Mass Transportation and Disintegration Maps

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1. L. Granieri, F. Maddalena, A Metric Approach to Elastic Reformations

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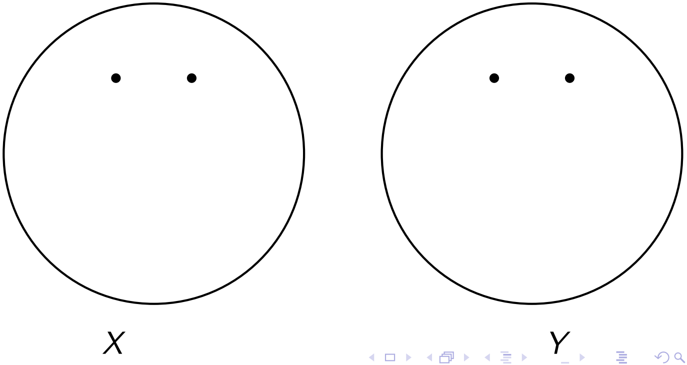

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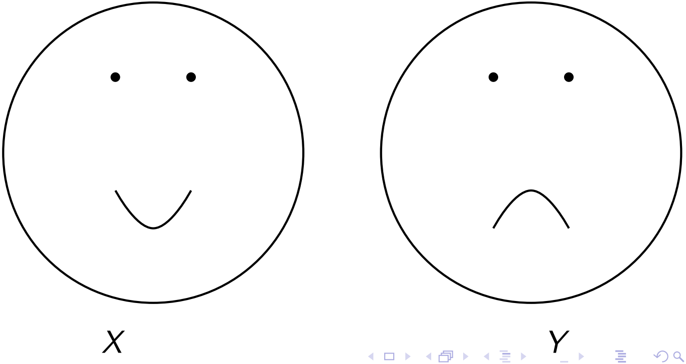

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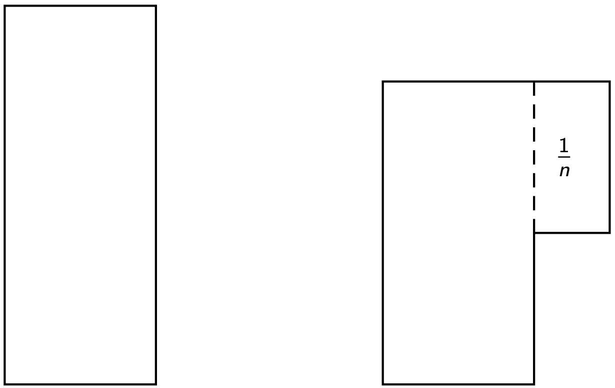
Therefore, the two shapes X, Y could be considered close to be *isometric* as the bi-Lipschitz constant L is close to one, so assuming the bi-Lipschitz constant as a quantifier of the closeness to the isometry.

This global approach has some disadvtanges. For instance, the shapes below



looks very close to be isometric but the bi-Lipschitz constant is quite large and far from $L = 1$, whatever the size of the bending part.

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Actually, under some regularity assumptions, by Liouville Rigidity Theorem the orthogonality of the Jacobian matrix characterizes the isometric maps.
Hence, a reasonable way to quantify how two shapes are isometric is that of measuring how ∇u is close to be an orthogonal matrix.

A Variational Approach

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This approach is pursued in

G. Wolansky, Incompressible, Quasi-Isometric Deformations of 2-Dimensional Domains, SIAM J. Imaging Sciences, **2**, No. 4 (2009), 1031-1048

where the admissible maps are incompressible diffeomorphisms, i.e. u such that $\det(\nabla u) = 1$.

In order to characterize the isometries, a polyconvex function W having minimal value at orthogonal matrices is selected.

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$$\text{minimize} \left\{ \int_{\Omega_1} W(\nabla u)\ dx \mid u(\Omega_1) = \Omega_2,\ u \in \mathcal{D} \right\},$$

where $\mathcal{D}$ denotes the set of incompressible diffeomorphisms.

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A main goal of our approach relies in exploiting possible extensions of this variational scheme of elasticity in order to compare more general shapes also allowing *fragmentations*.

