Geometry of the Cardioid

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Abstract

In this note, we discuss the cardiod. We give purely geometric proofs of its well-known properties.

Definitions and Basic Properties

The curve in Figure 1 is called a *cardioid*. The name is derived from the Greek word "$\kappa\alpha\rho\delta\iota\alpha$" meaning "heart". A cardioid has many interesting properties and very often appears in different fields of mathematics and physics. The study of geometric properties of remarkable curves is a classical topic in analytic and differential geometry. In this note, we focus mainly on the purely synthetic approach to the geometry of the cardioid.

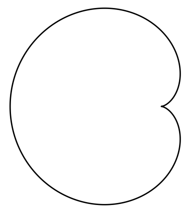

Fig. 1

In the polar coordinate system, the cardioid has the following equation:

$$r = 1 - \cos\varphi. \qquad (*)$$

In this article, we consider the geometric properties of a cardioid, so let us give a geometric definition. Take a circle of diameter 1 and let another circle of the same size roll around the exterior of the first one. Then the trace of a fixed point on the second circle will be a cardioid (Fig. 2).

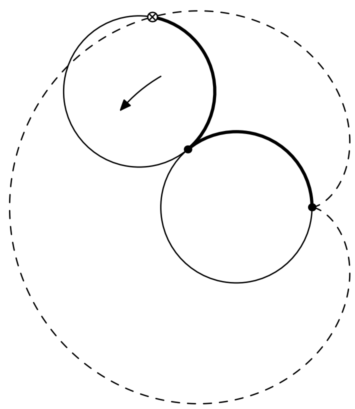

Fig. 2

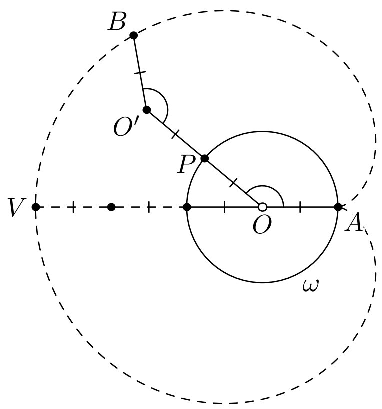

Fig. 3

This definition is not so useful for studying a cardioid (but actually it will help us later), so we restate the same definition in purely geometrical terms. Let ω be a circle with center O, A a point on it, and P a point moving along ω (Fig. 3). Suppose O' is the point symmetric to O in P. Let B be the reflection of A in the perpendicular bisector to OO'. Notice that $\angle BO'P = \angle POA$ and $O'B = OA$. Then, the locus of points B is a cardioid.

The point A is called the *cusp* of the cardioid, and the point V of the intersection of the cardioid and the ray AO is called the *vertex* (see Fig. 3).

Since the triangle AOP is isosceles and AB is parallel to OP, an easy angle count shows the following.

Lemma 1. *AP is the bisector of the angle OAB.*

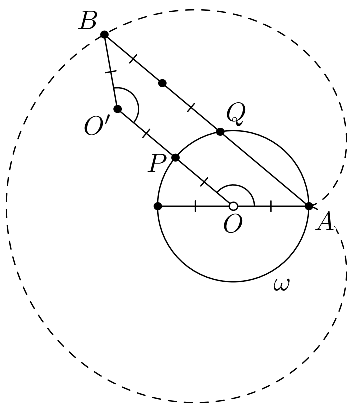

Fig. 4

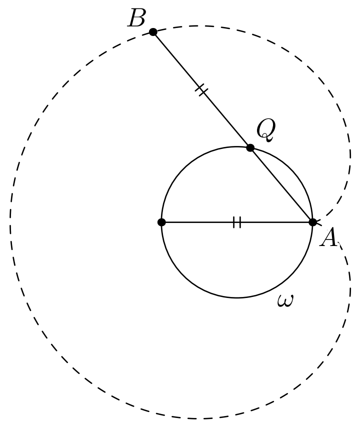

Fig. 5

Denote by Q the second point of intersection of AB and ω (Fig. 4). Note that in the quadrilateral $BO'OQ$, two of pair of opposite sides are parallel and other two are equal. Therefore it is a parallelogram or an isosceles trapezoid. But it is not the isosceles trapezoid, since in this case it will coincide with $AOO'B$. Therefore, BQ equals $O'O$, which, in turn, is equal to diameter of the circle ω. If the angle AOP is acute, then the point Q lies outside the segment AB and the construction looks a little different from Figure 4. But in this case, the proof is analogues to the considered case.

