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On maximum volume simplices in polytopes

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Abstract The paper is devoted to the volume interpretation of Radon's theorem about partitions. Here we give a new proof of the theorem, show how ideas of this proof can be applied to prisms and parallelotopes, and subsequently generalize several statements by M. Lassak.

Keywords Radon's theorem · Triangulations · Maximal volume simplex

1 Introduction

The theorem published by J. Radon [5] states that any $d+2$ point set in $\mathbb{R}^d$ can be partitioned into two subsets such that their convex hulls intersect. This partition is unique if points are in general position. This theorem, proved in 1921, is the cornerstone of discrete convex geometry. In particular, it implies Helly's theorem for convex bodies.

As a classical result, Radon's theorem has many miscellaneous proofs and various generalizations. One of the most famous generalization is the Tverberg theorem [7], which states that for any set of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ there is a partition into r subsets such that convex hulls of all subsets have a common point. The shortest proof of the Tverberg theorem belongs to K. Sarkaria [6]. He constructed a special $(d+1)(r-1)$-dimensional point set and showed that the Tverberg theorem follows from Barany's Colorful Caratheodory theorem [1] for this set.

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In this article we suggest another point of view on Radon's theorem that is quite similar to Sarkaria's approach but we use a volume argument rather than the combinatorial theorem by Barany. Also we use this approach to generalize several statements by M. Lassak [2].

The paper is organized in the following way. In the first section, we formulate a key lemma which is the main tool of the article. In the second section we discuss the Radon theorem and its connection to simplices of maximal volume. In the last section we discuss simplices of maximum volume in prisms and parallelotopes. There we give new proofs of some statements from Lassak's paper.

2 Key lemma

Lemma 2.1 *Let B be a centrally symmetric set of $2d + 2$ points in $\mathbb{R}^d$ whose center is at the origin such that no hyperplane through 0 contains $2d$ points of B. Let $\mathcal{T}$ be a family of simplices with vertices from B without a pair of symmetric points as vertices. Then there are exactly two simplices in $\mathcal{T}$ containing the origin in their respective interiors. They are symmetric to each other with respect to the origin and have the maximum Euclidean volume among all simplices in $\mathcal{T}$.*

Proof Denote the pairs of symmetric points of B by x_i, $-x_i$ for $i = 1, 2, \dots, d+1$. Consider a simplex Δ from $\mathcal{T}$ with the maximum volume. Its vertices are $\varepsilon_1 x_1, \dots, \varepsilon_{d+1} x_{d+1}$, where all $\varepsilon_i \in \{-1, 1\}$ for $i = 1, 2, \dots, d+1$. Note that 0 does not belong to a hyperlane containing a facet of Δ since this hyperlane contains $2d$ vertices of B.

Assume that 0 is outside of Δ. Then there is a facet of Δ such that the hyperplane carrying Δ separates Δ from 0. Without loss of generality this facet is $\varepsilon_1 x_1 \dots \varepsilon_d x_d$. Then the distance between $\varepsilon_{d+1} x_{d+1}$ and the hyperplane carrying this facet is less than the distance between $-\varepsilon_{d+1} x_{d+1}$ and this hyperplane. Therefore the volume of the simplex with vertices $\varepsilon_1 x_1, \dots, \varepsilon_d x_d, -\varepsilon_{d+1} x_{d+1}$ is greater than the volume of Δ. Contradiction.

Let us show that there is only one pair of such simplices. Without loss of generality we can assume that all ε_i, $i = 1, \dots, d + 1$, are $+1$. Note that the hyperplane passing through $0, x_3, \dots, x_{d+1}$ separates x_1 and x_2 since 0 belongs to Δ. Consider any other simplex from $\mathcal{T}$ with the origin in the interior. The hyperplane passing through $0, \pm x_3, \dots, \pm x_{d+1}$ is the same as for Δ and since it should separate two remaining vertices, these vertices must be either x_1 and x_2, or $-x_1$ and $-x_2$. Analogously for all pairs of x_i. □

Corollary 2.2 *In terms of Lemma 2.1, if a simplex from $\mathcal{T}$ with vertices $x_1, x_2, \dots, x_{d+1}$ has the volume greater than the volume of any simplex with vertices $x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_{d+1}$, where $j = 1, 2, \dots, d + 1$, then this simplex has the maximum volume among all simplices in $\mathcal{T}$.*

Proof From the first part of the proof of Lemma 2.1 it follows that this simplex contains 0 inside. From Lemma 2.1 it follows that this simplex has the maximum volume among all simplices in $\mathcal{T}$. □

From Lemma 2.1 it is easy to obtain the following well-known statement.