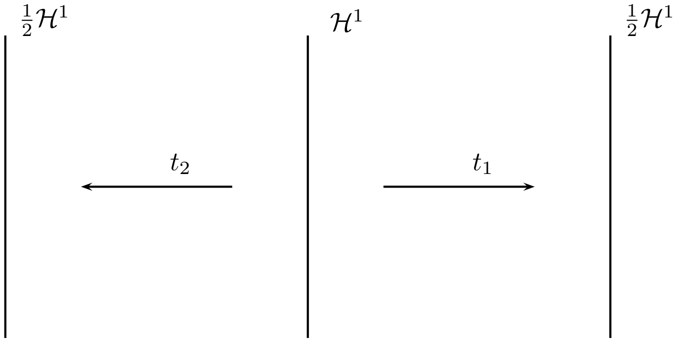

Figure: An isometric fractured reformation.

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The mass conservation property (1) is a generalized version of incompressibility and it can be always satisfied (provided μ has no atom) by some measurable map u. Actually, condition (1) is equivalent to the following change of variable formula

$$\int_X f(u(x)) \ d\mu = \int_Y f(y) \ d\nu, \tag{2}$$

for every continuous function $f : Y \to \mathbb{R}$.

Measurable is too weak

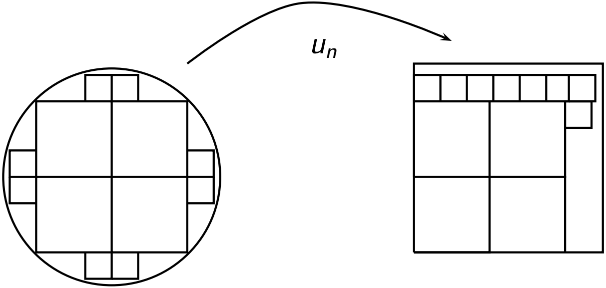

Figure: A piece-wise isometric map for the circle into a square.

A metric Formulation

The Lipschitz constant can be *localized* by

$$e_u(x_0) := \mathrm{Lip}(u)(x_0) = \limsup_{x \to x_0} \frac{|u(x) - u(x_0)|}{|x - x_0|}. \tag{3}$$

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Observe that Wolansky's approach cannot be pursued in a metric framework. Indeed the mapping $A \mapsto \varphi(\|A\|)$ is polyconvex only if φ is a positive convex and strictly increasing function, therefore the minimal value cannot be reached at orthogonal matrices A, since they have $\|A\| = 1$.

Reformation Maps

Similarly, we introduce the pointwise contraction energy of u at x_0 defined by

$$c_u(x_0) := \limsup_{x \longrightarrow x_0} \frac{|x - x_0|}{|u(x) - u(x_0)|}. \tag{4}$$

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The pointwise reformation energy of u at x_0 is defined by

$$r_u(x_0) = e_u(x_0) + c_u(x_0). \tag{5}$$

Definition (Reformation maps)

We shall call *reformation map* any map $u : X \to Y$ such that the following conditions hold true:

$$u_{\#}\mu = \nu, \tag{6}$$

$$\forall x \in X \;\; \exists\, H, K, r > 0 \text{ s.t. } e_u(y) \le K, \;\; c_u(y) \le H \tag{7}$$

$\forall y \in \overline{B}(x, r) \cap X$.

We shall denote by $\mathrm{Ref}(\mu; \nu)$ the set of reformation maps between μ and ν.

By the bounds (7), any $u \in \mathrm{Ref}(\mu;\nu)$ is continuous and, by Stepanov Theorem, for $X = \overline{\Omega}$, it is a.e. differentiable in Ω.

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In a mechanical perspective, the constraints stated in (7) could be considered as a bound on the maximum expansion or contraction experienced by the material Ω. In this setting, the assumption that of the constants H, K do not depend on the map u in (7) corresponds to a constitututive property of the material under consideration.

We point out that some bounds as in (7) are in some sense necessary to control the geometry of the reformations. For instance, in the case of $\nu = \delta_{y_0}$ we have $e_u = 0$, $c_u = +\infty$ for any map u satisfying (6).

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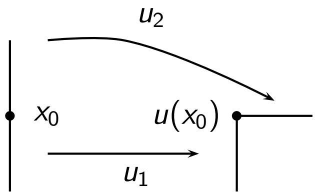

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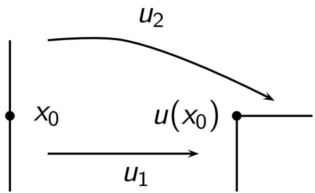

On the other hand, mapping a bar into a bended one (see Fig. 16) by two piecewise isometries u_1, u_2, we have $e_u(x_0) = +\infty$ at the discontinuity point.
Therefore, roughly speaking, the bound $c_u \leq H$ means no collapsing, while $e_u \leq K$ means no fractures.