Thus, we obtain a kinematic definition of cardioid. Let ω be a circle of the diameter 1 and A a point on it. Let us take a point Q that is moving along ω. Let B be the point on the line AQ such that BQ equals 1, the diameter of ω (Fig. 5). Then B moves along a cardioid path. The are two choices for point B (on either side of the point Q). Both of these points lie on the cardioid. From this construction it is easy to see that this curve is defined by the equation (*) in a polar coordinate system with center at A. Thus, all given definitions of a cardioid are equivalent.

Tangents to a Cardioid and its Image under Inversion

How can tangent lines to cardioids be constructed? Let us return to the very first definition and look at the situation informally. Suppose a circle ω' is rolling around ω and at some moment these circles touch each other at a point P (Fig. 6). How can we find the direction of the velocity vector of the marked point B? From mechanics, we know that the speed of the point P (on the circle ω') is equal to zero and the point is at rest. The speed of the point B is perpendicular to the segment BP, since the distance between the points B and P does not change. We obtain the following statement.

Lemma 2. *A tangent line to the cardioid at point B is perpendicular to BP.*

Thus, the cardioid touches all circles centered at points P on the circle ω' having radius $PB = PA$. In other words, the cardioid is the *envelope* of this family of circles (Fig. 7).

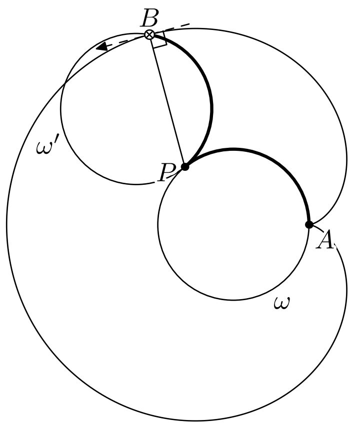

Fig. 6

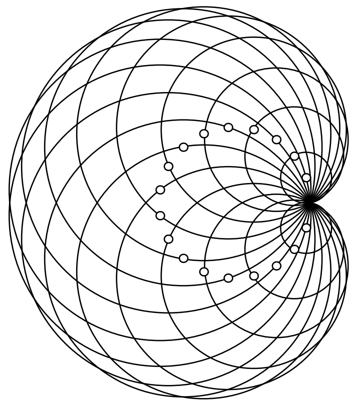
Fig. 7

Consider the images of these circles under the inversion with center A and with radius equal to the diameter of ω. The circle ω maps to the line ℓ, which is the perpendicular bisector of the segment with ends at the cusp and the vertex of the cardioid. Consider any circle ω^t from our family and let P be its center. The ray AP crosses ω^t at some point Y. Then, the image of the circle ω^t under the inversion is the line passing through the image of the point Y and perpendicular to AP. Since P lies on the circle ω, its image is the point X of intersection of lines AP and ℓ. Since Y is twice as far from A as P is, the distance between its image and A is half of the distance between A and X. Thus, we have proved that the inversion maps all circles from our family to the lines which are perpendicular bisectors of the segments with ends at A and at points X moving along the line ℓ (Fig. 8).

It is well-known that all the above lines touch the parabola with the focus A and directrix ℓ. Thus, our cardioid is its inversion image.

