Anisotropic congested transport

Lorenzo Brasco

LATP – Aix-Marseille Université
lbrasco@gmail.com — http://www.latp.univ-mrs.fr/~brasco/

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References

Part of the results here presented are contained in

1. L. B., G. Carlier, F. Santambrogio, *On certain anisotropic elliptic equations arising in optimal transport: local gradient bounds*, 90% completed

Outline

Introduction to the problem and goal of the talk

Some continuous models

Equilibrium issues

Regularity results

Congested transport: introduction

From an overall point of view...

Optimal Transport Problem, where the *infinitesimal cost* obeys

"*spreading the mass during the transport, we save cost*"

in a point, if our transport accumulates an amount of mass m, we pay

$$H(m) \qquad \text{where} \qquad H \textbf{ convex and superlinear } \text{(given)}$$

The **total cost** is something of the type $\displaystyle\int H(m(x))dx$

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$H(m)$ where H **convex and superlinear** (given)

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...and from an individual one

A **game** with many players going from certain **sources** to their **destinations** using a system of roads

"*travel time on a road increasingly depends on the traffic* "

i.e. my satisfaction is affected by **choices of the other players**

Anisotropic transport costs

The typical costs we will consider are of the form

$$\text{(C)} \qquad H(m) = H_1(m_1) + \cdots + H_N(m_N)$$

where m_i = mass transported in direction $\mathbf{e}_i$

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Motivation

The model we are going to present is a **continuous version** of a classical discrete model settled on **networks**. Question: do the discrete models "converge" to the continuous one, for very dense networks?

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Motivation

The model we are going to present is a **continuous version** of a classical discrete model settled on **networks**. Question: do the discrete models “converge” to the continuous one, for very dense networks? **Yes**, if the network is a regular grid of size $\varepsilon \ll 1$, with a cost that at each node distinguish between the mass entering with different directions (Baillon-Carlier)

The limit continuous model has a cost of the form (C), which “keeps memory” of the geometry of the approximating problems

Goals of the talk

From an overall point of view...

Prove **existence** of an **optimal transport**, i.e. existence of a way to accomplish the transport which minimizes the total cost and show **regularity properties** for this optimizer.

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...and from an individual one

Prove **existence** of **equilibrium** situations, i.e. existence of configurations where players have no interest in changing unilaterally their choice, in order to avoid congested routes

A continuous model for congested transport

We start with the overall optimization point of view

Data of the problem

- a "city" $\Omega \subset \mathbb{R}^N$
- $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$ probability measures
- "admissible couplings" (transport plans)

$$\Pi \subset \Pi(\rho_0, \rho_1) = \{\gamma \in \mathcal{P}(\Omega \times \Omega) \ : \ (\pi_x)_\# \gamma = \rho_0, \ (\pi_y)_\# \gamma = \rho_1\}$$

- a **density-cost function** $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^+$ smooth

$$\mathcal{H}(z) = H_1(z_1) + \cdots + H_N(z_N)$$

with H_i strictly convex, $H_i(0) = 0$ and $H_i(t) \simeq |t|^p$, for $p > 1$

The cost of transportation

Unknown of the problem: traffic assignments

$Q \in \mathcal{P}(\mathrm{Lip}([0,1];\Omega))$ such that $(e_0, e_1)_{\#}Q \in \Pi$

where $e_t(\sigma) = \sigma(t)$ for every curve σ

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Each Q gives rise to a **traffic intensity**

$$i_Q = (i_{Q,1}, \dots, i_{Q,N})$$

positive vector measure defined on Ω by

$$\int_\Omega \varphi(x)\, di_{Q,j}(x) = \int_{Lip([0,1];\Omega)} \left(\int_0^1 \varphi(\sigma(t))\, |\sigma_j'(t)|\, dt \right) dQ(\sigma)$$

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The problem

Total cost $= \int_\Omega \mathcal{H}(i_Q(x))\, dx$ if $i_Q \ll \mathscr{L}^N$ and $+\infty$ otherwise