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in a purely metric framework, such pointwise conditions are not enough to guarantee inversion properties. Consider for instance the map $u : \mathbb{R} \to \mathbb{R}, u(x) = |x|$ having $e_u = c_u = 1$ at every point.

We have the following inversion results

Theorem (Small reformations are invertible)

Let $u \in \mathrm{Ref}(\mu; \nu)$ be such that $e_u < \sqrt[N]{2}$. Then u is globally invertible.

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Let $u \in \mathrm{Ref}(\mu;\nu)$ be an open map such that $HK < 2$. Then $u_{|\Omega}$ is locally invertible.

Key tools: area formula and degree theory for maps in $\mathbb{R}^N$.

The Variational problem of Elastic Reformation

We define the total reformation energy $\mathscr{R}(u)$ of a reformation map u of μ into ν as follows

$$\mathscr{R}(u) := \int_X r_u(x)\, d\mu.$$

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Moreover, we have that $r_u(x)$ reaches its minimum value if $u : X \to Y$ is an isometric mapping. Therefore $\mathscr{R}(u)$ can be viewed as a measure detecting how u is far from being an isometric map.

Moreover, conversely assume $r_u(x_0) = 2$, then

$$2 = e_u(x_0) + c_u(x_0) \geq e_u(x_0) + \frac{1}{e_u(x_0)} \geq 2,$$

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taking $x = x_0 + \delta v$ and sending $\delta \to 0$, we get

$$c_u(x_0) = e_u(x_0) = 1 \Rightarrow \frac{|\nabla u(x_0) \cdot v|}{|v|} = 1 \Rightarrow \nabla u(x_0) \in O(N).$$

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Therefore, a Rigidity theorem (in Sobolev spaces) could be applied

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We expect to characterize isometric maps as those having the smallest reformation energy. The question is now to establish conditions in order the infimum is attained.

Theorem

Let $\mu \in \mathcal{P}(\Omega)$ *and* $\nu \in P(Y)$ *so that* $\mu = \mathcal{L}^N \llcorner \Omega$, $\nu = \mathcal{L}^N \llcorner Y$. *Then the variational problem*

$$\mathrm{minimize}\{\mathscr{R}(u) \mid u \in \mathrm{Ref}(\mu;\nu),\ e_u < \sqrt[N]{2}\} \tag{8}$$

admits solutions whenever $\{u \in \mathrm{Ref}(\mu;\nu), e_u < \sqrt[N]{2}\} \neq \emptyset$.

Sketch of the proof

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$$\liminf_{n\to+\infty} \left(\int_X \mathrm{Lip}(u_n)(x) d\mu + \int_Y \mathrm{Lip}(u_n^{-1})(y)\ d\nu \right) = \liminf_{n\to+\infty} \mathscr{R}(u_n).$$

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$$\liminf_{n\to+\infty} \left(\int_X \mathrm{Lip}(u_n)(x) d\mu + \int_Y \mathrm{Lip}(u_n^{-1})(y)\ d\nu \right) = \liminf_{n\to+\infty} \mathscr{R}(u_n).$$

We also obtain existence for the variational problem over the set $\{u \in \mathrm{Ref}(\mu;\nu), u \text{ incompressible}\}$ or the set $\{u \in \mathrm{Ref}(\mu;\nu), u \text{ open } s.t.\ HK < 2\}$.

Since $c_u(x) = e_{u^{-1}}(u(x))$, we compute

$$\mathscr{R}(u) = \int_X (e_u + c_u)\, d\mu = \int_X (e_u(x) + e_{u^{-1}}(u(x)))\, d\mu =$$

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This proof could be considered also in a metric framework.

We then characterize isometric measures by the following

Theorem

Let $\mu \in \mathcal{P}(\Omega)$ and $\nu \in \mathcal{P}(Y)$, so that $\mu = \mathcal{L}^N \llcorner \Omega$, $\nu = \mathcal{L}^N \llcorner Y$, for a given bounded set Y. Then, $\mathcal{E}(\mu, \nu) = 2$ if and only if there exists an isometry u such that $u_{\#}\mu = \nu$.