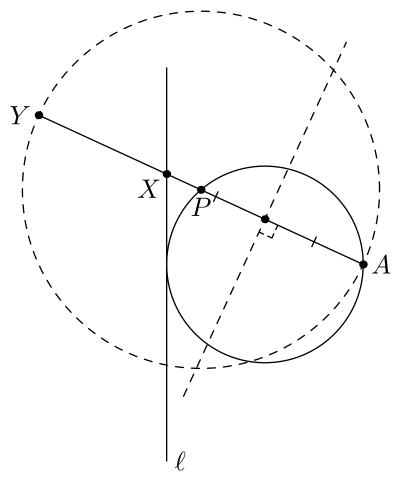

Fig. 8

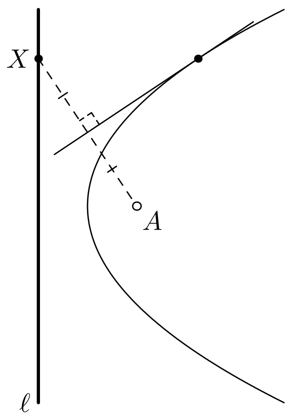

Fig. 9

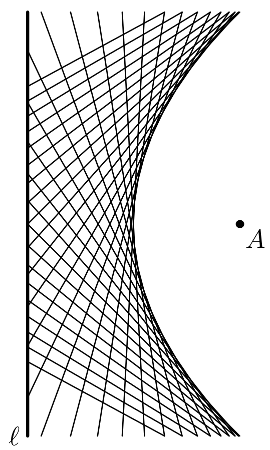

Fig. 10

We formulate this statement as a theorem and provide a rigorous proof.

Theorem 3. *Let κ be a cardioid with the cusp A and the vertex V. Let ℓ be the perpendicular bisector of the segment AV. Then, the inversion with the center at A and the radius $\frac{AV}{2}$ maps κ to the parabola with the focus at A and directrix ℓ.*

Proof. We show directly that a point on the cardioid is mapped to a point on the parabola. Here, our proof follows from the construction described above.

Denote the midpoint of AV by M and the intersection of AP and ℓ by P' (Fig. 11). Let P be a point on the circle with diameter MA and let B be any point of the cardioid κ. Suppose B' is the image of the point B. Then, it follows from the properties of the inversion that the quadrilateral $P'PB'B$ is cyclic, since the points P and P' are inverse image of each other.

Note that the following equalities on the angles hold:

$$\angle B'P'P = \angle B'BP = \angle PAB = \angle MAP = 90^\circ - \angle MP'A.$$

Thus, the angle $MP'B'$ is right angle and the triangle $AB'P'$ is isosceles. As a consequence, the point B' is equidistant to the line ℓ and to the point A, i.e., B' lies on the parabola from the statement of the Theorem. □

Since inversions map tangent curves to tangent curves, we obtain another proof of Lemma 2.

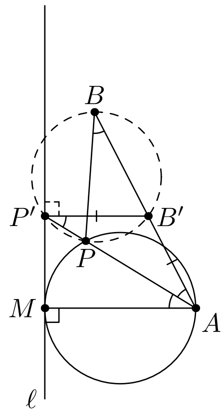

Fig. 11

Some Properties of a Cardioid

Consider again Figure 4. From Lemma 1, and symmetry we see that the angle between BP and BA is equal to the half of $\angle OAB$. Since the tangent line to the cardioid at the point B is perpendicular to BP, simple calculation of angles give us the following Lemma.

Lemma 4. *The oriented angle between the tangent line to the cardioid at a point B and the line AV equals $90^\circ - \frac{3}{2}\angle BAV$, where A and V are the cusp and the vertex of the cardioid.*

By the oriented angle we mean the angle needed to rotate the tangent line clockwise so that it becomes parallel to AV (if the angle is negative, we rotate counterclockwise).

We see that the angle of slope of the tangent line changes one and a half times faster than the angle of slope of BA. Here are two nice corollaries of this observation.

Corollary 5. *Suppose X and Y are two points on a cardioid and the segment XY passes through the cusp. Then, tangent lines at points X and Y are perpendicular (Fig. 12).*

Corollary 6. *Let A be the cusp of a cardioid and X, Y, and Z three points on it. If*

$$\angle XAY = \angle YAZ = \angle ZAX = 120^\circ,$$

then the tangent lines at X, Y, and Z are parallel (Fig. 13).

