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Combinatorial Generalizations of Jung's Theorem

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Abstract We consider combinatorial generalizations of Jung's theorem on covering a set by a ball. We prove the "fractional" and "colorful" versions of the theorem.

Keywords Jung's theorem · Helly's theorem · Covering by a ball

1 Introduction

The famous theorem of Jung states that any set with diameter 1 in $\mathbb{R}^d$ can be covered by the ball of radius $R_d = \sqrt{\frac{d}{2(d+1)}}$ (see [3]).

The proof of this theorem is based on Helly's theorem:

Theorem 1 (Helly's theorem) *Let $\mathscr{P}$ be a family of convex compact sets in $\mathbb{R}^d$ such that an intersection of any $d+1$ of them is not empty. Then the intersection of all of the sets from $\mathscr{P}$ is not empty.*

Helly's theorem has many generalizations. Katchalski and Liu in 1979 [8] proved a "fractional" version of Helly's theorem and Kalai in 1984 [7] gave the strongest version of it. In 1979, Lovász suggested a "colorful" version of Helly's theorem. In this paper, we give analogous generalizations of Jung's theorem.

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2 The Fractinal Version of Jung's Theorem

Let us recall the fractional version of Helly's theorem.

Theorem 2 (Katchalski and Liu [8]) *For every $d \geq 1$ and every $\alpha \in (0, 1]$ there exists a $\beta = \beta(d, \alpha) > 0$ with the following property. Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ be convex sets in $\mathbb{R}^d$ such that $\bigcap \mathcal{X}_i \neq \emptyset$ for at least αC_n^{d+1} index sets $I \subseteq [n]$ of size $(d+1)$. Then there exists a point contained in at least βn sets among $\mathcal{X}_i$.*

The best possible value of $\beta(d, \alpha)$ is $1-(1-\alpha)^{1/(d+1)}$ [7] and, in particular, $\beta \to 1$ as $\alpha \to 1$.

Using this we can prove the fractional version of Jung's theorem.

Theorem 3 *For every $d \geq 1$ and every $\alpha \in (0, 1]$ there exists a $\beta = \beta(d, \alpha) > 0$ with the following property. Let $\mathcal{V}$ be an n-point set in $\mathbb{R}^d$ such that for at least αC_n^2 of pairs $\{x, y\}$ $(x, y \in \mathcal{V})$ the distance between x and y is less than 1. Then there exists a ball of radius R_d, which covers βn points of $\mathcal{V}$. Moreover, $\beta \to 1$ as $\alpha \to 1$.*

Proof Let us show that $\beta > 0$ exists.

We construct a graph G on points of V as vertices. Two vertices of G are connected if and only if the distance between the points is not greater than 1.

Then the degree of some vertex v of G is not less than $\alpha(n-1)$. The ball with center at the point corresponding to v and radius 1 contains at least $\alpha(n-1)+1$ points of $\mathcal{V}$. Any ball of radius 1 can be covered by c_d balls of radius R_d, where c_d is a constant depending only on the dimension d. Then one of these balls should cover more than $\frac{\alpha(n-1)}{c_d}$ points of the set $\mathcal{V}$. Therefore the statement of the theorem holds for $\beta = \alpha/2c_d$, because

$$\frac{\alpha}{2c_d} n < \frac{\alpha(n-1)}{c_d}.$$

Note that this argument works for any radius R not necessarily equal to R_d.

Let us show that $\beta \to 1$ as $\alpha \to 1$. Any pair of vertices belongs to C_{n-2}^{d-1} subsets of $d+1$ vertices. Therefore, if there are at most $(1-\alpha)C_n^2$ empty edges in G, then the number of incomplete subgraphs on $d+1$ vertices is not greater than

$$(1-\alpha)C_n^2 C_{n-2}^{d-1} = (1-\alpha)C_n^{d+1} C_{d+1}^2.$$

We see that among the C_n^{d+1} subgraphs of $d+1$ vertices at least $\alpha' C_n^{d+1}$ are full, where $\alpha' = 1-(1-\alpha)C_{d+1}^2$.