Corollary 2.3 *A simplex of maximum volume inscribed into a centrally symmetric convex body contains the center of this body.*

Remark 2.4 For each simplex with vertices $0, \varepsilon_1 x_1, \dots, \varepsilon_{j-1} x_{j-1}, \varepsilon_{j+1} x_{j+1}, \dots, \varepsilon_{d+1} x_{d+1}$, where j is fixed, its volume is the same for all possible sets of ε_i, $1 \le i \le d+1$, $i \ne j$, since it is equal to $\frac{1}{d!}$ multiplied by the same determinant. For each j denote this volume by V_j. The volume of any simplex on $\varepsilon_1 x_1, \ldots, \varepsilon_{d+2} x_{d+2}$ (i.e. any simplex of $\mathcal{T}$) is $\pm V_1 \pm V_2 \pm \ldots \pm V_{d+2}$ depending on a position of 0 with respect to this simplex. In case 0 is inside this simplex the volume is $V_1 + V_2 + \ldots + V_{d+1}$. And again we get that the maximum volume of simplices in $\mathcal{T}$ is realized if and only if 0 is inside a simplex.

3 Radon's theorem and basic ideas

Lemma 2.1 can be applied for the proof of Radon's theorem.

Theorem 3.1 [Radon's theorem] Any set S of $d + 2$ points in general position in $\mathbb{R}^d$ has a unique partition into two disjoint sets whose convex hulls intersect.

Such partitions are called *Radon's partitions*.

The main idea under finding Radon's partitions is in the spirit of Sarkaria's proof of Tverberg's theorem. We consider two parallel hyperplanes R_1 and R_2 in $\mathbb{R}^{d+1}$ with congruent copies $B_1 \subset R_1$ and $B_2 \subset R_2$ of S. For each partition of S into two disjoint subsets S_1 and S_2 we consider vertices corresponding to S_1 in B_1 and vertices corresponding to S_2 in B_2. These $d + 2$ points are vertices of a $(d + 1)$-dimensional simplex. Thus for each partition we defined a simplex corresponding to it. Simplices that can be obtained from partitions of S will are called *acceptable*.

Now the problem is to define hyperplanes and copies of S in them properly and to characterize Radon's partitions in terms of simplices. We consider two close constructions. The first one is a special case of Sarkaria's proof of Tverberg's theorem. Hyperplanes R_1 and R_2 and copies of S are symmetric with respect to 0. We will call this case *centrally symmetric* and denote the family of acceptable simplices by $\mathcal{T}_c$. The second construction is for two parallel hyperplanes R_1 and R_2 and copies of S congruent up to a translation. We will call this case *parallel* and denote the family of acceptable simplices by $\mathcal{T}_p$.

Lemma 3.2 *A partition is Radon's if and only if in the centrally symmetric case the simplex corresponding to it contains* 0 *in the interior.*

Proof Assume that the simplex Δ corresponding to a partition of S into S_1 and S_2 does not contain 0 in the interior. Denote $\Delta \cap R_1$ by Δ_1 and $\Delta \cap R_2$ by Δ_2. A simplex centrally symmetric to Δ_2 is denoted by Δ_2'. Consider any hyperplane through 0 which does not intersect Δ. Then this hyperplane separates Δ_1 and Δ_2', so Δ_1 and Δ_2' do not intersect. Since the partition of B into Δ_1 and Δ_2' is congruent to the partition of S into S_1 and S_2, it is not Radon's.