A pair of (not congested) example

Anisotropic, not congested

If we take the density-cost $\mathcal{H}(z) = |z_1| + \cdots + |z_N|$ then

$$\textbf{Total cost} = \int_\Omega d\|i_Q\|_{\ell^1} = \int \mathrm{length}_{\ell^1}(\sigma)\, dQ(\sigma)$$

and the minimization is equivalent to Monge problem with cost $c(x, y) = \|x - y\|_{\ell^1}$

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Neither anisotropic, nor congested

The traffic intensity i_Q is a scalar measure, if we take $\mathcal{H}(z) = |z|$

$$\textbf{Total cost} = \int_\Omega d\, i_Q = \int \mathrm{length}(\sigma)\, dQ(\sigma)$$

and we are back to the standard Monge problem with cost $c(x, y) = |x - y|$. The optimal i_Q is given by the **transport density**

Existence of an optimal transport

Theorem (Carlier-Jimenez-Santambrogio)

The problem

$$(\mathcal{W}) = \min \left\{ \int_\Omega \mathcal{H}(i_Q)\, dx \;:\; Q \quad s.t. \quad (e_0, e_1)_\# Q \in \Pi\,,\; i_Q \in L^p \right\}$$

admits a solution $\widetilde{Q}$

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Sketch of the proof:

- for a minimizing sequence

$$C \geq \int_\Omega \mathcal{H}(i_{Q_n}) \gtrsim \int_\Omega |i_{Q_n}| \simeq \int \mathrm{length}(\sigma)\, dQ_n(\sigma)$$

- up to a time reparametrization, $\{Q_n\}_{n\in\mathbb{N}}$ is compact and each i_{Q_n} is unchanged
- the weak L^p limit of i_{Q_n} is “greater” than i_Q and $\mathcal{H}$ is “increasing”

For the case $\Pi = \Pi(\rho_0, \rho_1)$

a more comfortable formulation is available...

Beckmann's continuos model of trasportation

Transportation activities are described by $\Phi : \Omega \to \mathbb{R}^N$, s.t.

- $|\Phi(x)| =$ amount of **mass** passing from x
- $\Phi(x)\,|\Phi(x)|^{-1} =$ **direction** of transportation in x
- $\mathrm{div}\,\Phi = \rho_0 - \rho_1$, i.e. the transport is ruled by the balance demand/offer
- $\mathcal{H}(\Phi) =$ cost for transporting $|\Phi|$, with direction $\Phi/|\Phi|$

Beckmann's Optimization problem

$$(\mathcal{B}) = \min_{\Phi \in L^p} \left\{ \int_\Omega \mathcal{H}(\Phi(x))\,dx \;:\; \mathrm{div}\,\Phi = \rho_0 - \rho_1,\ \langle \Phi, \nu_\Omega \rangle = 0 \right\}$$

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For $\mathcal{H}(z) = |z|$, this is the dual formulation of Kantorovich problem $\max_{\varphi\ 1-Lip} \langle \varphi, \rho_0 - \rho_1 \rangle$

What is the relation between Beckmann's model and the previous one?

The two models are equivalent

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Restriction We consider the case $\Pi = \Pi(\rho_0, \rho_1)$

Theorem (B.-Carlier-Santambrogio)

Let $\Omega \subset \mathbb{R}^N$ bounded with smooth boundary. Assume that:

- $\rho_i = f_i \cdot \mathscr{L}^N$, *with* $f_i \in L^p(\Omega)$.