Consider the family $\mathscr{B}$ of n balls of radius R_d with centers at the points of $\mathcal{V}$. If the distances between the centers of some set of balls is no greater than 1, then these balls have nonempty intersection (because Jung's theorem implies that these centers can be covered by a ball of radius R_d). Therefore, among C_n^{d+1} subsystems of $d+1$ balls from $\mathscr{B}$, at least $\alpha' C_n^{d+1}$ have nonempty intersection. From fractional Helly's theorem, it follows that there is a point which belongs to $\beta(d, \alpha')n$ balls. The ball with the center at this point and radius R_d covers βn points of $\mathcal{V}$. To conclude the proof, we note that α' tends to 1, as $\alpha \to 1$. The result of Kalai now implies that $\beta(d, \alpha')$ also tends to 1. □

Note that using an approximation arguments it is possible to prove the same theorem for a measure.

3 Close Sets

We will use the following definition.

Definition 1 We call two nonempty sets $\mathcal{V}_1$ and $\mathcal{V}_2$ *close* if for any points $x \in \mathcal{V}_1$ and $y \in \mathcal{V}_2$, the distance between x and y is not greater than 1.

It is easy to see that if two close sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are given, the diameter of each of them is not greater than 2. Moreover, the following theorem holds.

Theorem 4 *The union of several pairwise close sets in* $\mathbb{R}^d$ *can be covered by a ball of radius* 1.

Proof Denote by $\mathcal{X}$ one of the sets and by $\mathcal{Y}$ union of other sets. Without loss of generality, we may assume that $\mathcal{X}$ and $\mathcal{Y}$ are convex closed sets, because the condition of the theorem also holds for cl (conv$\mathcal{X}$) and cl (conv$\mathcal{Y}$). If $\mathcal{X}$ and $\mathcal{Y}$ have nonempty intersection, then a unit ball with center at any point from the intersection covers $\mathcal{X}$ and $\mathcal{Y}$.

Suppose they do not intersect. Choose points $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ so that the length of $[x, y]$ is minimal. Let m be the midpoint of the segment $[x, y]$. We will show that the unit ball with center at m covers $\mathcal{X}$ and $\mathcal{Y}$. It is known that the hyperplane perpendicular to $[x, y]$ and passing through m separates $\mathcal{X}$ and $\mathcal{Y}$ (see [6]). Take any point z from $\mathcal{X}$. Note that the angle zmy is obtuse (Fig. 1). Therefore the segment $[m, z]$ is shorter than $[y, z]$, which is no longer than 1. The same argument works for any point z from $\mathcal{Y}$. □

It is clear that the diameter of the covering ball in this theorem could not be decreased. The following two questions arise naturally.

Suppose that a family of pairwise close sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ *in* $\mathbb{R}^d$ *is given.*

1. *What is the minimal* R *so that at least one of the sets* $\mathcal{V}_i$ *can be covered by a ball of radius* R?

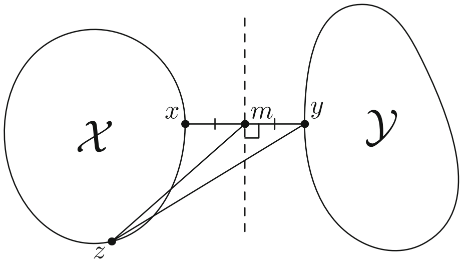

Fig. 1 Covering of two bodies

2. *What is the minimal D so that at least one of the sets $\mathcal{V}_i$ has diameter no greater than D?*

For $n = 3$ and $d = 2$, the author learned the answer to the first question from Vladimir Dol'nikov. His arguments work well for all $n > d$. In Theorem 5, we find the exact value of R for all pairs d and n in the first question. In Theorem 7, we show that the second question is equivalent to the well-known problem about spherical antipodal codes.

4 Colorful Jung's Theorem

Theorem 5 *Let $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_n$ be pairwise close sets in $\mathbb{R}^d$. Then one of the sets $\mathcal{V}_i$ can be covered by a ball of radius R.*

$$R = \frac{1}{\sqrt{2}} \quad \text{if } n \le d;$$
$$R = R_d = \sqrt{\frac{d}{2(d+1)}} \quad \text{if } n > d.$$

Proof First, let us show that R from the statement of the Theorem is minimal. Suppose $n \le d$. Consider a crosspolytope with $2n$ vertices and with edge length 1. Let the sets $\mathcal{V}_i$ be the pairs of opposite vertices of the crosspolytope. Then the distance between any two points from different sets is equal to 1, and distance between two points from the same set is exactly $\sqrt{2}$. Therefore the radius of the minimal cover ball for each of the sets equals $\frac{1}{\sqrt{2}}$.