The converse is very similar so we just describe it briefly. If the partition is not Radon's we can divide the corresponding subsets of B by a hyperplane in $\mathbb{R}^d$. Connecting this hyperplane with 0 we obtain a hyperplane in $\mathbb{R}^{d+1}$ that does not intersect the simplex corresponding to the partition and therefore 0 is outside of it. □

Proposition 3.3 *In the centrally symmetric case, Δ is a simplex of maximal volume from $\mathcal{T}_c$ if and only if a partition corresponding to Δ is Radon's.*

Proof This proposition follows immediately from Lemmas 2.1 and 3.2. □

By proving this proposition we simultaneously proved Radon's theorem. Since by Lemma 2.1 there exist two maximal simplices and they correspond to the same partition, Radon's partition exists and is unique.

Lemma 3.4 *Consider two simplices Δ and Δ' in $\mathbb{R}^{d+1}$ with vertices in two parallel hyperplanes H_1 and H_2 so that $\Delta \cap H_1 = \Delta' \cap H_1$ and $\Delta \cap H_2$ is centrally symmetric to $\Delta' \cap H_2$. Then Δ and Δ' have equal volumes.*

Proof Since parallel translations of simplex vertices in H_2 do not change the volume, we may consider the case when the center of symmetry in H_2 is a vertex of both Δ and Δ'. The rest follows from the volume formula through determinants. □

Proposition 3.5 *In the parallel case, Δ is a simplex of maximal volume from $\mathcal{T}_p$ if and only if a partition corresponding to Δ is Radon's.*

Proof Applying the previous lemma we see that the volumes in the parallel and centrally symmetric cases are actually the same. Therefore, maximal volumes in two cases occur for simplices corresponding to the same partitions and, because of Proposition 3.3, this proposition is proved. □

Remark 3.6 It also follows from Corollary 2.2 that if we cannot increase the volume of the simplex by changing one vertex to its corresponding translate then this simplex is maximal by volume.

Remark 3.7 It is interesting whether a similar approach can be applied to prove the Tverberg theorem. Our attempts to apply the previously mentioned construction of Sarkaria are unsuccessful.

4 Generalization of Lassak's lemmas

Consider a convex polytope $Q \subset \mathbb{R}^d$ and a non-zero vector $\vec{v}$.

Definition 4.1 A simplex Δ inside Q is called $\vec{v}$-*maximal* if by moving any vertex of Δ (but only one) inside Q parallel to $\vec{v}$ the volume of Δ cannot be increased.

Definition 4.2 A simplex Δ inside Q is called *vertex maximal* if by moving any vertex of Δ (but only one) inside Q the volume of Δ cannot be increased.

Remark The following implications are obvious: Δ is maximal (with respect to the volume) inside $Q \Rightarrow \Delta$ is vertex maximal inside $Q \Rightarrow \Delta$ is $\vec{v}$-maximal inside Q.

For the next lemmas we consider a prism Q with bases B_1 and B_2 and a translation vector $\vec{v}$ from B_1 to B_2.

Theorem 4.3 For a prism Q with a translation vector $\vec{v}$ between its bases, any $\vec{v}$-maximal simplex $\Delta \subset Q$ contains some points y_1 and y_2 such that $\overrightarrow{y_1 y_2} = \vec{v}$ and this pair of points is unique for this simplex.

Proof Firstly we prove that such a pair exists. The proof is by induction on dimension. The base of induction for $d = 1$ is obvious. Now for the inductive step we have two cases.

In the first case there is a vertex of Δ that is strictly between two bases of Q. Since any move of this vertex along $\vec{v}$-direction does not increase the volume of Δ, the facet of Δ opposite to this vertex is parallel to $\vec{v}$. Consider the section of Q by the hyperplane containing this facet. By the induction hypothesis such pair exists for this section and this case is done.

