Last Cases of Dejean's Conjecture

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Repetitions

(p, q) is a *repetition* in a word w if :

- pq is a factor of w,
- $p \neq \epsilon$ and
- q is a prefix of pq.

The *exponent* of the repetition is $\frac{|pq|}{|p|}$.
Squares are repetitions of exponent 2.

A word is said x-*free* (resp. x^+-free) if it does not contain a repetition of exponent y with $y \geq x$ (resp. $y > x$).

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Let $\text{RT}(k)$ be the smallest x such that there is an infinite x^+-free word over a k-letter alphabet ($k \geq 2$).

Conjecture (Dejean's conjecture, 1972)

$$\text{RT}(k) = \begin{cases} \frac{7}{4} & \textit{if } k = 3 \\ \frac{7}{5} & \textit{if } k = 4 \\ \frac{k}{k-1} & \textit{otherwise.} \end{cases}$$

Already proved for:

- $k = 2$ [Thue 1906]
- $k = 3$ [Dejean 1972]
- $k = 4$ [Pansiot 1984]
- $5 \leq k \leq 11$ [Moulin Ollagnier 1992]
- $12 \leq k \leq 14$ [Currie, Mohammad-Noori 2004]
- $k \geq 33$ [Carpi 2007]
- $k \geq 27$ [Currie, Rampersad 2008,2009]
- $8 \leq k \leq 38$ [R. 2009]
- $15 \leq k \leq 26$ [Currie, Rampersad 2009]

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$k = 2$ and $k = 3$

Theorem (Thue 1906)

Thue-Morse word (i.e. fixed point of $0 \to 01$, $1 \to 10$) is 2^+-free.

$$f(a) = abcacbcabcbacbcacba$$
$$f(b) = bcabacabcacbacabacb$$
$$f(c) = cabcbabcabacbabcbac$$

Theorem (Dejean 1972)

A fixed point of f is $\frac{7}{4}^+$-free.

Theorem (Brandenburg 1983)

Fixed point method does not work for $k \geq 4$.

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Pansiot's Coding

If a word w on a k-letter alphabet is $\frac{k-1}{k-2}$-free, then every factor of length $k-1$ consists of $k-1$ different letters.

$\rightarrow$ w can be encoded by a binary word $P_k(w)$:

$$P_k(w)[i] = \begin{cases} 0 & \text{if } w[i+k-1] = w[i] \\ 1 & \text{if } w[i+k-1] \notin \{w[i], \dots, w[i+k-2]\}. \end{cases}$$

$$\begin{array}{lcccccccccccccc} w & = & 1 & 2 & 3 & 4 & 5 & 1 & 6 & 3 & 2 & 4 & 1 & 5 & \dots \\ P_6(w) & = & & & & & & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \dots \end{array}$$

Remark: If w validates Dejean's conjecture then $P_k(w)$ is $\{00, 111\}$-free.

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Let $M_k()$ be inverse of $P_k()$. (s.t. $M_k(w)[1..k-1] = 1 \ \ldots \ k-1$)

i.e. $P_k(M_k(w)) = w$ for every binary word w.

Let w_4 be a fixed point of $h_4 : 0 \longrightarrow 101101, 1 \longrightarrow 10$.

Theorem (Pansiot 1984)

$M_4(w_4)$ is $\frac{7}{5}^+$-free.

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Pansiot's coding can also be viewed by the way of an action on the symmetric group $\mathbb{S}_k$:

Let Ψ be the morphism between $\{0,1\}^*$ and $\mathbb{S}_k$ such that:

- $\Psi(0) = (1\ \ 2\ldots k-1)$ and $\Psi(1) = (1\ \ 2\ldots k-1\ \ k)$

For all $i \geq 0$ and $1 \leq j \leq k-1$, $M_k(w)[i+j] = \Psi(w[1..i])(j)$.

$M_k(w)[i\ ..\ i+k-1] = M_k(w)[j\ ..\ j+k-1]$ (for $j > i$) iff $\Psi(w[i+1\ ..\ j]) = \mathrm{Id}_k$.

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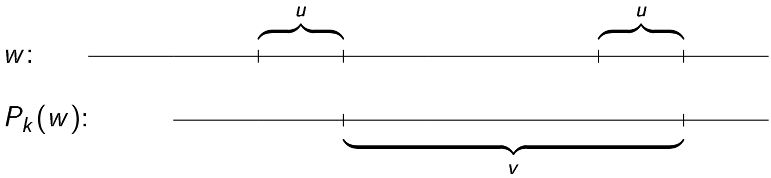
A picture...

