2+\big(y(x,t)-y(x',t)\big)^2}\,dx', \tag{123}$$

$$v_2(t,x) = \frac{1}{2\pi}\,\mathrm{p.\,v.}\int_{\mathbb{R}} \frac{(x-x')\omega(x',t)}{(x-x')^2+\big(y(x,t)-y(x',t)\big)^2}\,dx', \tag{124}$$

where $(v_1, v_2) = v$ is the average velocity given in (107). Therefore, the above evolution equations involve the two unknown functions $y(x,t)$ and $\omega(x,t)$, which is also the case for the Birkhoff–Rott equation (112), where the two unknowns are the two components of $r(s,t) = \big(x(s,t), y(s,t)\big)$, or of $z(s,t) = x(s,t) + iy(s,t)$ in the complex notation. In fact, since the Birkhoff–Rott equation was obtained by choosing the vorticity density as a parameter, one recovers this vorticity by the formula

$$\omega(s,t) = \frac{1}{|\partial_s z(s,t)|}.$$

Because the system is a local perturbation of a second-order elliptic equation, imposing two constraints at $t = 0$ is similar to solving this elliptic equation simultaneously with both Neumann and Dirichlet boundary conditions. It is known that in the absence of stringent compatibility of the conditions (they are related by the so-called Dirichlet to Neumann operator) such a problem can be solved only locally and with analytic data. This is why the solution of (122)–(124) is obtained locally in time under the assumption that the functions $y(x,0)$ and $\omega(x,0)$ are analytic.

7.2.2. *Singularities.* For the construction of solutions with singularities one follows the same idea, and uses the time reversibility of the $2D$ Euler equations. More precisely, if one constructs solutions which are singular at $t = 0$ and regular on the interval $(0,T]$, then just by changing the time variable t to $T - t$ one has smooth solutions at $t = 0$ that blow up at $t = T$. The first result in this direction was obtained by Duchon and Robert [64]; the initial condition on the vorticity at $t = 0$ is relaxed, and one assumes that the solution $y(x,t)$ goes to zero as $t \to \infty$. Then one can regard the system (122)–(124) as a two-point Dirichlet boundary-value problem with $y(x,t)$ given for $t = 0$ and required to tend to zero as $t \to \infty$. By a perturbation method one proves the following proposition.

Proposition 7.2. *There exists an $\varepsilon > 0$ such that for any initial data for which*

$$y(x,0) = \int e^{ix\xi} g(\xi)\, d\xi, \qquad \textit{where} \quad \int |g(\xi)|\, d\xi \leqslant \varepsilon, \tag{125}$$

the problem (122)–(124) *has a unique solution which goes to zero as $t \to \infty$. Furthermore, this solution is analytic* (*with respect to* (x,t)) *for all $t > 0$.*

As mentioned above, this is a result about the formation of a singularity. It exhibits (by changing the time variable t to $T - t$) an example of solutions which are analytic for some time, but with no more regularity at a later time than what is allowed by the equation (125). In fact, it was observed in some numerical experiments in [65] and [66] that the first breakdown of regularity appears as a cusp on the curve $r(\lambda,t)$. This motivated Caflisch and Orellana [22] to introduce the function

$$f_0(\lambda,t) = (1-i)\big\{(1-e^{-\frac{t}{2}-i\lambda})^{1+\sigma} - (1-e^{-\frac{t}{2}+i\lambda})^{1+\sigma}\big\} \tag{126}$$

which enjoys the following properties:

i) for any $t > 0$ the map $\lambda \mapsto f(\lambda,t)$ is analytic;
ii) for $t = 0$ the map $\lambda \mapsto f(0,\lambda)$ does not belong to the Hölder space $C^{1+\sigma}$, but it belongs to every Hölder space $C^{1+\sigma'}$ with $0 < \sigma' < \sigma$;

iii) the function

$$z_0(\lambda,t)=\lambda+\varepsilon f_0(\lambda,t)$$

is an exact solution of the linearized Birkhoff–Rott equation, or more precisely, one has

$$\partial_t\overline{f(\lambda,t)}=\frac{1}{2\pi}\,\mathrm{p.\,v.}\int\frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}\,d\lambda'. \tag{127}$$

Therefore, by using the ellipticity of this linear operator, one can prove by a perturbation method the following proposition.

