# Conjugation of lines with respect to a triangle 

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#### Abstract

Isotomic and isogonal conjugate with respect to a triangle is a well-known and well studied map frequently used in classical geometry. In this article we show that there is a reason to study conjugation of lines. This conjugation has many interesting properties and relations to other objects of a triangle.


## 1 Introduction

Isotomic conjugation with respect to a triangle $A B C$ is a map which maps any point $P$ with barycentric coordinates $(x, y, z)$ to the point $P^{*}$ with coordinates $(1 / x, 1 / y, 1 / z)$. Isogonal conjugation is defined similarly, but instead barycentric coordinates the trilinear coordinates are used. The following property of isogonal conjugation is important: If $P$ and $P^{*}$ are isogonal conjugates, then

$$
\angle P A B=\angle C A P^{*}, \quad \angle P B C=\angle A B P^{*}, \quad \angle P C A=\angle B C P^{*}
$$

This property is often used as a definition of isogonal conjugation.


Fig. 1. Isotomic conjugation


Fig. 2. Isogonal conjugation

Other properties of isogonal and isotomic conjugation and its applications to triangle geometry can be found in [2].

Isotomic and isogonal conjugations are projectively equivalent. It means that there is a projective transformation of the plane which preserves vertices of a triangle and any pair of isotomically conjugate points is mapped to a pair of isogonally conjugate points.

In the general case, define conjugation with respect to a triangle $A B C$ and a point $S$ as follows.

Definition 1. Let $A^{\prime} B^{\prime} C^{\prime}$ is a cevian triangle of a point $S$ with respect to a triangle $A B C$. Let $P$ be any point and $A_{1} B_{1} C_{1}$ be its cevian triangle. Choose on the sides of the
triangle points $A_{2}, B_{2}$ and $C_{2}$ such that

$$
\begin{aligned}
& {\left[A, C_{1} ; B, C^{\prime}\right]=\left[B, C_{2} ; A, C^{\prime}\right],} \\
& {\left[B, A_{1} ; C, B^{\prime}\right]=\left[C, A_{2} ; B, A^{\prime}\right],} \\
& {\left[C, B_{1} ; A, C^{\prime}\right]=\left[A, B_{2} ; C, B^{\prime}\right],}
\end{aligned}
$$

where $[X, Y ; Z, T]$ means a cross-ratio of four collinear points $X, Y, Z$ and $T$. It is easy to see that the cevians $A A_{2}, B B_{2}$ and $C C_{2}$ are concurrent. The point of intersection of these three lines called a conjugate point of the point $P$ with respect to the triangle $A B C$ and the point $S$.


Fig. 3. Conjugation
If point $S$ is the centroid of the triangle then this map is an isotomic conjugation. If $S$ coincides with the incenter of the triangle then this map is an isogonal conjugation.

Correctness of the definition 1 could be easily proven by the Ceva's theorem or by means of projective transformation and correctness of the definition of isotomic or isogonal conjugation.

There is a well known theorem about isogonal and isotomic conjugation.
Theorem 1. Under any conjugation with respect to a triangle $A B C$ an image of any line is a conic passing through the vertices of this triangle.

For example, image of a line at infinity under isogonal conjugation is a circumcircle of a triangle and under isotomic conjugation is a circumscribed Steiner ellipse of the triangle.

## 2 Conjugation of lines

Let's apply a dual transformation to the construction in the definition 1. In this case we obtain another map that still makes sense. Let's call this map a conjugation of lines with respect to a triangle and a line.

Let us give a strict definition.
Definition 2. Let a line $s$ intersects the sides of a triangle $A B C$ at points $A^{\prime}, B^{\prime}$ and $C^{\prime}$. Suppose a line $\ell$ intersects the sides of the triangle at points $A_{1}, B_{1}$ and $C_{1}$. Choose
on the sides of the triangle points $A_{2}, B_{2}$ and $C_{2}$ such that

$$
\begin{aligned}
& {\left[A, C_{1} ; B, C^{\prime}\right]=\left[B, C_{2} ; A, C^{\prime}\right],} \\
& {\left[B, A_{1} ; C, B^{\prime}\right]=\left[C, A_{2} ; B, A^{\prime}\right],} \\
& {\left[C, B_{1} ; A, C^{\prime}\right]=\left[A, B_{2} ; C, B^{\prime}\right] .}
\end{aligned}
$$

Then the points $A_{2}, B_{2}$ and $C_{2}$ lie of the same line. Call this line a conjugate line of the line $\ell$ with respect to the triangle $A B C$ and the line $s$.