$|u| = k - 1.$



Then $\Psi(v) = \mathrm{Id}_k$.

Moulin Ollagnier's ideas

- A repetition (p, q) is *short* if $|q| < k - 1$.
  (Note: if forbidden $\Rightarrow$ bounded size.)
- A repetition is a *kernel repetition* if $|q| \geq k - 1$.

On Pansiot's codes:

- (p, q) is a Ψ-*kernel repetition* if (p, q) is a repetition, and $\Psi(p) = \mathrm{Id}_k$.

Lemma (Moulin Ollagnier)

$M_k(w)$ has a kernel repetition (p, q) $\Leftrightarrow$ w has a Ψ-kernel repetition (p', q') with $|p| = |p'|$ and $|q'| = |q| - k + 1$.

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Moulin Ollagnier's ideas

To prove Dejean's conjecture for $k \geq 5$, find a morphism h s.t.:

- $M_k(w)$ has no forbidden short repetition,
- w has no Ψ-kernel repetition (p, q) with $\frac{|pq|+k-1}{|p|} > \frac{k}{k-1}$.

where w is a fixed point of h.

Moulin Ollagnier's ideas

Idea: Limit us to h such that:

- $\Psi(h(0)) = \sigma \cdot \Psi(0) \cdot \sigma^{-1}$ and
- $\Psi(h(1)) = \sigma \cdot \Psi(1) \cdot \sigma^{-1}$,

for a $\sigma \in \mathbb{S}_k$.

Lemma (Moulin Ollagnier)

Let (p, q) be a Ψ-kernel repetition in w. If q is long enough, then (p, q) is an image by h of a smaller Ψ-kernel repetition in w.

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Moulin Ollagnier's results

Let w be a fixed point of h.

Checking if $M_k(w)$ is $\frac{k}{k-1}^+$-free is decidable:

- check if $M_k(w)$ has no small forbidden repetition,
- check if iterated images of "small" kernel repetitions in w are not forbidden.

Moulin Ollagnier gives morphisms h_k for $5 \leq k \leq 11$ such that $M_k(w_k)$ is $\frac{k}{k-1}^+$-free (where w_k is a fixed point of h_k).
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$$h_5: \begin{cases} 0 \longrightarrow 010101101101010110110 \\ 1 \longrightarrow 101010101101101101101 \end{cases}$$

$$h_6: \begin{cases} 0 \longrightarrow 010101101101011010110 \\ 1 \longrightarrow 101011010110110101101 \end{cases}$$

$$h_7: \begin{cases} 0 \longrightarrow 0110110110110101101101101010 \\ 1 \longrightarrow 1010110110110101101101101101 \end{cases}$$

$$h_8: \begin{cases} 0 \longrightarrow 1011010101101011010110101010 \\ 1 \longrightarrow 1011010101011011011010101101 \end{cases}$$

$$h_9: \begin{cases} 0 \longrightarrow 101011010110110101011010110110101010 \\ 1 \longrightarrow 101010110110110101011011010110101101 \end{cases}$$

$$h_{10}: \begin{cases} 0 \longrightarrow 1010101011011011011010101011011011010110 \\ 1 \longrightarrow 1010101011011011010110101010101011010101 \end{cases}$$

$$h_{11}: \begin{cases} 0 \longrightarrow 1010101010101101101011010110101010110110 \\ 1 \longrightarrow 1010101010101011011011011011011010101101 \end{cases}$$

New results

Image of the Thue-Morse sequence

Moulin Ollagnier ideas can be extended to HDOLs.

It is sufficient to work of a special case of HDOLs:
Image by a morphism of the Thue-Morse sequence.

Let w_{TM} be the Thue-Morse sequence on a, b
i.e. fixed point of $g : a \longrightarrow ab, b \longrightarrow ba$.

Let $h : \{a, b\}^* \longrightarrow \{0, 1\}^*$ be a morphism.

Question

Is $M_k(h(w_{TM}))$ $\frac{k}{k-1}^+$-free ?

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Let $\sigma_a = \Psi(h(a))$ and $\sigma_b = \Psi(h(b))$,

Let $\Psi' : \{a, b\}^* \rightarrow \mathbb{S}_k$ s.t. $\Psi'(a) = \sigma_a$ and $\Psi'(b) = \sigma_b$.

Idea behind this:

Find a h such that:

- $h(w_{TM})$ has to avoid forbidden "short" Ψ-kernel repetitions
- w_{TM} has to avoid forbidden "long" Ψ'-kernel repetitions.