Proposition 7.3. *For sufficiently small $\varepsilon>0$ there exists a function $r_\varepsilon(\lambda,t)$ with the following properties*:

i) *the function $\lambda\mapsto r_\varepsilon(\lambda,t)$ is analytic for $t>0$;*

ii) *the function $\lambda\mapsto\varepsilon\big(f_0(\lambda,t)+r_\varepsilon(\lambda,t)\big)$ is a solution of the Birkhoff–Rott equation* (112);

iii) *the function $\lambda\mapsto r_\varepsilon(\lambda,t)$ is (for $\lambda\in\mathbb{R}$, $t\in\mathbb{R}_+$) uniformly bounded in C^2.*

As a consequence of Proposition 7.3 (and of the reversibility in time) one can establish the existence of analytic solutions to the Birkhoff–Rott equation (112), say on the interval $0\leqslant t<T$, such that at time $t=T$ the map $\lambda\mapsto z(\lambda,t)$ does not belongs to $C^{1+\sigma}$ at the point $\lambda=0$.

7.2.3. *Analyticity and pathological behaviour after the breakdown of regularity.* The local reduction of the KH problem to the equation

$$\varepsilon|\beta|^2\partial_t\bar f(\lambda,t)=\frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{z(\lambda',t)\in\mathscr{U}}\frac{d\lambda'}{(\lambda-\lambda')\big(1-\varepsilon\frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'}\big)}+E\big(z(\lambda,t)\big) \tag{128}$$

obviously requires *some assumptions* about the regularity of the function $z(\lambda,t)$ near the point $(0,0)$. However, when this reduction is valid it will, thanks to the ellipticity, imply that the solution is C^∞, and even analytic. Therefore, there appears to be a threshold (say $\mathbf{T}$) in the behaviour of the solutions of the KH problem. Existence of such a *regularity threshold* is common in the study of free-boundary problems. This threshold is characterized by the fact that any function with regularity stronger than $\mathbf{T}$ is in fact analytic, and that there may exist solutions with less regularity than $\mathbf{T}$. This has the following practical and important consequence: regularity of the solutions that are smooth for $t<T$ and singular after the time $t=T$ cannot be extended to $t\geqslant T$ by solutions which are more regular than the threshold $\mathbf{T}$. Otherwise, the above theorem would lead to a contradiction. This fact explains why after the breakdown of regularity the solution becomes very singular.

For instance, it was shown by Lebeau [4] (and Kamotski and Lebeau [67] for the local version) that any solution belonging to C^σ_t ($C^{1+\sigma}_\lambda$) near a point must be analytic. As a consequence, if a solution constructed (by changing t to $T-t$) according to the method of Caflisch and Orellana [22] could be continued after time $t=T$, it would not be in any Hölder space $C^{1+\sigma'}$.

Therefore, the challenge (and an open problem) is to determine this threshold of regularity that will imply analyticity. Up to now, the best (to the best of our knowledge) known result is due to Wu [5], [6]. The $C^{\alpha}_{\mathrm{loc}}\big(\mathbb{R}_t; C^{1+\beta}_{\mathrm{loc}}(\mathbb{R}_\lambda)\big)$ hypothesis is replaced by $H^1_{\mathrm{loc}}(\mathbb{R}_t \times \mathbb{R}_\lambda)$. The estimates are obtained by explicitly using theorems of David [42] saying that, for all chord arc curves $\Gamma\colon s \mapsto \xi(s)$ parameterized by their arc length, the Cauchy integral operator

$$C_\Gamma(f) = \mathrm{p.\,v.}\int \frac{f(s')}{\xi(s) - \xi(s')}\, d\xi(s')$$

is bounded in $L^2(ds)$.

The importance of this improvement is illustrated by a numerical experiment in [68]. It is interesting to note that these results will apply to logarithmic spirals $r = e^{\theta}$, $\theta \in \mathbb{R}$, but not to infinite-length algebraic spirals. What is observed is that from a cusp singularity the solution evolves into a spiral which behaves like an algebraic spiral, and therefore has infinite length. *The results of* [69] *provide an explanation of the fact that the spiral has to have infinite length.*

After the appearance of the first singularity the solution becomes very irregular. This leads to the issue of the definition of *weak solution* (a solution less regular than the threshold $\mathbf{T}$) *not of the Euler equations themselves but of the Birkhoff–Rott equation.* For instance, Wu [5], [6] proposed the following definition.