Correctness of this definition follows from correctness of the definition 1. Also it could be independently proven by the Menelaus theorem.


Fig. 4. Conjugation of a line
If a line $s$ is the line at infinity then it is reasonable to call this conjugation isotomic conjugation of lines. Also call this map isogonal conjugation if $s$ is the line passing through the feet of the three external angle bisectors of the triangle. In these cases this map has a simpler definition (fig. 5 and 6 ).


Fig. 5. Isotomic conjugation of a line


Fig. 6. Isogonal conjugation of a line

The dual theorem to Theorem 1 is the following.
Theorem 2. The set of lines passing through a fixed point is conjugate with respect to $\triangle A B C$ to the set of tangent lines to some inscribed conic.

Considering "critical lines" of the conjugation we obtain the following corollaries of Theorem 1.

Corollary 3. Isotomically (or isogonally) conjugate line to any line passing through the center of an equilateral triangle is a tangent line to the inscribed circle of the triangle.

Corollary 4. Isotomically conjugate line to any line passing through the Nagel point of a triangle is a tangent line to the inscribed circle of the triangle.

The proofs follow from the fact that Gergonne point and Nagel point are isotomically conjugate.

Corollary 5. Isotomically conjugate line to any line passing through the centroid of a triangle is a tangent line to the inscribed Steiner ellipse of the triangle.

Corollary 6. Isogonally conjugate line to any line passing through the internal similitude center of the incircle and the circumcircle of a triangle is a tangent line to the inscribed circle of the triangle.

The proof follows from the fact that the Gergonne point and the internal similitude center are isogonally conjugate.

Corollary 7. Isogonally conjugate line to any line passing through the Lemoine point of a triangle is a tangent line to the inscribed Steiner ellipse of the triangle.

The proof follows from the fact that the Lemoine point and the centroid of the triangle are isogonally conjugate. Also, since the Brocard ellipse is tangent to the sides of a triangle in the feet of its semidians we obtain the following.

Corollary 8. Isogonally conjugate line to any line passing through the centroid of a triangle is a tangent line to the Brocard ellipse.

## 3 Concrete lines

From corollaries of the previous section we can formulate some statements about conjugate lines of well-known lines of a triangle. Let us recall that Nagel point, centroid and incenter of a triangle lie on the same line and this line is called a Nagel line of a triangle.

Theorem 9. 1. Isotomically conjugate line of the Euler line is tangent to the inscribed Steiner ellipse.
2. Isogonally conjugate line of the Euler line is tangent to the Brocard ellipse.
3. Isogonally conjugate line of the line IO (line through the incenter and the circumcenter of a triangle) is tangent to the incribed circle of a triangle.
4. Isotomically conjugate line of the Nagel line is tangent to the inscribed Steiner ellipse and the incircle of a triangle.
5. Isogonally conjugate line of the Nagel line is tangent to the Brocard ellipse.

It is possible to continue this list for other lines and centers of a triangle.
Let us prove, that the isotomically conjugate line of the Nagel line possesses another interesting property.

Theorem 10. Isotomically conjugate line to the Nagel line of a triangle is tangent to the inscribed circle at the Feuerbach point.

Beacuse of Theorem 9 it is sufficient to prove the following lemma.
Lemma 11. Common tangent line to the incircle and the inscribed Steiner ellipse of a triangle (which doesn't coincide with any side of this triangle) is tangent to the inscribed circle at the Feuerbach point.


Fig. 7.
Proof. Denote vertices of a triangle by $A, B, C$ and this common tangent line by $\ell$. Let $A_{1}, B_{1}$ and $C_{1}$ be points of intersection of $\ell$ and the sides of $\triangle A B C$. For the proof of the lemma it is sufficient to prove that the points $A_{1}, B_{1}$ and $C_{1}$ are on the radical axis of the inscribed circle and the nine-point circle of the triangle $A B C$. We calculate powers of these points with respect to both circles.