Then we have

$$(\mathcal{W}) = (\mathcal{B})$$

and for every optimal Q, we can construct an optimal Φ with the same cost (and viceversa)

Proof of the equivalence: $(\mathcal{B}) \le (\mathcal{W})$

Given Q optimal, construct a vector field Φ_Q such that

$$\langle \varphi, \Phi_Q \rangle = \int_{Lip([0,1];\Omega)} \left(\int_0^1 \langle \varphi(\sigma(t)), \sigma'(t) \rangle \, dt \right) dQ(\sigma)$$

then

- Φ_Q is admissibile for $(\mathcal{B})$
- $|\Phi_{Q,j}(x)| \le i_{Q,j}(x)$ (since the traffic defined in a vectorial way allows for some "mass cancellations")
- $\mathcal{H}$ is increasing in each variable

so in conclusion

$$(\mathcal{B}) \le \int_\Omega \mathcal{H}(\Phi_Q) \le \int_\Omega \mathcal{H}(i_Q) = (\mathcal{W})$$

Proof of the equivalence: $(\mathcal{B}) \geq (\mathcal{W})$

Idea

If Φ optimal, construct Q_Φ by following the flow of $\lambda_t \Phi$ for a suitable scalar λ_t such that

$$(\star) \quad i_{Q_\Phi} = (|\Phi_1|, \ldots, |\Phi_N|)$$

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Heuristics

- set $\mu_t = (1-t)\,\rho_0 + t\,\rho_1$ and take Q concentrated on the flow X_t of the field Φ/μ_t
- $(X_t)_\# \rho_0$ and μ_t **coincide**, since by the **method of characteristics** they both solve the same continuity equation...
- ...Q transports ρ_0 to ρ_1, since $X_0 = \mathrm{Id}$ and $(X_1)_\# \rho_0 = \rho_1$
- finally, we have $(\star)$

To give sense to the previous heuristics, we need the following "**probabilistic method of characteristics**"

Theorem (Ambrosio-Crippa, Maniglia)

Let $\mu : [0,1] \to \mathscr{P}(\Omega)$ *a curve of measures and* $\mathbf{v} : [0,1] \times \Omega \to \mathbb{R}^N$ *such that*

$$\int_0^1 \int_\Omega |\mathbf{v}(t,x)|\, d\mu_t(x)\, dt < \infty$$

with $(\mu, \mathbf{v})$ *solving the continuity equation (in distributional sense). Then there exists a* $Q \in \mathscr{P}(C([0,1];\Omega))$ *such that*

$$\mu_t = (e_t)_\# Q \qquad \text{and} \qquad \sigma'(t) = \mathbf{v}(t, \sigma(t)) \text{ for } Q\text{−a.e. } \sigma$$

Remark

The choice $\mu_t = (1-t)\,\rho_0 + t\,\rho_1$ and $\mathbf{v} = \Phi/\mu_t$ verifies the hypothesis

Some remarks on this procedure

Remark 1

Not only we have **equality of the minima**, but the two models **describe the same optimal structures** (using two complementary point of views)

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Remark 2

The deterministic flow construction becomes feasible if the optimal Φ is smooth enough (i.e. Lipschitz or Sobolev) and the data ρ_0, ρ_1 are smooth and bounded from below. In this case, we can construct an optimal traffic assignment **supported on a real flow**, not just on a probabilistic one

The latency functions

We have to **quantify** the effects of congestion on the routes

Latency functions

Increasing functions $h_j \geq 1$ such that

$$h_j(i_{Q,j}) = \quad \text{cost (per unit length) of passing from a point where the traffic in direction } \mathbf{e}_j \text{ is } i_{Q,j}$$

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Some important comments

1. the cost expressed by h_j should be thought as a **time**, i.e.

$$[h_j] = \frac{\text{time}}{\text{length}} = \frac{1}{\text{speed}}$$

in fact "*the higher the congestion, the slower we can move*"

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   in fact "*the higher the congestion, the slower we can move*"

2. why do we require $h_j \geq 1$? because

   "*you can not move with infinite speed on an empty road*"

Equilibrium issues

Individual cost for using the road $\sigma \in C^{x,y}$

$$c_h(\sigma) := \sum_{j=1}^{N} \int_0^1 h_j \circ i_{Q,j}(\sigma(t)) \, |\sigma_j'(t)| \, dt$$

Finsler length, averaged according the traffic, i.e. **congestion effects compensate the difference of length**

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Some individuals could decide to change their path, taking a less crowded one. This change of strategy alters the traffic distribution Q and so the cost paid by the others **and so on and on...**

Goal

Does a **Nash equilibrium** exist? What does it mean here?