A weak solution is a function $\alpha \mapsto z(\alpha, t)$ from $\mathbb{R}$ into $\mathbb{C}$ such that

$$\partial_t\left(\int \bar{z}(\alpha, t)\eta(\alpha)\, d\alpha\right) = \frac{1}{4\pi i}\iint \frac{\eta(\alpha) - \eta(\beta)}{z(\alpha, t) - z(\beta, t)}\, d\alpha\, d\beta$$

for any $\eta \in C_0^\infty(\mathbb{R})$.

However, the problem is basically open, because we have no theorem on the existence of such a solution. Furthermore, for physical reasons weak solutions of the Birkhoff–Rott equation should provide weak solutions of the incompressible Euler equations, but in fact this is not always the case, as is illustrated by the Prandtl–Munk example (cf. [70]) with initial vortex sheet

$$\omega_0(x_1, x_2) = \frac{x_1}{\sqrt{1 - x_1^2}}\big(\chi_{(-1,1)}(x_1) \otimes \delta(x_2)\big), \tag{129}$$

where $\chi_{(-1,1)}$ is the characteristic function of the interval $(-1, 1)$. By the Biot–Savart law, the velocity v is constant:

$$v = \left(0, -\frac{1}{2}\right). \tag{130}$$

The solution of the Birkhoff–Rott equation is given by the formula

$$x_1(t) = x_1(0), \qquad x_2(t) = \frac{t}{2}, \qquad \omega(x_1, x_2, t) = \omega_0\left(x_1, x_2 + \frac{t}{2}\right).$$

On the other hand, it was observed in [70] that the velocity u associated with this vorticity is *not* even a weak solution of the Euler equations. In fact, one has

$$\nabla \cdot u = 0 \qquad \text{and} \qquad \partial_t u + \nabla_x \cdot (u \otimes u) + \nabla p = F, \tag{131}$$

where F is given by the formula

$$F = \frac{\pi}{8}\left(\delta\left(x_1 + 1, x_2 + \frac{t}{2}\right) - \delta\left(x_1 - 1, x_2 + \frac{t}{2}\right), 0\right). \tag{132}$$

This has led Lopes Filho, Nussenweig Lopes, and Schochet [71] to propose a weaker definition which contains more freedom with respect to the parameter and may be more adaptable.

The first author thanks the organizers of the Workshop on Mathematical Hydrodynamics held in Moscow in June 2006, where he had the opportunity to lecture on the subject. The authors are thankful for the kind hospitality of the Wolfgang Pauli Institute (WPI, Vienna) and the Bernoulli Centre of the École Polytechnique Fédéral de Lausanne, where part of this work was completed.

Bibliography

[1] S. Montgomery-Smith, "Finite time blow up for a Navier–Stokes like equation", *Proc. Amer. Math. Soc.* **129**:10 (2001), 3025–3029.

[2] H. Bellout, S. Benachour, and E. S. Titi, "Finite-time singularity versus global regularity for hyper-viscous Hamilton–Jacobi-like equations", *Nonlinearity* **16**:6 (2003), 1967–1989.

[3] T. Kato, "Remarks on zero viscosity limit for non stationary Navier–Stokes flows with boundary", *Seminar on nonlinear partial differential equations* (Berkeley, 1983) (S. S. Chern, ed.), Math. Sci. Res. Inst. Publ., vol. 2, Springer, New York 1984, pp. 85–98.

[4] G. Lebeau, "Régularité du problème de Kelvin–Helmholtz pour l'équation d'Euler $2d$", *ESAIM Control Optim. Calc. Var.* **8** (2002), 801–825.

[5] S. Wu, "Recent progress in mathematical analysis of vortex sheets", *Proceedings of the international congress of mathematicians.* vol. III. *Invited lectures* (Beijing, China, 2002), Higher Ed. Press, Beijing 2002, pp. 233–242.

[6] S. Wu, "Mathematical analysis of vortex sheets", *Comm. Pure Appl. Math.* **59**:8 (2006), 1065–1206.

[7] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts Appl. Math., vol. 27, Cambridge Univ. Press, Cambridge 2002.

[8] C. Marchioro and M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*, Appl. Math. Sci., vol. 96, Springer-Verlag, New York 1994.