- We obtain results for smaller h.

Image of the Thue-Morse sequence

Following Moulin Ollagnier's idea:

Limit us to h such that:

- $\Psi'(g(a)) = \sigma_a \cdot \sigma_b = \sigma \cdot \sigma_a \cdot \sigma^{-1}$ and
- $\Psi'(g(b)) = \sigma_b \cdot \sigma_a = \sigma \cdot \sigma_b \cdot \sigma^{-1}$

for a $\sigma \in \mathbb{S}_k$.

Remark: σ_a and σ_b have to be conjugate.

Moreover: h is *uniform*, *synchronizing*, and the last letters differ.

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- $\Psi'(g(b)) = \sigma_b \cdot \sigma_a = \sigma \cdot \sigma_b \cdot \sigma^{-1}$

for a $\sigma \in \mathbb{S}_k$.

Remark: σ_a and σ_b have to be conjugate.

Moreover: h is *uniform*, *synchronizing*, and the last letters differ.

Let (p, q) be a Ψ-kernel repetition of $h(w_{TM})$.

- If q is long enough, then (p, q) is an image by h of a Ψ'-kernel repetition in w_{TM}.

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An image by g of a Ψ'-kernel repetition has a smaller exponent.

This is decidable :

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Results

Theorem

For every $k \in \{8, \ldots 38\}$, *there is a uniform morphism* h_k *such that* $M_k(h_k(w_{TM}))$ *is* $\frac{k}{k-1}^{+}$*-free.*

Corollary

Dejean's conjecture holds for $8 \leq k \leq 38$.

Results

Example: $k = 18$.

$$h_{18} : \begin{cases} a \rightarrow 1010110101011010110101010110110101011010110 \\ b \rightarrow 1010101101011010110101011010110101011010101. \end{cases}$$

Size of $h_k(x)$

From 30 (for $k = 8$) up to 74 (for $k = 38$).

Computation time on a 2.4 Ghz processor

- Few seconds/minutes for "small" k.
- Couple of hours for $k = 38$.

Results

This technique does not work for $k \in \{3,5,6,7\}$ since for every $\sigma_a, \sigma_b, \sigma \in \mathbb{S}_k$ such that:

- $\sigma_a \cdot \sigma_b = \sigma \cdot \sigma_a \cdot \sigma^{-1}$ and
- $\sigma_b \cdot \sigma_a = \sigma \cdot \sigma_b \cdot \sigma^{-1}$,

w_{TM} has a Ψ'-kernel repetition of exponent at least $\mathrm{RT}(k)$.

Works for $k = 4$ ($|h_4(x)| = 80$).

Conjecture

For every $k \geq 8$ there is a morphism h_k such that $M_k(h_k(w_{TM}))$ is $\frac{k}{k-1}^+$-free.

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Morphisms

$$h_4: \begin{cases} a \longrightarrow 1011010101101101011011010101101101010110\ldots \\ \qquad\ldots 1101010110101011011010110110101011011010 \\ b \longrightarrow 1011010101101101011011010101101010110110\ldots \\ \qquad\ldots 1010110110101011011010110110101011010101 \end{cases}$$

$$h_8: \begin{cases} a \longrightarrow 101101101101010101011010101101 \\ b \longrightarrow 101101010101011011011011011010 \end{cases}$$

$$h_9: \begin{cases} a \longrightarrow 1011011010101010101011011010110101 \\ b \longrightarrow 1010110110101010101011011010110101 10 \end{cases}$$

$$h_{10}: \begin{cases} a \longrightarrow 1010110101010110101010110101011010110 1 \\ b \longrightarrow 1010110101011010110110101101010101011 0 \end{cases}$$

$$h_{11}: \begin{cases} a \longrightarrow 10101011010110101011010110101101011010101010110110 \\ b \longrightarrow 10101101010110101101011010101101011010101010110101 \end{cases}$$

$$h_{12}: \begin{cases} a \longrightarrow 1011010110110110101010101011010110110110110 \\ b \longrightarrow 1011010101010110101101101101011010110101101 \end{cases}$$

$$h_{13}: \begin{cases} a \longrightarrow 101011010101101010101101011010101010110110101 01 \\ b \longrightarrow 101010110110101010110101011011010101010110101 10 \end{cases}$$

$$h_{14}: \begin{cases} a \longrightarrow 10101101101011011010110101101010110101101101 10 \\ b \longrightarrow 10110101101011011010110101101010110101101101 01 \end{cases}$$