In the second case all vertices of Δ are situated on the bases of Q. If there are two vertices x_1, x_2 of Δ with $\overrightarrow{x_1x_2} = \vec{v}$ that is the vector we are looking for. If there are no such vertices, for each vertex we consider its pair from the other base of Q forming a vector congruent to $\vec{v}$. In the terms of the previous sections Δ belongs to the set of acceptable simplices $\mathcal{T}_p$. From Proposition 3.5 and Remark 3.6 it follows that Δ is maximal in $\mathcal{T}_p$ and represents the corresponding Radon's partition. The point of intersection for this Radon's partition belongs to both subsets of the partition. Consider a point y_1 in Δ corresponding to the point of intersection in the first subset of the partition and a point y_2 in Δ corresponding to the point of intersection in the second subset of the partition. Then $\overrightarrow{y_1y_2} = \vec{v}$ or $-\vec{v}$. In both cases we are done.

Now we prove uniqueness. If there are two such pairs $\overrightarrow{y_1y_2} = \overrightarrow{y_3y_4} = \vec{v}$, then $\overrightarrow{y_1y_3} = \overrightarrow{y_2y_4}$ which is impossible due to linear independence of simplex vectors lying in B_1 and B_2. □

Following Lassak [2] by $(-d\Delta)$ we denote the homothetic copy of Δ with ratio $(-d)$ and homothety center at the center of mass of Δ.

Proposition 4.4 Δ is vertex maximal inside $Q \Rightarrow \Delta \subset Q \subset -d\Delta \Rightarrow \Delta$ is $\vec{v}$-maximal inside Q.

Proof The first implication follows directly from the fact that the hyperplane through a vertex of Δ parallel to its opposite face touches Q.

For the second implication we assume the contrary, i.e. there is a vertex a of Δ such that we can move it parallel to $\vec{v}$ inside Q and increase the volume of the simplex. Consider the hyperplane H through all other vertices of Δ. It divides $-d\Delta$ into two parts, one of which is a simplex Δ' homothetic to Δ. Moving a inside the other part cannot increase the volume so a must be moved to this simplex.

Now for each vertex of Δ consider a vector congruent to $\vec{v}$ going through this vertex and lying in Q. One of endpoints of each vector lies between the hyperplane of $-d\Delta$ containing a and H. The second endpoint of such vector for a lies in the other open halfspace of H. Since a is the farthest vertex of Δ the ends of other vectors lie in the other halfspace too. Therefore when we slightly move all vertices of Δ in the direction of $\vec{v}$ the whole simplex must remain inside Q and subsequently inside $-d\Delta$ which is of course impossible. □

Applying Theorem 4.3 for a parallelotope as a prism for many directions we get the following lemma.

Lemma 4.5 For a d-dimensional parallelotope P with edge vectors $\vec{v}_1, \ldots, \vec{v}_d$ and a d-dimensional simplex Δ inside P, if Δ is $\vec{v}_i$-maximal inside P for each i, $1 \le i \le d$, then for each i, $1 \le i \le d$, there exists a unique pair of points y_1^i, y_2^i inside Δ such that $\overrightarrow{y_1^i y_2^i} = \vec{v}_i$.

From three previous statements we immediately get two lemmas proved in [2] by Lassak (also see [3,4]).

Lemma 4.6 If $\Delta \subset Q \subset -d\Delta$ then Δ contains a unique pair of points y_1, y_2 inside such that $\overrightarrow{y_1y_2} = \vec{v}$.

Lemma 4.7 If P is a d-dimensional parallelotope with edge vectors $\vec{v}_1, \ldots, \vec{v}_d$ and Δ is a d-dimensional simplex such that $\Delta \subset P \subset -d\Delta$ then for each i, $1 \le i \le d$, there exists a unique pair of points y_1^i, y_2^i inside Δ such that $\overrightarrow{y_1^i y_2^i} = \vec{v}_i$.

Also Lemma 4.5 immediately implies the following fact (see Final Remarks in [2]).

Proposition 4.8 For a d-dimensional parallelotope P with edge vectors $\vec{v}_1, \ldots, \vec{v}_d$ and a d-dimensional simplex Δ inside P, if Δ is vertex maximal inside P then for each i, $1 \leq i \leq d$, there exists a unique pair of points y_1^i, y_2^i inside Δ such that $\overrightarrow{y_1^i y_2^i} = \vec{v}_i$.

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