[9] L. Lichtenstein, "Über einige Existenzprobleme der Hydrodynamik homogener, unzusammendrückbarer, reibungsloser Flüssigkeiten und die Helmholtzschen Wirbelsätze", *Math. Z.* **23**:1 (1925), 89–154; "Über einige Existenzprobleme der Hydrodynamik. Zweite Abhandlung. Nichthomogene, unzusammendrückbare, reibungslose Flüssigkeiten", *Math. Z.* **26**:1 (1927), 196–323; "Über einige Existenzprobleme der Hydrodynamik. III. Permanente Bewegungen einer homogenen, inkompressiblen, zähen Flüssigkeit", *Math. Z.* **28** (1928), 387–415; "Über einige Existenzprobleme der Hydrodynamik. IV. Stetigkeitssätze. Eine Begründung der Helmholtz–Kirchhoffschen Theorie geradliniger Wirbelfäden", *Math. Z.* **32**:1 (1930), 608–640.

[10] C. Bardos and S. Benachour, "Domaine d'analyticité des solutions de l'équation d'Euler dans un ouvert de $\mathbb{R}^n$", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **4**:4 (1977), 647–687.

[11] J. T. Beale, T. Kato, and A. Majda, "Remarks on the breakdown of smooth solutions for the 3-D Euler equations", *Comm. Math. Phys.* **94**:1 (1984), 61–66.

[12] A. Ferrari, "On the blow-up of solutions of the 3-D Euler equations in a bounded domain", *Comm. Math. Phys.* **155**:2 (1993), 277–294.

[13] H. Kozono, T. Ogawa, and Y. Taniuchi, "Navier–Stokes equations in the Besov space near L^∞ and BMO", *Kyushu J. Math.* **57**:2 (2003), 303–324.

[14] Y. Meyer, *Wavelets and operators*, vol. I, Cambridge Stud. Adv. Math., vol. 37, Cambridge Univ. Press, Cambridge 1992.

[15] G. Ponce, "Remarks on a paper "Remarks on the breakdown of smooth solutions for the 3-D Euler equations" by J. T. Beale, T. Kato and A. Majda", *Comm. Math. Phys.* **98**:3 (1985), 349–353.

[16] P. Constantin, C. Fefferman, and A. Majda, "Geometric constraints on potential singular solutions for the 3-D Euler equations", *Comm. Partial Differential Equations* **21**:3–4 (1996), 559–571.

[17] P. Constantin, "Euler equations, Navier–Stokes equations and turbulence", *Mathematical foundation of turbulent viscous flows* (Martina Franca, Italy, 2003) (M. Cannone and T. Miyakawa, eds.), Lecture Notes in Math., vol. 1871, Springer, Berlin 2006, pp. 1–43.

[18] В. И. Юдович, "Нестационарные течения идеальной несжимаемой жидкости", *ЖВМиМФ* **3** (1963), 1032–1066; English transl., V. I. Yudovich, "Non-stationary flows of an ideal incompressible fluid", *USSR Comput. Math. and Math. Phys.* **3** (1963), 1407–1456.

[19] J.-M. Delort, "Existence de nappes de tourbillon en dimension deux", *J. Amer. Math. Soc.* **4**:3 (1991), 553–586.

[20] V. Scheffer, "An inviscid flow with compact support in space-time", *J. Geom. Anal.* **3**:4 (1993), 343–401.

[21] A. Shnirelman, "On the nonuniqueness of weak solution of the Euler equation", *Comm. Pure Appl. Math.* **50**:12 (1997), 1261–1286.

[22] R. Caflisch and O. Orellana, "Singular solutions and ill-posedness for the evolution of vortex sheets", *SIAM J. Math. Anal.* **20**:2 (1989), 293–307.

[23] K. Ohkitani and J. D. Gibbon, "Numerical study of singularity formation in a class of Euler and Navier–Stokes flows", *Phys. Fluids* **12**:12 (2000), 3181–3194.

[24] T. Hou and C. Li, *Dynamic stability of the* 3*D axi-symmetric Navier–Stokes equations with swirl*, `arXiv: math/0608295`, 2006.

[25] J. Leray, "Étude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique", *J. Math. Pures Appl.* (9) **12** (1933), 1–82.

[26] J. Leray, "Essai sur les mouvements plans d'une liquide visqueux que limitent des parois", *J. Math. Pures Appl.* (9) **13** (1934), 331–418.

[27] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace", *Acta Math.* **63**:1 (1934), 193–248.

[28] E. Hopf, "Ein allgemeiner Endlichkeitssatz der Hydrodynamik", *Math. Ann.* **117**:1 (1941), 764–775.

[29] E. Hopf, "Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen", *Math. Nachr.* **4** (1951), 213–231.