Generalized Reformations

The notion of reformation map corresponds to the notion of the so-called transport map, i.e. $u : X \to Y$ such that $u_{\#}\mu = \nu$.

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Disintegrations

The key tool is the following

Theorem (Disintegration theorem)

Let $\gamma \in \mathcal{P}(X \times Y)$ be given and let $\pi^1 : X \times Y \to X$ be the first projection map of $X \times Y$, we set $\mu = (\pi^1)_\# \gamma$. Then for $\mu -$ a.e. $x \in X$ there exists $\nu_x \in \mathcal{P}(Y)$ such that

(i) *the map $x \mapsto \nu_x$ is Borel,*

(ii) $\forall \varphi \in \mathcal{C}_b(X \times Y) : \int_{X \times Y} \varphi(x, y) d\gamma = \int_X \left(\int_Y \varphi(x, y) d\nu_x(y) \right) d\mu(x).$

Moreover the measures ν_x are uniquely determined up to a negligible set with respect to μ.

As usual we will write $\gamma = \nu_x \otimes \mu$, assuming that ν_x satisfy the condition (i) and (ii) of Disintegration Theorem.

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Definition

Let $\mu, \nu \in \mathcal{P}(X)$, the 1-Wasserstein distance between μ and ν is defined by

$$W(\mu, \nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \int_X d(x, y)\, d\gamma(x, y). \tag{9}$$

Finding reformation plans

In the following examples we show that it is possible to compare shapes with regular disintegration maps despite no regular transport map does exist.

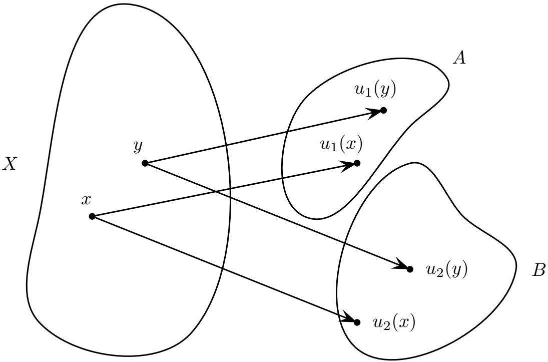

Figure: A disconnected target reformation

By results of Dacorogna-Moser, we find two diffeomorphisms $u_1 : X \to A$, $u_2 : X \to B$ so that $|\det(\nabla u_1)| = \mathcal{L}^N(A)$, $|\det(\nabla u_2)| = \mathcal{L}^N(B)$.

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$$W(\nu_x, \nu_{x_0}) = \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)|.$$

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Since u_1, u_2 are diffeomorphisms, we find constants $K_{1,2}, H_{1,2}, H, K$ such that

$$\frac{1}{H}|x - x_0| \leq \frac{\mathcal{L}^N(A)}{H_1}|x - x_0| + \frac{\mathcal{L}^N(B)}{H_2}|x - x_0| \leq$$

$$\mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)| =$$

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Namely, for any reformation plan $\gamma = \nu_x \otimes \mu$ of μ into ν we define the pointwise expansion energy

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and the pointwise compression energy

$$c_\gamma(x_0) = \limsup_{x \to x_0} \frac{|x - x_0|}{W(\nu_x, \nu_{x_0})}. \tag{12}$$

Since $W(\delta_x, \delta_y) = d(x, y)$, for $\gamma = (I \times u)_{\#}\mu$ we have

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We define the reformation energy of γ as follows

$$\mathscr{R}(\gamma) = \int_X (e_\gamma + c_\gamma)\, d\mu. \tag{13}$$

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We prove the following

Theorem

Let $\gamma = f(x) \otimes \mu \in \mathrm{GRef}(\mu;\nu)$ *be such that* $\mathscr{R}(\gamma) = 2$, μ *absolutely continuous with respect to the Lebesgue measure. Then there exists an open dense subset of* X *on which the disintegration map* f *is a local isometry (with respect to the Wasserstein distance).*

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Definition

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$$\forall x_0 \in X : \ \exists\, r > 0, H, K \ s.t. \ e_\gamma(x) \leq K, \ c_\gamma(x) \leq H \qquad (14)$$

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Definition

Let us define the set of small reformation plans between μ and ν as follows

$$\mathrm{GRef}_0(\mu,\nu) = \{\gamma \in \Pi(\mu,\nu) \mid e_\gamma \leq K, c_\gamma \leq H, HK < \sqrt[N]{2}\}. \quad (15)$$

By the metric area formula, small reformations yields globally invertible disintegration maps.