Let the incircle be tangent to the sides of the triangle at points $A_{2}, B_{2}$ and $C_{2}$. Denote midpoints of the triangle by $A_{3}, B_{3}$ and $C_{3}$ and feet of altitudes by $A_{4}, B_{4}$ and $C_{4}$. Lengths of the sides are denoted by $a, b$ and $c$.

Suppose $P$ is the point of intersection of lines $A_{1} B$ and $B_{1} A$. From the Brianchon theorem it follows that the point $P$ lies on the lines $A_{2} B_{2}$ and $A_{3} B_{3}$, therefore it is a point of intersection of these lines.

It is known that the second point of intersection of the line $C_{2} P$ and the incircle is the Feuerbach point (See [3], the problem 22). From the Brianchon theorem it follows that this point is the point of tangent line $A_{1} B_{1}$ and incircle. So, this line is tangent to the incircle in the Feuerbach point.

Since the proof of the mentioned property of the Feuerbach point is little bit complicated and based on other properties of Feuerbach point, we will give another independent proof.

Let us show that the points $A_{1}, B_{1}$ and $C_{1}$ lie on the radical axis of incircle and nine-point circle of the triangle.


Fig. 8.
Let's apply the Menelaus theorem for the line $A_{2} B_{2}$ and the triangle $C A_{3} B_{3}$.

$$
\frac{A_{3} P}{B_{3} P}=\frac{A_{3} A_{2}}{A_{2} C} \cdot \frac{C B_{2}}{B_{2} B_{3}}=\frac{c-b}{a+b-c} \cdot \frac{a+b-c}{c-a}=\frac{c-b}{a-c} .
$$

Now apply the Menelaus theorem for the line $A A_{1}$ and the triangle $C A_{3} B_{3}$.

$$
\frac{A_{1} A_{3}}{A_{1} C}=\frac{A_{3} P}{B_{3} P} \cdot \frac{B_{3} A}{A C}=\frac{c-b}{2(a-c)}
$$

Since $C A_{3}=a / 2$ we have

$$
A_{1} C=\frac{2(a-c)}{c-b+2 a-2 c} \cdot \frac{a}{2}=\frac{a(a-c)}{2 a-b-c} .
$$

Now it is easy to find lengths of the segments $A_{1} A_{2}, A_{1} A_{3}$ and $A_{1} A_{4}$.

$$
\begin{gather*}
A_{1} A_{2}=\frac{a+b-c}{2}-\frac{a(a-c)}{2 a-b-c}  \tag{1}\\
A_{1} A_{3}=\frac{a}{2}-\frac{a(a-c)}{2 a-b-c}  \tag{2}\\
A_{1} A_{4}=\frac{a^{2}+b^{2}-c^{2}}{2 b}-\frac{a(a-c)}{2 a-b-c} . \tag{3}
\end{gather*}
$$

We check the equation $A_{1} A_{2}^{2}=A_{1} A_{3} \cdot A_{1} A_{4}$.

$$
\begin{align*}
& A_{1} A_{2}^{2}-A_{1} A_{3} \cdot A_{1} A_{4}=\left(\frac{a+b-c}{2}\right)^{2}-2 \frac{a+b-c}{2} \cdot \frac{a(a-c)}{2 a-b-c}- \\
& -\frac{a}{2} \cdot \frac{a^{2}+b^{2}-c^{2}}{2 b}+\frac{a^{2}+b^{2}-c^{2}+a b}{2 b} \cdot \frac{a(a-c)}{2 a-b-c}= \\
& =\frac{c-a-b}{4} \cdot \frac{2 a^{2}-a b-a c+b^{2}-c^{2}}{2 a-b-c}+\frac{2 a^{3}+a^{2} b-3 a^{2} c+b^{3}-b^{2} c-c^{2} b+c^{3}}{4(2 a-b-c)}=0 . \tag{4}
\end{align*}
$$

We obtain that the point $A_{1}$ lies on the radical axis of the inscribed circle and the nine-point circle of the triangle. The same argument works for the points $B_{1}$ and $C_{1}$.