[30] E. Hopf, "On nonlinear partial differential equations", *Lecture series of the symposium on partial differential equations* (Berkeley, 1955), Univ. Kansas Press, Lawrence 1957, pp. 1–29.

[31] О. А. Ладыженская, "Шестая проблема тысячелетия: уравнения Навье–Стокса, существование и гладкость", *УМН* **58**:2 (2003), 45–78; English transl., O. A. Ladyzhenskaya, "Sixth problem of the millennium: Navier–Stokes equations, existence and smoothness", *Russian Math. Surveys* **58**:2 (2003), 251–286.

[32] L. Caffarelli, R. Kohn, and L. Nirenberg, "Partial regularity of suitable weak solutions of the Navier–Stokes equations", *Comm. Pure Appl. Math.* **35**:6 (1982), 771–831.

[33] B. E. Launder and D. B. Spalding, *Lectures in mathematical models of turbulence*, Academic Press, London–New York 1972.

[34] B. Mohammadi and O. Pironneau, *Analysis of the k-epsilon turbulence model*, RAM Res. Appl. Math., Masson, Paris; Wiley, Chichester 1994.

[35] S. B. Pope, *Turbulent flows*, Cambridge Univ. Press, Cambridge 2000.

[36] J. S. Smagorinsky, "General circulation experiments with the primitive equations", *Monthly Weather Rev.* **91**:3 (1963), 99–164.

[37] L. Onsager, "Statistical hydrodynamics", *Nuovo Cimento* **6**:2 (suppl.) (1949), 279–287.

[38] G. L. Eyink, "Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer", *Phys. D* **78**:3–4 (1994), 222–240.

[39] P. Constantin, W. E, and E. S. Titi, "Onsager's conjecture on the energy conservation for solutions of Euler's equation", *Comm. Math. Phys.* **165**:2 (1994), 207–209.

[40] P.-L. Lions, *Mathematical topics in fluid mechanics. vol. 1. Incompressible models*, Oxford Lecture Ser. Math. Appl., vol. 3, Clarendon Press, Oxford 1996.

[41] P. Constantin and C. Foias, *Navier–Stokes equations*, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL 1988.

[42] G. David, "Courbes corde-arc et espaces de Hardy généralisés", *Ann. Inst. Fourier* (*Grenoble*) **32**:3 (1982), 227–239.

[43] C. Bardos, F. Golse, and D. Levermore, "Fluid dynamic limits of kinetic equations. I. Formal derivations", *J. Statist. Phys.* **63**:1–2 (1991), 323–344.

[44] Z. R. DiPerna and P.-L. Lions, "On the Cauchy problem for Boltzmann equations: global existence and weak stability", *Ann. of Math.* (2) **130**:2 (1989), 321–366.

[45] F. Golse and L. Saint-Raymond, "The Navier–Stokes limit of the Boltzmann equation for bounded collision kernels", *Invent. Math.* **155**:1 (2004), 81–161.

[46] L. Saint-Raymond, "Convergence of solutions to the Boltzmann equation in the incompressible Euler limit", *Arch. Ration. Mech. Anal.* **166**:1 (2003), 47–80.

[47] C. Cheverry, "Propagation of oscillations in real vanishing viscosity limits", *Comm. Math. Phys.* **247**:3 (2004), 655–695.

[48] C. Bardos, "Existence et unicité de la solution de l'équation d'Euler en dimension deux", *J. Math. Anal. Appl.* **40**:3 (1972), 769–790.

[49] P. Constantin and J. Wu, "The inviscid limit for non-smooth vorticity", *Indiana Univ. Math. J.* **45**:1 (1996), 67–81.

[50] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and Z. Xin, "Existence of vortex sheets with reflection symmetry in two space dimensions", *Arch. Ration. Mech. Anal.* **158**:3 (2001), 235–257.

[51] P. Gérard, P. Markowich, N. Mauser, and F. Poupaud, "Homogenization limits and Wigner transforms", *Comm. Pure Appl. Math.* **50**:4 (1997), 323–379.

[52] М. И. Вишик, А. В. Фурсиков, *Математические проблемы статистической механики*, Наука, М. 1980; English transl., M. Vishik and A. Fursikov, *Mathematical problems of statistical hydromechanics*, Math. Appl. (Soviet Ser.), vol. 9, Kluwer, Dordrecht 1988.