Wardrop equilibrium

Definition

Q is a **Wardrop equilibrium for** $h = (h_1, \ldots, h_N)$ if it gives full mass to the geodesics of the *traffic-dependent metric*

$$d_Q(x,y) = \inf \left\{ \sum_{j=1}^{N} \int_0^1 h_j \circ i_{Q,j}(\sigma(t)) |\sigma_j'(t)| \, dt \; : \; \begin{array}{l} \sigma(0) = x \\ \sigma(1) = y \end{array} \right\}$$

Important remark

The metric d_Q can be defined when $h_j \circ i_{Q,j} \in L^s(\Omega)$, with $s > N$

Why? Because...

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Why? Because...

If $\xi \in C(\Omega; \mathbb{R}^+)$, the metric $\quad d_\xi(x,y) = \inf \int_0^1 \xi(\sigma) \, |\sigma'(t)| \, dt$

has an Hölder estimate in terms of the L^s norm of $\xi \Longrightarrow$ define d_Q as the supremum of $\quad d_{\xi_n} \quad$ as $\quad \xi_n \to h \circ i_Q \quad$ in L^s

Given the data ρ_0, ρ_1 and Π and the latency functions h_j, does a Wardrop Equilibrium exist?

Existence via convex optimization

Theorem (Carlier-Jimenez-Santambrogio)

Let Π *be* **convex** *and suppose that*

$$\nabla \mathcal{H} = (h_1, \dots, h_N)$$

Then $\widetilde{Q}$ *minimizes* $(\mathcal{W})$ **if and only if**

1. $\widetilde{Q}$ *is a Wardrop equilibrium for* $(h_1, \dots, h_N)$
2. $\widetilde{\gamma} = (e_0, e_1)_{\#} \widetilde{Q} \in \Pi$ *solves the MK problem*

$$\min \left\{ \int_{\Omega \times \Omega} d_{\widetilde{Q}}(x, y)\, d\gamma(x, y) \ : \ \gamma \in \Pi \right\}$$

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Proof: some hints

- Convex perturbations to derive Euler-Lagrange inequality, i.e.

$$\int_\Omega \langle \nabla\mathcal{H}(i_{\widetilde{Q}}), i_Q \rangle \geq \int_\Omega \langle \nabla\mathcal{H}(i_{\widetilde{Q}}), i_{\widetilde{Q}} \rangle \qquad \text{for every } Q$$

- necessary conditions are sufficient as well

Some comments

- A global optimum for the cost $\mathcal{H}$, gives a Wardrop equilibrium for the marginal costs $(\partial_{x_1}\mathcal{H}, \ldots, \partial_{x_N}\mathcal{H})$

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$$\int_\Omega \langle \nabla \mathcal{H}(i_{\widetilde{Q}}), i_{\widetilde{Q}} \rangle$$

- Open problem: how large can be the ratio

$$\frac{\int_\Omega \langle \nabla \mathcal{H}(i_{\widetilde{Q}}), i_{\widetilde{Q}} \rangle}{\min \int_\Omega \langle \nabla \mathcal{H}(i_Q), i_Q \rangle} := \quad \textbf{price of anarchy}$$

A significant choice for the cost

Basic requirement

We want costs $\mathcal{H}$ such that

"*marginal costs* $\partial_{x_j}\mathcal{H}$ *are latency functions, i.e.* $\partial_{x_j}\mathcal{H} \geq 1$"

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"*congestion effects are negligible for small masses*"

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Restriction

We require $p < N/(N-1)$, so that $\partial_{x_j}\mathcal{H} \circ i_Q \in L^s(\Omega) \quad s > N$

Optimization and optimality for $(\mathcal{B})$

Beckmann's dual

$$\sup\left\{\langle\varphi,\rho_0-\rho_1\rangle-\int_\Omega \mathcal{H}^*(\nabla\varphi(x))\,dx\ :\ \varphi\in W^{1,q}(\Omega)\right\}$$