For $n > d$, let the sets $\mathcal{V}_i$ coincide with regular simplices with edge length 1. In this case, the sets $\mathcal{V}_i$ are close to each other and the radius of the minimal cover ball for each set equals R_d.

Now let us show that one of the sets $\mathcal{V}_i$ can be covered by a ball of radius R. Without loss of generality, we may assume that the sets $\mathcal{V}_i$ are all closed, since the condition of the theorem holds for the closure of these sets.

Suppose $n \le d$. For any set $\mathcal{V}_i$, consider the minimal ball $B(o_i, r_i)$ covering this set. Let r_1 be the minimal radius among r_i. Note that there exists a point x of the set $\mathcal{V}_2$ which does not belong to the interior of the ball $B(o_1, r_1)$. Indeed, if all the points of the set $\mathcal{V}_2$ belong to the interior of the ball $B(o_1, r_1)$, then $\mathcal{V}_2$ can be covered by a ball of radius less than r_1. This contradicts the assumption that r_1 is minimal among r_i. Let s be a hyperplane passing through the point o_1 and perpendicular to the segment $[o_1, x]$ (Fig. 2). This hyperplane divides the sphere $S(o_1, r_1)$ onto two hemispheres.

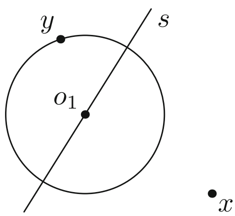

Fig. 2 The minimal covering ball and a point outside

Denote by $S'(o_1, r_1)$ the hemisphere, which lies in the halfspace generated by s not containing x. We need the following lemma (see [3] statements 2.6 and 6.1 or [5] Lemma 2).

Lemma 1 *If a sphere S is the boundary of the minimal ball which covers a closed set $\mathcal{V}$, then the center of S belongs to* $\operatorname{conv}(\operatorname{cl}\mathcal{V} \cap S)$.

Therefore the hemisphere $S'(o_1, r_1)$ contains at least one point y from the set $\mathcal{V}_1$. Note that distance from any point of the hemisphere $S'(o_1, r_1)$ to x is no less than $\sqrt{2}r_1$. Thus the distance between the points x and y is no less than $\sqrt{2}r_1$. Since $\mathcal{V}_1$ and $\mathcal{V}_2$ are close and the distance between x and y is no greater than 1, we have $r_1 \le \frac{1}{\sqrt{2}}$.

Consider the second case: $n \ge d + 1$. This argument is due to V. L. Dol'nikov.

We will use the following theorem of Lovász [1].

Theorem 6 (Colorful Helly's theorem) *Let $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_{d+1}$ be $d+1$ finite families of convex sets in $\mathbb{R}^d$. If $\cap_{i=1}^{d+1} \mathcal{X}_i \neq \emptyset$ for all choices of $\mathcal{X}_1 \in \mathscr{F}_1$, $\mathcal{X}_2 \in \mathscr{F}_2$, ..., $\mathcal{X}_{d+1} \in \mathscr{F}_{d+1}$, then for some i the tersection of all sets from $\mathscr{F}_i$ is not empty.*

Let $\mathscr{B}_i$ be a family of balls of radius R_d with centers at the points of the set $\mathcal{V}_i$. Note that this set of families satisfies condition of colorful Helly's theorem. Indeed, if we choose one ball from each of the sets $\mathcal{B}_i \in \mathscr{B}_i$, then the distance between its centers will not be greater than 1. From Jung's theorem, it follows that they can be covered by a ball $\mathcal{B}$ of radius R_d. This means that the center of $\mathcal{B}$ is contained in all sets $\mathcal{B}_i$, $i = 1, 2, \ldots, n$. Thus they have nonempty intersection.

Applying colorful Helly's theorem to the families $\mathscr{B}_i$, we obtain that for some i, the balls of the family $\mathscr{B}_i$ have nonempty intersection. Therefore, all points of the set $\mathcal{V}_i$ can be covered by a ball of radius R_d centered at any point of intersection of all balls from the family $\mathscr{B}_i$. □

Remark 1 Note that, by the same arguments, it can be shown that for $n \le d$ all sets except one can be covered by a ball of radius $1/\sqrt{2}$. For the proof, one considers the minimal ball which covers all sets except one.