$$h_{15}: \begin{cases} a \longrightarrow 101101101101010110101010101101011011011011010101010110 \\ b \longrightarrow 101011011011010110110101010101011011010110110101010101 \end{cases}$$

$$h_{16}: \begin{cases} a \longrightarrow 101010110101101101011010110101101101011010110101011011010110 \\ b \longrightarrow 101010110110101101011010101101101101011011010101011011010101 \end{cases}$$

$$h_{17}: \begin{cases} a \longrightarrow 10110101101101010110110110110101010110110110101 \\ b \longrightarrow 10110101101101010110110110101101010110110110110 \end{cases}$$

$$h_{18}: \begin{cases} a \longrightarrow 10101101010110101101010110110101011010110 \\ b \longrightarrow 10101011010110101101011010110101011010101 \end{cases}$$

$$h_{19}: \begin{cases} a \longrightarrow 101010110110110101101011010101010110101011010110110110 \\ b \longrightarrow 101010110110110101101011010110110110101010110101010101 \end{cases}$$

$$h_{20}: \begin{cases} a \longrightarrow 1010110101010110101101101010101101011010110110110101101 01 \\ b \longrightarrow 1010110101010110110101101010101101010110110110110101101 10 \end{cases}$$

$$h_{21}: \begin{cases} a \longrightarrow 10101011010101101101101101011010110101010101 \\ b \longrightarrow 10101101010101101101101011011010110101010110 \end{cases}$$

$$h_{22}: \begin{cases} a \longrightarrow 101011010110101101101101011010110101011010110 \\ b \longrightarrow 101011010110101101101101101010110101011010101 \end{cases}$$

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$$h_{24}: \begin{cases} a \longrightarrow 10101010101010110110110101010110110101010101 \\ b \longrightarrow 10101010101010110110101101010101101101010110 \end{cases}$$

$$h_{25}: \begin{cases} a \longrightarrow 1010101101010110101101010110101010110101010110101 \\ b \longrightarrow 1010101101010110101010110101101010101101010110110 \end{cases}$$

$$h_{26}: \begin{cases} a \longrightarrow 10110101010101010101011011011010101011011011 01 \\ b \longrightarrow 10110101010101010110110110110101010110110101 0 \end{cases}$$

$$h_{27}: \begin{cases} a \longrightarrow 1010101010101101101011011010101010101011011011010101 \\ b \longrightarrow 1010101010101101101011010110101010101011011011010110 \end{cases}$$

$$h_{28}: \begin{cases} a \longrightarrow 1010101010101010110101101101010101010101010110110101 01 \\ b \longrightarrow 1010101010101010110101101011010101010101010110110101 10 \end{cases}$$

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$$h_{30}: \begin{cases} a \longrightarrow 101010101010101101101101101101010101011011011010101010101 \\ b \longrightarrow 101010101010101101101101101011010101011011011010101010110 \end{cases}$$

$$h_{31}: \begin{cases} a \longrightarrow 1011011011011011011010101101011011010110110101011011010101 \\ b \longrightarrow 1011011011011011011010101011011011010110110101011011011010110 \end{cases}$$

$$h_{32}: \begin{cases} a \longrightarrow 10110101101101101010110101010110110101101101010101101010101101 \\ b \longrightarrow 10110101101101101010110110110110101101101010101101010101010 \end{cases}$$

$$h_{33}: \begin{cases} a \longrightarrow 1010101010101011010101101011011010101010101010110101010110110101 \\ b \longrightarrow 1010101010101011010101101011010110101010101010110101010110110110 \end{cases}$$

$$h_{34}: \begin{cases} a \longrightarrow 10101101101010101010101010101011010110101011010101010101010101010101 \\ b \longrightarrow 10101101101010101010101010101010110110101011010101010101010101010110 \end{cases}$$

⋮

In a Dejean word (with $k \geq 5$), each letter has frequency at least $\frac{1}{k+1}$ and at most $\frac{1}{k-1}$.

Conjecture (Ochem 2005)

(1) *For every $k \geq 5$, there exists an infinite $\frac{k}{k-1}^+$-free word over k-letter with letter frequency $\frac{1}{k+1}$.*

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(Does not work for $k = 8$.)

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Further Researches

Generalized Repetition Threshold (next two talks).

The *growth rate* of L is $g(L) = \lim_{n\to\infty} \sqrt[n]{|L \cap \Sigma^n|}$.

Question

Compute $g(D_k)$ for $k \geq 3$.

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Thank you !