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Primal-dual optimality conditions

$$\nabla\varphi_0\in\partial\mathcal{H}(\Phi_0)\quad\text{or}\quad\Phi_0=\nabla\mathcal{H}^*(\nabla\varphi_0)$$

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Key point

Regularity of $\Phi_0 \rightsquigarrow$ regularity of solutions to

$$(BVP) \qquad \operatorname{div} \nabla \mathcal{H}^*(\nabla u) = f \quad + \quad \begin{pmatrix} \text{homogeneous Neumann} \\ \text{conditions} \end{pmatrix}$$

Wide degeneracy $\quad \mathcal{H}^*(\xi) = \sum_{i=1}^{N} \frac{(|\xi_i| - 1)_+^q}{q} \qquad q = p/(p-1)$

Regularity estimates for $(\mathcal{B})$

Local "almost" L^∞ estimate (B.-Carlier-Santambrogio)

Let $q \geq 2$ *and* $f \in L^\infty(\Omega)$ *with zero-mean. If* $\varphi_0 \in W^{1,q}(\Omega)$ *is a weak solution of* (BVP) *then*

$$\nabla \varphi_0 \in L^r_{loc}(\Omega), \text{ for every } r \geq q$$

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Local Sobolev estimate (B.-Carlier-Santambrogio)

Let $q \geq 2$ and $f \in W^{1,p}(\Omega)$ with zero-mean. If $\varphi_0 \in W^{1,q}(\Omega)$ is a weak solution of (BVP) then

$$(|\partial_{x_j}\varphi_0| - 1)_+^{\frac{q}{2}} \frac{\partial_{x_j}\varphi_0}{|\partial_{x_j}\varphi_0|} \in W^{1,2}_{loc}(\Omega), \qquad j = 1, \dots, N$$

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Local Sobolev estimate (B.-Carlier-Santambrogio)

Let $q \geq 2$ and $f \in W^{1,p}(\Omega)$ with zero-mean. If $\varphi_0 \in W^{1,q}(\Omega)$ is a weak solution of (BVP) then

$$(|\partial_{x_j}\varphi_0| - 1)_+^{\frac{q}{2}} \frac{\partial_{x_j}\varphi_0}{|\partial_{x_j}\varphi_0|} \in W^{1,2}_{loc}(\Omega), \qquad j = 1, \ldots, N$$

Corollary (Regularity of Beckmann's optimizer)

$$\Phi_0 = \nabla\mathcal{H}^*(\nabla\varphi_0) \in W^{1,s}(\Omega; \mathbb{R}^N) \cap L^r(\Omega; \mathbb{R}^N)$$

for every $s < 2$ and every $r \geq q$

A sketch of the proof: higher integrability of the gradient

First of all, we try a quick review of the standard theory

- **First step**: equation for the gradient

$$\mathrm{div}\left(D^2\mathcal{H}^*(\nabla\varphi_0)\,\nabla\partial_{x_j}\varphi_0\right) = \partial_{x_j} f$$

  this is linear and degenerate elliptic
- **usually**, convex increasing functions $g(|\nabla\varphi_0|)$ (ex. power functions) are subsolutions and this would suffice to produce an **iterative scheme of reverse Hölder inequalities**
- **how does it work?**:

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- **usually**, convex increasing functions $g(|\nabla\varphi_0|)$ (ex. power functions) are subsolutions and this would suffice to produce an **iterative scheme of reverse Hölder inequalities**

- **how does it work?**: suppose that

$$c\,|z|^{q-2}\,\mathrm{Id} \le D^2\mathcal{H}^*(z) \le C\,|z|^{q-2}\,\mathrm{Id} \qquad \text{for} \quad M \le |z|$$

  then use test functions like $(|\nabla\varphi_0|^k - (2M)^k)_+$ and get the **unnatural inequality** (Caccioppoli)

$$\int_{B_\varrho} \left|\nabla\left(|\nabla\varphi_0|^{\beta_k}\right)\right|^2 \lesssim (R-\varrho)^{-2}\int_{B_R}\left(|\nabla\varphi_0|^{\beta_k}\right)^2$$