If $n > d$, then all but d sets can be covered by a ball of radius R_d. Here one should use the modified version of colorful Helly's theorem: *Suppose a collection of families $\{\mathscr{F}_i\}$ of convex sets is given and the intersection of any $d+1$ sets from different families is not empty. Then there is a point which pierces all but d sets from the families $\{\mathscr{F}_i\}$.*

The proof is the same as the classical proof of colorful Helly's theorem (see [9]): consider d sets from different families whose intersection has the lowest higher point among all intersections of d sets from different families. This point should be contained in all but the chosen d sets of the families $\{\mathscr{F}_i\}$.

5 The Bound on the Diameter of a Set

In this section, we study a bound on the diameter of $\mathcal{V}_i$.

Definition 2 A *spherical code* is a finite set of points in $\mathbb{S}^d$. A spherical code is called *antipodal* if it is symmetric with respect to the center of the sphere.

A spherical code $\mathcal{W}$ of n points is called *optimal* if it has a maximal *minimal diameter* ($\min|x-y|$, where $x, y \in \mathcal{W}$ and $x \neq y$) among all spherical antipodal codes of the cardinality n.

By $D_d(n)$, we denote the minimal diameter of the optimal spherical antipodal code of cardinality $2n$ on the unit sphere $\mathbb{S}^{d-1}$.

Theorem 7 *Let $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ be pairwise close sets in $\mathbb{R}^d$. Then one of the sets $\mathcal{V}_i$ has diameter no greater than*

$$D = \frac{2}{\sqrt{4 - D_d(n)^2}}.$$

Proof Suppose otherwise. Then the diameter of each $\mathcal{V}_i$ is no less than $D' > D$. Again, we may assume that all sets $\mathcal{V}_i$ are closed convex sets. Then in each set $\mathcal{V}_i$, it is possible to choose two points a_i and b_i with distance between them equal to D'. So, we can assume that each set $\mathcal{V}_i$ is a two-point set: $\mathcal{V}_i = \{a_i, b_i\}$.

Let us show, without loss of generality, we may assume that the midpoints of the segments $[a_i, b_i]$ coincide.

Indeed, denote the origin by o.

Suppose

$$a_i' = o + \frac{\overrightarrow{b_i a_i}}{2}, \quad b_i' = o + \frac{\overrightarrow{a_i b_i}}{2}.$$

Then

$$\begin{aligned} |a_i' - a_j'| &= |\overrightarrow{a_i' a_j'}| \\ &= \frac{|\overrightarrow{a_i b_i} + \overrightarrow{b_j a_j}|}{2} = \frac{|\overrightarrow{a_i b_j} + \overrightarrow{a_j b_i}|}{2} \\ &\le \tfrac{1}{2}(|a_i - b_j| + |a_j - b_i|) \le 1. \end{aligned}$$

The same inequality holds for all other pairs of points (a_i', b_j') and (b_i', b_j').

We may assume that the points of the sets $\mathcal{V}_i$ lie on the sphere of radius $D'/2$ and form the antipodal code. Since the distance between the points, say, a_i and a_j is not greater than 1, the distance between a_i and b_j should be greater than $\sqrt{D'^2 - 1}$.

Therefore,

$$\begin{aligned} &\frac{2\sqrt{D'^2 - 1}}{D'} < D_d(n) \\ &\Leftrightarrow 4D'^2 - 4 < D_d(n)^2 D'^2 \\ &\Leftrightarrow 4D'^2 - D_d(n)^2 D'^2 < 4 \\ &\Leftrightarrow D'^2 \le \frac{4}{4 - D_d(n)^2} \\ &\Leftrightarrow D' \le \frac{2}{\sqrt{4 - D_d(n)^2}} = D. \end{aligned}$$

This contradicts the assumption $D' > D$ and concludes the proof. □

It is clear that the corresponding optimal antipodal code yields equality for D.

Unfortunately, the precise value of $D_d(n)$ is known only for a few cases. In particular, the following numbers are known (see [2,4]):

$$\begin{aligned} &D_2(n) = 2\sin\frac{\pi}{2n};\\ &D_d(n) = \sqrt{2} \text{ for } n \le d;\\ &D_3(6) = \frac{2}{\sqrt{10+2\sqrt{5}}};\\ &D_4(12) = D_8(120) = D_{24}(98280) = 1. \end{aligned}$$

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