[53] C. Foias, O. Manely, R. Rosa, and R. Temam, *Navier–Stokes equations and turbulence*, Encyclopedia Math. Appl., vol. 83, Cambridge Univ. Press, Cambridge 2001.

[54] C. Foias, "What do the Navier–Stokes equations tell us about turbulence?", *Harmonic analysis and nonlinear differential equations* (Riverside, CA, USA, 1995), Contemp. Math., vol. 208, Providence, RI, Amer. Math. Soc. 1997, pp. 151–180.

[55] U. Frisch, *Turbulence. The legacy of A. N. Kolmogorov*, Cambridge Univ. Press, Cambridge 1995.

[56] E. Grenier, "On the nonlinear instability of Euler and Prandtl equations", *Comm. Pure Appl. Math.* **53**:9 (2000), 1067–1109.

[57] О. А. Олейник, "К математической теории пограничного слоя для нестационарного течения несжимаемой жидкости", *ПММ* **30**:5 (1966), 801–821; English transl., O. A. Oleinik, "On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid", *J. Appl. Math. Mech.* **30**:5 (1966), 951–974.

[58] W. E and B. Engquist, "Blowup of solutions to the unsteady Prandtl's equation", *Comm. Pure Appl. Math.* **50**:12 (1997), 1287–1293.

[59] W. E, "Boundary layer theory and the zero-viscosity limit of the Navier–Stokes equation", *Acta Math. Sin.* (*Engl. Ser.*) **16**:2 (2000), 207–218.

[60] K. Asano, "Zero-viscosity limit of the incompressible Navier–Stokes equations. II", *Mathematical analysis of fluid and plasma dynamics*, I (Kyoto, 1986), Sūrikaisekikenkyūsho Kōkyūroku, vol. 656, Kyoto Univ., Kyoto 1988, pp. 105–128.

[61] R. Caflisch and M. Sammartino, "Navier–Stokes equations on a circular domain: construction of the solution and zero viscosity limit", *C. R. Acad. Sci. Paris. Sér. I Math.* **324**:8 (1997), 861–866.

[62] M. Cannone, M. C. Lombardo, and M. Sammartino, "Existence and uniqueness for Prandtl equations", *C. R. Acad. Sci. Paris. Sér. I Math.* **332**:3 (2001), 277–282.

[63] C. Sulem, P.-L. Sulem, C. Bardos, and U. Frisch, “Finite time analyticity for the two and three dimensional Kelvin–Helmholtz instability”, *Comm. Math. Phys.* **80**:4 (1981), 485–516.

[64] J. Duchon and R. Robert, “Global vortex sheet solutions of Euler equations in the plane”, *J. Differential Equations* **73**:2 (1988), 215–224.

[65] D. W. Moore, “The spontaneous appearance of a singularity in the shape of an evolving vortex sheet”, *Proc. Roy. Soc. London. Ser. A* **365**:1720 (1979), 105–119.

[66] D. Meiron, G. Baker, and S. Orszag, “Analytic structure of vortex sheet dynamics. I. Kelvin–Helmholtz instability”, *J. Fluid Mech.* **114** (1982), 283–298.

[67] V. Kamotski and G. Lebeau, “On 2D Rayleigh–Taylor instabilities”, *Asymptot. Anal.* **42**:1–2 (2005), 1–27.

[68] R. Krasny, “Computation of vortex sheet roll-up in Trefftz plane”, *J. Fluid Mech.* **184** (1987), 123–155.

[69] S. Wu, “Well-posedness in Sobolev spaces of the full water wave problem in 2-D”, *Invent. Math.* **130**:1 (1997), 39–72.

[70] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and M. O. Souza, “On the equation satisfied by a steady Prandtl–Munk vortex sheet”, *Commun. Math. Sci.* **1**:1 (2003), 68–73.

[71] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and S. Schochet, *A criterion for the equivalence of the Birkhoff–Rott and Euler description of vortex sheet evolution*, **arXiv: math/0502215**, 2005.

[72] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 1st edn., Grundlehren Math. Wiss., vol. 325, Springer-Verlag, Berlin 2000.

C. Bardos
Université Paris 7 – Denis Diderot;
Université Pierre et Marie Curie, Paris, France
E-mail: bardos@ann.jussieu.fr

Received 02/APR/07
Translated by THE AUTHORS

E. S. Titi
University of California, Irvine, USA;
Weizmann Institute of Science, Rehovot, Israel
E-mail: etiti@math.uci.edu