Transport problems with gradient penalization

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International conference
Monge-Kantorovich optimal transportation problem, transport metrics and their applications

June 7, 2012

Outline

The more general formulation

Let be $\Omega \subset \mathbb{R}^d$ a bounded open set, $\mu \in \mathcal{P}(\Omega)$, $\nu \in \mathcal{P}(\mathbb{R}^d)$; we investigate the problem

$$\inf \left\{ \int_\Omega c(x, T(x), \nabla T(x)) \mathrm{d}\mu(x) \right\}$$

among the functions $T : \Omega \to \mathbb{R}^d$, ∇T being the Jacobian matrix of T, such that $T_{\#}\mu = \nu$.

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- link with the incompressible elasticity :
  - minimization of the stress tensor, quatratic in ∇T
  - the constraint involves $|\det \nabla T|$, which is equivalent to conditions on the image measure $T_\#\mu$ for regular and injective T

The quadratic case, if μ has a density f :

$$\inf \left\{ \int_\Omega \left(|T(x) - x|^2 + |\nabla T(x)|^2 \right) f(x) \mathrm{d}x \right\} \qquad (1)$$

If $0 < c \leq f \leq C < +\infty$, the constraint is

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Let $(T_n)_n$ be a minimizing sequence ; there exists $T \in H^1(\Omega)$ and $(n_k)_k$ such that

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$\Rightarrow$ The problem (1) admits at least one solution.

It is well-known (Brenier, 1987) that for the quadratic cost $c(x,y) = |y-x|^2$ and $\mu \ll \mathcal{L}^1$, the Monge problem admits a unique solution, which has the form $T = \nabla\phi$ where ϕ is convex.

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- the nondecreasing T such that $T_{\#}\mu = \nu$ is not optimal in general ;
- it is optimal if $\mu = \mathcal{L}^1$.

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Theorem (L.-Santambrogio '11)

Let $I \subset \mathbb{R}$ be a bounded interval. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be convex, nondecreasing, nonnegative. Let U, $T \in W^{1,1}(I)$ such that

- $\int_I f(|U'(x)|)\mathrm{d}x < +\infty$
- *T is nondecreasing and $T_{\#}\mathcal{L}^1 = U_{\#}\mathcal{L}^1$*

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Then $\int_I f(T'(x))\mathrm{d}x < +\infty$ with the inequality

$$\int_I f(|U'(x)|)\mathrm{d}x \geq \int_I f(n(x)T'(x))\mathrm{d}x \tag{2}$$

where $n(x) = \#U^{-1}(T(x))$, $x \in I$.

Inequality (2) : sketch of the proof

The proof is elementary if U is piecewise C^1 and monotone, using the formula

$$\frac{1}{T'(x)} = \sum_{y:U(y)=T(x)} \frac{1}{|U'(y)|}$$

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We generalize to the case T, $U \in W^{1,1}(I)$ considering a sequence $(U_k)_k$ of such functions, verifying more :

- $U_k \to U$ in $W^{1,1}(I)$ and $f \circ |U_k'| \to f \circ |U'|$ in $L^1(I)$
- the sequence of corresponding monotone transport maps T_k (i.e. such that $(T_k)_\# \mathcal{L}^1 = (U_k)_\# \mathcal{L}^1$) is uniformly convergent to T.

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We take the limit of the inequality at the rank k by semi-continuity and Γ-convergence techniques.

A counter-example in the non-Lebesgue case

We would like to get $\mu \in \mathcal{P}([0,1])$ and U, T with T nondecreasing, U non-injective, $T_{\#}\mu = U_{\#}\mu$ and the inequality (2) false.

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We take for U the triangle function :

$$U(x) = 2x \text{ on } [0,1/2],\ 1-2x \text{ on } [1/2,1]$$

and to each μ, we associate the unique T nondecreasing such that $T_{\#}\mu = U_{\#}\mu$.

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We take $\mu \ll \mathcal{L}^1$ with

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1} = \begin{cases} \alpha \text{ on } [0,1/4] \cup [3/4,1] \\ 1 \text{ otherwise} \end{cases}$$

(α will be fixed later, and μ has to be renormalized)

We compute $\nu = U_{\#}\mu$ and T. This gives $T' = \alpha$ on $[1 - \frac{1}{2\alpha}, 1]$, thus

$$\int_I T'^p \mathrm{d}\mu \geq \frac{\alpha^p}{2}$$

while $\int_I |U'|^p \mathrm{d}\mu = 2^p(\alpha + 1)$. Taking α large enough, the inequality (2) becomes false.

(This stays true if we consider $U \mapsto \int_I f(|U'|)$ with $f(x)/x \to +\infty$).

Introduction to the general case

We consider the functional

$$J : T \mapsto \int_I \left((T(x) - x)^2 + T'(x)^2 \right) \mathrm{d}\mu(x)$$

Problem : which is the suitable functional space X to consider the problem

$$\inf\{J(T) : T \in X, T_{\#}\mu = \nu\}?$$

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- If we do not assume μ to be regular, the condition $T \in L^2_\mu(I)$ does not guarantee the existence of T' even at the weak sense
- We should ideally get the implication

$$(T_n)_n \text{ bounded in } X \Rightarrow \exists T, (n_k)_k, T_{n_k} \to T \mu - \text{a.e.}$$

Notion of tangential gradient

In any dimension, let $u \in L^2_\mu(\Omega)$.