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$$\mathcal{E}_G(\mu, \nu) = \mathrm{inf}\{\mathscr{R}(\gamma) \mid \gamma \in \mathrm{GRef}_0(\mu; \nu)\}. \tag{16}$$

We prove the following

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We prove the following

Theorem

If $\mathcal{E}_G(\mu,\nu) = 2$, with μ absolutely continuous with respect to the Lebesgue measure, then the infimum is attained at a local isometric reformation plan.

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Lemma

Let $u, v : X \to Y$ *be two given Borel maps,* $\mu \in \mathcal{P}(X)$ *and let* $f, g : X \to \mathcal{P}(Y)$ *defined by* $f(x) = \delta_{u(x)}$, $g(x) = \delta_{v(x)}$. *Then*

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Corollary

Let $u, v : X \to Y$ be two given Borel maps, $\mu \in \mathcal{P}(X)$, let $f, g : X \to \mathcal{P}(Y)$ defined by $f(x) = \delta_{u(x)}$, $g(x) = \delta_{v(x)}$ and let $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$. Then

$$\pi^2_{\#}\gamma = \pi^2_{\#}\eta \Leftrightarrow f_{\#}\mu = g_{\#}\mu. \tag{18}$$

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$$f_{\#}\mu = g_{\#}\mu \Rightarrow \pi^2_{\#}\gamma = \pi^2_{\#}\eta. \tag{19}$$

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part of the proof of the above Lemma works for general transport plans $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$, yielding the following

Lemma

Let $\mu \in \mathcal{P}(X)$, $f, g : X \to \mathcal{P}(Y)$, $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$ be given. Then the following implication holds true

$$f_{\#}\mu = g_{\#}\mu \Rightarrow \pi^2_{\#}\gamma = \pi^2_{\#}\eta. \tag{19}$$

Therefore, also for transport plans, the second marginal can be fixed by fixing the push forward of disintegration maps.in general the converse of (19) is not true as for $f : X \to \mathcal{P}(Y)$ defined by $f(x) = \nu$ and $\gamma = f(x) \otimes \mu$. Let $\eta = g(x) \otimes \mu$ where $g(x) = \delta_{u(x)}$ for a given transport map $u : X \to Y$ with $u_{\#}\mu = \nu$

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Let $\gamma, \eta \in \Pi(\mu, \nu)$ with $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$ be given. We shall say that γ and η are equivalent (by disintegration), in symbols $\gamma \approx \eta$, if $f_{\#}\mu = g_{\#}\mu$.
For any $\eta \in \Pi(\mu, \nu)$ with $\eta = g(x) \otimes \mu$, we shall call transport class any equivalence class of a transport plan η and it will be denoted by $[\eta]$, i.e.

$$[\eta] = \{\gamma = f(x) \otimes \mu \mid f_{\#}\mu = g_{\#}\mu\}. \tag{20}$$

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it follows that all transport plans induced by transport maps belong to the same transport class.

To better explain the notion of transport class, consider the case of a discrete first marginal $\mu = \sum_i \alpha_i \delta_{x_i}$.

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Therefore, transport classes are fixed by the range of f.

Consider the discrete marginals

$$\mu = \frac{1}{3}\delta_{x_1} + \frac{1}{3}\delta_{x_2} + \frac{1}{3}\delta_{x_3}, \quad \nu = \frac{1}{6}\delta_{y_1} + \frac{5}{6}\delta_{y_2}.$$

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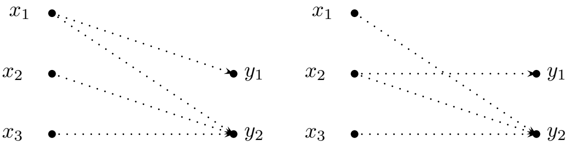

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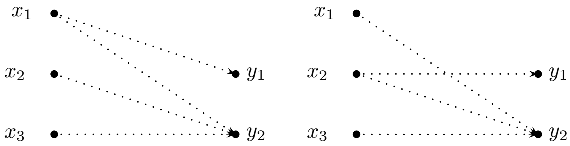

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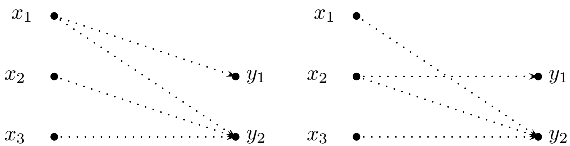