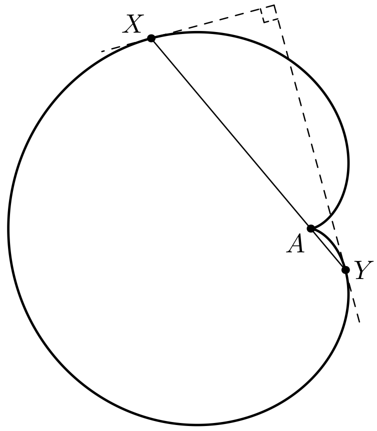

Fig. 12

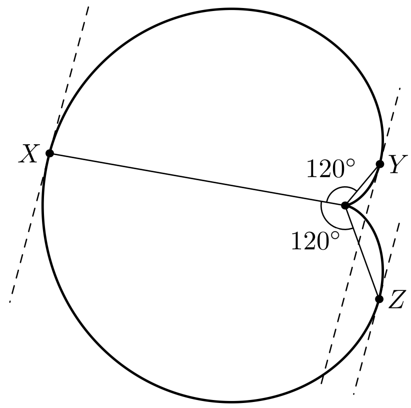

Fig. 13

We continue studying the construction from Figure 4. Denote by C the second point of intersection of the line AB and the cardioid. Let Ω be the circle with center at O and the radius OV. Let M and N be the points of intersection of the rays OP and OQ with Ω (Fig. 14).

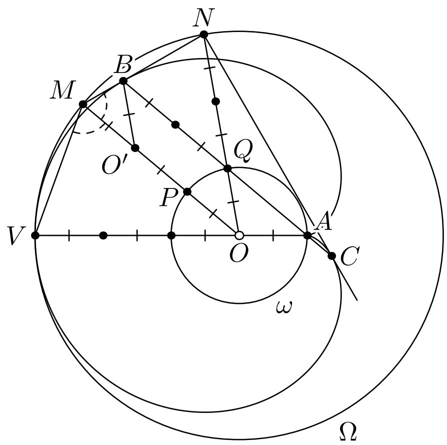

Fig. 14

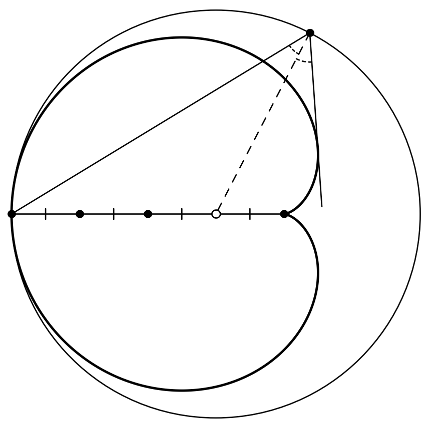

Fig. 15

From the second definition of a cardioid, we know that Q is the midpoint of the segment BC. Therefore, QN equals half of BC and the angle BNC is right angle. Let us show that BN and NC are tangent lines to the cardioid. Since $\angle PBQ = \frac{1}{2}\angle QAO$ and the triangle BQN is isosceles, we have $\angle QBN = 90° - \frac{1}{2}\angle BQN = 90° - \frac{1}{2}\angle QAO$. Therefore, the angle PBN is right angle and BN is a tangent line (analogously for CN). We showed, that *the tangent lines from Corollary 5 are not only perpendicular, but also intersect on the circle* Ω.

Since the angles POV and POQ are equal (and equal to $\angle QAO$), we get that

$$\angle VMO = \angle OMN = 90° - \frac{1}{2}\angle QAO.$$

So, we have shown that *a ray from the vertex* V *after reflecting in the interior of the circle* Ω *travels along a tangent to the cardioid* (Fig. 15). In other words, the cardioid is a *caustic* of the circle if a light source lies on it. So it is possible to meet it in everyday life (Fig. 16, photo by Gérard Janot). In fact, you can try this experiment yourself. Take a regular metal kitchen pan and a source of light such as a lighter, and hold the light source near the edge of the pan. The light reflecting on the walls of the pan will form a cardioid-like shape on the bottom of the pan.

Another interesting fact about the cardioid is that it is the central part of the well-known *Mandelbrot set* (Fig. 17, picture by Wolfgang Beyer). For more details we recommend the book [4].

Fig. 16

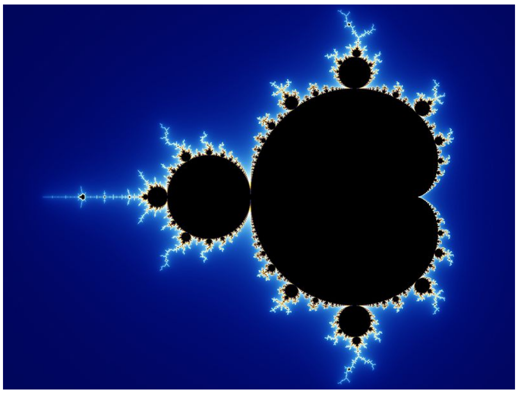

Fig. 17

Acknowledgments

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