Definition (Bouchitté-Buttazzo-Seppecher, Zhikov)

We say $v \in L^2_\mu(\Omega)^d$ to be a gradient of u if :

$$\exists (u_n)_n \in \mathcal{D}(\Omega)^{\mathbb{N}} : \begin{cases} u_n \longrightarrow u \\ \nabla u_n \longrightarrow v \end{cases} \quad \text{in } L^2_\mu$$

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We denote by $\Gamma(u)$ these set. We call tangential gradient of u, and we denote by $\nabla_\mu u$, the element of $\Gamma(u)$ with minimal L^2_μ-norm. We denote by H^1_μ the space of $u \in L^2_\mu$ such that $\Gamma(u) \neq \emptyset$.

Definition

There exists $x \mapsto T_\mu(x)$ a multifunction, called tangent space to μ such that, for $v \in (L^2_\mu)^d$, we have the equivalence :

$$v \in \Gamma(0) \Leftrightarrow v(x) \perp T_\mu(x) \text{ for } \mu\text{-a.e. } x$$

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Then for $u \in H^1_\mu$ *and* $v \in \Gamma(u)$ *we have :*

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Examples :

- if μ is uniform on $[0,1] \times \{0\}^{d-1}$, $\nabla_\mu u = \left(\frac{\partial u}{\partial x_1}, 0, ..., 0\right)$
- if M is a k-dimensional manifold and $\mu = \mathcal{H}^k|_M$, then $T_\mu = T_M$.

Caracterization in dimension 1

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We show that :

$$T_\mu(x) = \begin{cases} \{0\} \text{ if } x \in M \cup A \\ \mathbb{R} \text{ otherwise} \end{cases}$$

Arguments for the caracterization

$T_\mu = \mathbb{R}$ outside of $M \cup A$: we want :

$$\begin{pmatrix} u_n \to 0 \\ u_n' \to v \end{pmatrix} \quad \Rightarrow \quad v = 0 \quad \mathcal{L}^1 - \text{a.e. outside of } M$$

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If J verifies $\int_J (1/f) < +\infty$ and $\phi \in \mathcal{D}(J)$:

$$\left| \int_J u_n' \phi \right| \leq \left(\int_J u_n^2 f \right)^{1/2} \left(\int_J \frac{(\phi')^2}{f} \right)^{1/2} \to 0$$

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$T_\mu = \{0\}$ on M : we want :

$$(\forall u \in H^1_\mu) \quad (\exists v, (u_n)_n) \quad \left(\begin{array}{l} u_n \longrightarrow u \\ u_n' \longrightarrow v \\ v\big|_M = 0 \end{array} \right)$$

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For any interval J containing an element of M, we have $\int_J 1/f = +\infty$ and the injection $L^2_\mu \hookrightarrow L^1$ is false ; thus

$$\inf \left\{ \int_J |v'|^2 f : v = u \text{ at the bounds of } J \right\} = 0 \ ;$$

we use this proprety to approach u in L^2_μ by regular functions which the derivatives on M are arbitrary small (for the L^2_μ-norm).

Corollary

The problem

$$\inf \left\{ \int_I \left((T(x) - x)^2 + (\nabla_\mu T(x))^2 \right) \mathrm{d}\mu(x) : T \in H^1_\mu(I),\, T_\# \mu = \nu \right\}$$

has at least one solution.

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The problem

$$\inf\left\{\int_I \left((T(x)-x)^2+(\nabla_\mu T(x))^2\right) \mathrm{d}\mu(x) : T\in H^1_\mu(I),\, T_\#\mu=\nu\right\}$$

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Idea : let $(T_n)_n$ be a minimizing sequence.

- Outside of $M\cup A$, we have the injection $L^2_\mu \hookrightarrow L^1_{loc}$ and thus $H^1_\mu \hookrightarrow BV \Rightarrow$ convergence μ-a.e.
- On $M\cup A$, $\nabla_\mu T=0$. We substitute T_n by the nondecreasing map which maps $\mu|_{M\cup A}$ on the same measure.
- We verify that the limit function satisfies $T_\#\mu=\nu$.

Partial results in any dimension

- We still have $T_\mu = \mathbb{R}^d$, a.e. for the regular part of μ, outside of the set

$$M = \left\{ x \in \Omega : \forall \varepsilon > 0, \int_{B(x,\varepsilon)\cap\Omega} \frac{1}{f} = +\infty \right\}$$

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- We can build f with $\int 1/f = +\infty$ on any open set, but

$$\inf \left\{ \int_\Omega |\nabla u|^2 f \,:\, u = \phi \text{ on } \partial\Omega \right\} > 0$$

  The construction that we performed in the one-dimensional case to get $T_\mu = 0$ on M does not work if $d \geq 2$ in that case
- However, it is possible to show that $T_\mu = \{0\}$ on each atom of the measure μ

More or less short term :

- precise description of T_μ in any dimension ;
- result of "pointwise compactness" in H^1_μ ;
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More longer term :

- optimality conditions on T ;
- behavior with respect to the measure μ ; link between

$$\inf\{||T(x) - x||_{H^1_\mu} : T_\# \mu = \nu\}$$

and a "Benamou-Brenier" formulation

$$\inf\left\{\int_0^1 ||v_t||^2_{H^1(\rho_t)} \mathrm{d}t : \rho_0 = \mu, \rho_1 = \nu, \partial_t \rho_t + \mathrm{div}(\rho_t v_t) = 0\right\}$$

Thank you for your attention !

Спасибо за внимание !