- combining with Sobolev inequality, we get the reverse Hölder inequalities

$$\|\nabla\varphi_0\|_{L^{2^*\beta_k}(B_\varrho)} \lesssim (R-\varrho)^{-\frac{1}{2^*\beta_k}} \|\nabla\varphi_0\|_{L^{2\beta_k}(B_R)}$$

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...and for our $\mathcal{H}^*$?

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Problems

- this is **not uniformly convex**, neither globally nor "at infinity"
- ellipticity fails each time **a component** of $\nabla \varphi$ is small
- $D^2 \mathcal{H}^*$ has a diagonal structure, with

$$h_i''(\partial_{x_j} \varphi_0) \simeq |\partial_{x_j} \varphi_0|^{q-2} \qquad \text{on the diagonal}$$

  imitating the previous proof and choosing test functions that "try to mimick the Hessian", ex. $|\partial_{x_j} \varphi|^k$, we end up with...

- ...partial derivatives are mixed! i.e. **surrogate of Caccioppoli inequality**

$$\sum_{i=1}^{N} \int h_i''(\partial_{x_i}\varphi_0)\, \left|\partial_{x_i}(\partial_{x_j}\varphi_0)^{\beta+1}\right|^2 \lesssim \int |\nabla\varphi_0|^{q+2\,\beta}$$

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- key point: a **surrogate of Sobolev inequality** for the LHS, something of the type

$$\sum_{i=1}^{N} \int h_i''(\partial_{x_i}\varphi_0)\, |\partial_{x_i}\varphi_0|^2\, |\partial_{x_j}\varphi_0|^\alpha \le \sum_{i=1}^{N} \int h_i''(\partial_{x_i}\varphi_0) \left|\partial_{x_i}(\partial_{x_j}\varphi_0)^{\beta+1}\right|^2 + \text{(lower order terms)}$$

with $\alpha > 2\,\beta$

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- key point: a **surrogate of Sobolev inequality** for the LHS, something of the type

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- **Di Benedetto's trick**: the latter is obtained by inserting the test function $\varphi_0\,|\nabla\varphi_0|^{\alpha}\,\xi^2$ in the equation (**not** in the derived equation) — for this we need to know that $\varphi_0 \in L^{\infty}$ (easy)

A sketch of the proof: Sobolev estimate

- **first of all:** in general $\varphi_0 \notin W^{2,q}$
- we use Nirenberg's method (i.e. the method of incremental ratios), to differentiate the equation in a discrete sense...
- ...and the monotonicity and growth properties of $\nabla \mathcal{H}^*$, i.e.

$$\langle \nabla \mathcal{H}^*(z) - \nabla \mathcal{H}^*(w), z - w \rangle \gtrsim |G(z) - G(w)|^2$$

where

$$G(z) = \sum_{i=1}^{N} (|z_i| - 1)_+^{\frac{q}{2}} \frac{z_i}{|z_i|} \, \mathbf{e}_i$$

- finally, observe that $\Phi_0 = f(G)$, with f locally Lipschitz

Regularity estimates for $(\mathcal{W})$

Hypothesis

Let $\rho_0, \rho_1 \in \mathscr{P}(\Omega)$ be such that $\rho_i = f_i \cdot \mathscr{L}^N$ with $f_i \in L^\infty$

Using the equivalence $(\mathcal{W}) = (\mathcal{B})$, we get:

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- if ρ_0 and ρ_1 are bounded from below by δ, we can estimate

$$\int \text{length}(\sigma)^s \, dQ(\sigma) \leq C_{s,\delta} \qquad \text{for every } s \geq 1$$

i.e. "*optimal routes have almost uniformly bounded lengths*"

Thanks for your attention

"Discipline is never an end in itself, only a means to an end"

Further readings

Discrete and continuous models

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