Note that we also proved the Feuerbach theorem, because we showed that the radical axis of the incirbed circle and the nine-point circle is tangent to the first one.

The following theorem follows from Lemma 11.
Theorem 12. Isogonally conjugate line to the line passing through the Lemoine point and the internal similitude center of the incircle and the circumcircle of a triangle is tangent to the inscribed circle at the Feuerbach point.

## 4 Related results

We mention another result related to construction on the figure 8 .
Theorem 13. Suppose triangles $A_{1} B_{1} C_{1}$ and $A_{2} B_{2} C_{2}$ are cevian triangles in a triangle $A B C$. Correspondent sides of these triangles intersect each other at points $A_{3}, B_{3}$ and $C_{3}$. Then

1) The lines $A_{1} A_{3}, B_{1} B_{3}$ and $C_{1} C_{3}$ are concurrent. Denote the point of intersection of these lines by $P$.

The lines $A_{2} A_{3}, B_{2} B_{3}$ and $C_{2} C_{3}$ are concurrent. Denote the point of intersection of these lines by $Q$.
2) The following triples of points are collinear: $\left(A, B_{3}, C_{3}\right),\left(B, A_{3}, C_{3}\right)$ and $\left(C, A_{3}\right.$, $B_{3}$ ). Denote intersection of this lines and the correspondent sides of triangle by $A_{4}, B_{4}$ and $C_{4}$.
3) The points $A_{4}, B_{4}$ and $C_{4}$ lie on the line $P Q$.

Proof. Inscribe into the triangle $A B C$ two conics $\alpha_{1}$ and $\alpha_{2}$, which touch the sides of $A B C$ in the vertices of the triangles $A_{1} B_{1} C_{1}$ and $A_{2} B_{2} C_{2}$. Then fourth common tangent of $\alpha_{1}$ and $\alpha_{2}$ will be line from Item 3 of the Theorem. The points of tangent of these conics and the line are the points $P$ and $Q$. Items 1 and 2 follows from Brianchon theorem.


The Nagel triangle is a triangle with vertices in points of tangency of excircles and correspondent sides of a triangle $A B C$. The ellipse tangent to sides of $A B C$ at these points is called the Nagel ellipse.

The recent result F. Ivlev [5] stats that if $A_{1} B_{1} C_{1}$ (in terms of Theorem 13) is a medial triangle and $A_{2} B_{2} C_{2}$ is a Nagel triangle then $Q$ is the Feuerbach point of the triangle $A B C$. Reformulating Ivlev's theorem in terms of this article we obtain the following.

Theorem 14. The Nagel ellipse of a triangle passes through the Feuerbach point. The tangent line to the Nagel ellipse at this point is also tangent to the inscribed Steiner ellipse.

Corollary 15. Isotomically conjugate line to the line passing through centroid of a triangle and the Gergonne point passes through the Feuerbach point.

Corollary 16. Isogonally conjugate line to the line passing through the Lemoine point and the external similitude center of the incircle and the circumcircle of a triangle passes through the Feuerbach point.

Ivlev's proof is based only on the fact that the Nagel point and the Gergonne point are isotomically conjugate (actually his theorem is more general). So, it is possible to generalize Theorem 14.

Theorem 17. Points $P$ and $Q$ are conjugate with respect to the point $T$ in the triangle $A B C$. Conjugation of the lines is given. Let $\alpha_{1}, \alpha_{2}$ and $\alpha_{0}$ three inscribed in the triangle $A B C$ conics which correspond to the points $P, Q$ and $T$ (see Theorem 2). Then fourth common tangents to the pairs of conics $\alpha_{0}, \alpha_{1}$ and $\alpha_{0}, \alpha_{2}$ intersect at one of the points of intersection $\alpha_{1}$ and $\alpha_{2}$.

Note that for any conjugation there are four points such that this conjugation can be defined with respect to any of these four points (for example isogonal conjugation is also a conjugation with respect to any excenter of a triangle). For each of these points there is a different point of intersection of $\alpha_{1}$ and $\alpha_{2}$ in Theorem 17.

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## References

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