The Above transport plans belong to the same transport class since

$$f(x_1) = 3(a\delta_{y_1} + b\delta_{y_2}), \quad f(x_2) = \delta_{y_2}, \quad f(x_3) = \delta_{y_2}, \quad a = b = \frac{1}{6}.$$

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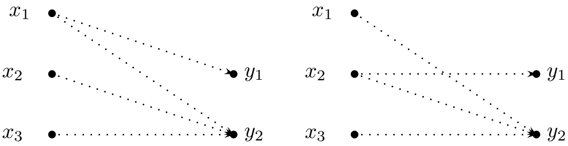

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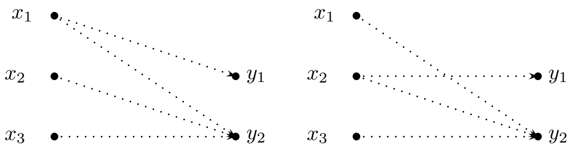

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Hence, all the transport plans with one mass splitted belong to the same transport class.

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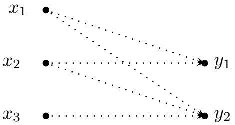

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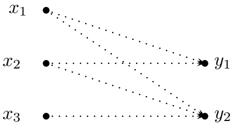

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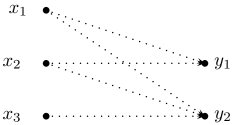

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$$a'' = \frac{1}{30},\ b' = \frac{9}{30},\ c' = \frac{4}{30},\ d' = \frac{6}{30}.$$

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Proposition

Let $u : X \to Y$, be a transport map, i.e. such that $u_{\#}\mu = \nu$, with μ non-atomic, and let $\eta = (I \times u)_{\#}\mu = \delta_{u(x)} \otimes \mu$. If $\gamma \in [\eta]$ then there exists a transport map $v : X \to Y$ such that $\gamma = \delta_{v(x)} \otimes \mu$, i.e. the transport plan γ is concentrated on the graph of v.

Monge transport problem can be reformulated as follows

$$\text{Minimize}\left\{\int_X c(x,u(x))d\mu \;:\; u_{\#}\mu=\nu\right\}=$$

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for a transport map v. Therefore, Monge problem correspond to minimization of the functional $\int c \ d\gamma$ in a fixed transport class of $\Pi(\mu,\nu)$, while the Kantorovich one corresponds to the minimization on the whole $\Pi(\mu,\nu)$.

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$$\int_{\mathcal{P}(Y)} I_\varphi(\lambda)\, d\Lambda(\lambda) = \int_{\mathcal{P}(Y)} \left(\int_Y \varphi(y)\, d\lambda \right) d\Lambda.$$

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$$MK_{\Lambda}(c, \mu, \nu) := \inf_{\gamma} \left\{ \int_{X \times Y} c(x, y) d\gamma \;\; : \;\; \gamma = f \otimes \mu, f_{\#}\mu = \Lambda \right\} \tag{22}$$

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Proposition

For every transport class Λ we have

$$M(\tilde{c},\mu,\Lambda) = MK_\Lambda(c,\mu,\nu).$$

Proof.

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Observe that by the above proof it follows that f is a solution of $M(\tilde{c}, \mu, \Lambda)$ if and only if $f \otimes \mu$ is a solution of $MK_{\Lambda}(c, \mu, \nu)$.

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Observe that by the above proof it follows that f is a solution of $M(\tilde{c}, \mu, \Lambda)$ if and only if $f \otimes \mu$ is a solution of $MK_\Lambda(c, \mu, \nu)$. Therefore, every existence result for the Monge problem $M(\tilde{c}, \mu, \Lambda)$ in the abstract setting corresponds to an existence result for the Monge-Kantorovich problem in the transport class Λ

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Theorem

(Existence of optimal reformation plans) Let $\eta \in \mathrm{GRef}_0(\mu;\nu)$ *be given. Then, for every* $p > 1$ *the variational problem*

$$\mathrm{minimize}_{\mathrm{GRef}_0(\mu;\nu)} \left\{ \mathscr{R}^p(\gamma) := \int_X (c_\gamma^p + e_\gamma^p) d\mu \mid \gamma \in [\eta] \right\} \tag{24}$$

admits solutions.

Sketch of the proof

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Thanks for the Attention