A SHORT PROOF OF THE COLLAPSING WALLS LEMMA

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Abstract. In this short note we give a simple geometric proof of the *Collapsing Walls Lemma* given by I. Pak and R. Pinchasi in [1]. Our arguments work in spaces of constant curvature.

1. Introduction

In [1] Igor Pak and Rom Pinchasi proved a theorem which can be formulated in the following way: If the facets of a pyramid with a convex base are "collapsed" then they cover the base of the pyramid. In [2] they prove the more general result, which they call the *Collapsing Walls Lemma*.

For a formulation of the statement, we need one definition. Let $\mathcal{P}$ be a convex polytope in Euclidean space $\mathbb{R}^d$, spherical space $\mathbb{S}^d$ or hyperbolic space $\mathbb{H}^d$. For a facet F of $\mathcal{P}$, we denote by H_F the hyperplane supporting $\mathcal{P}$ at the facet F. If H_F and H_G intersect, we denote by $\Phi_{F,G}$ the rotation about $H_F \cap H_G$ such that the half-hyperplanes H_G containing G are rotated onto H_F containing F. If these planes are parallel or ultraparallel (in hyperbolic case) then we define $\Phi_{F,G}$ as a reflection in the plane which bisects planes H_F and H_G[1]. We call $\Phi_{F,G}(G)$ the collapse of the facet G.

The following lemma was proved in [2] for the Euclidean case.

Collapsing Walls Lemma. *Let $\mathcal{P} \subset \mathbb{R}^d$ ($\mathbb{S}^d$ or $\mathbb{H}^d$) be a convex polytope and let F be a fixed facet of $\mathcal{P}$. Then*

$$F \subseteq \bigcup_{G \neq F} \Phi_{F,G}(G),$$

where the union is over all facets G of $\mathcal{P}$ different from F.

2. Proof of the Lemma

For any facet G_i of $\mathcal{P}$ denote by G'_i the hyperplane which is the bisector of H_F and H_G. That means that if H_F and H_{G_i} intersect then it is the hyperplane which bisects the dihedral angle between them into two equal parts and intersects the polytope $\mathcal{P}$ (actually there are two bisecting hyperplanes and the second one does not intersect $\mathcal{P}$). If H_F and H_{G_i} do not intersect then denote the bisecting hyperplane by G'_i; it is unique

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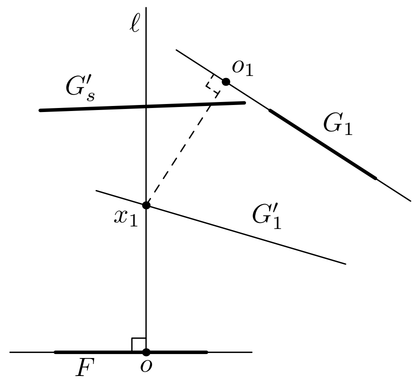
[1] It is more natural to define this transformation as a rotation about a $(d-2)$-plane which is lies beyond the absolute but this requires additional definitions.

in this case. It is important to note that H_F and H_{G_i} are symmetric with respect to the hyperplane G_i'.

Let o be any point of the facet F, and ℓ be a ray from the point o which is perpendicular to F and goes to the halfspace of $\mathcal{P}$. For each facet G_i of $\mathcal{P}$ denote by x_i the intersection point of ℓ and G_i' (in the hyperbolic case this point sometimes does not exist; in this case for any point on ℓ the distance to G_i is greater than the distance to F). Without loss of generality, the distance $d(x_i, o)$ is minimal for $i = 1$. Let us show that after collapsing the facet G_1 covers the point o.

Firstly, note that x_1 lies inside P because for any hyperplane H_{G_i} it lies on the same half-space with the point o.



Denote by o' the perpendicular projection of the point x_1 onto the hyperplane H_{G_1}. It is easy to see that $d(x_1, o') = d(x_1, o)$ and the point o' goes to o after the collapsing of G_1. For the proof it is sufficient to show that o' belongs to G_1.

Suppose it is not. Then there is a facet G_s separating the point o_1 from $\mathcal{P}$. Note that in this case

$$d(x_1, G_s) < d(x_1, o_1) = d(x_1, o).$$

So, we see that the point x_s should exist and x_1 belongs to the segment $[x_s, o]$. We have

$$d(x_s, o) = d(x_s, x_1) + d(x_1, o) = d(x_s, x_1) + d(x_1, o_1) > d(x_s, x_1) + d(x_1, G_s) \geqslant d(x_s, G_s),$$

a contradiction.

References

[1] I. Pak and R. Pinchasi. *The collapsing walls theorem*, Amer. Math. Monthly, 2012, to appear.

[2] I. Pak and R. Pinchasi. *How to cut out a convex polyhedron*, to appear in Contributions to Discrete Mathematics (2012), to appear.

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