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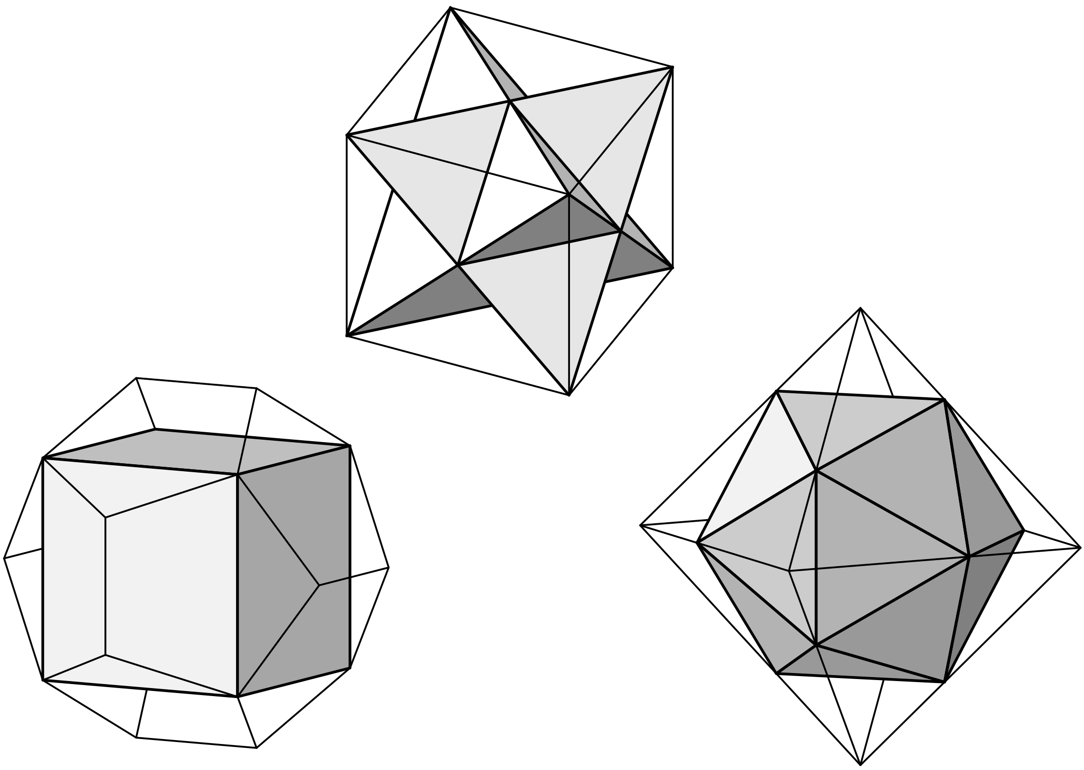

Mathematics via Problems

PART 2: Geometry

Alexey A. Zaslavsky & Mikhail B. Skopenkov

MSRI Mathematical Sciences Research Institute
American Mathematical Society

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Translated from Russian by Paul Zeitz and Sergei G. Shubin

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Contents

Foreword

Problems, exercises, circles, and olympiads

This is a translation of Part 2 of the book *Mathematics via Problems* by A. B. Skopenkov, M. B. Skopenkov, and A. A. Zaslavsky, and is part of the AMS/MSRI Mathematical Circles Library series. The goal of this series is to build a body of works in English that help to spread the "math circle" culture.

A *mathematical circle* is an eastern European notion. Math circles are similar to what most Americans would call a math club for kids, but with several important distinguishing features.

First, they are *vertically integrated*: young students may interact with older students, college students, graduate students, industrial mathematicians, professors, and even world-class researchers, all in the same room. The circle is not so much a classroom as a gathering of young initiates with elder tribespeople, who pass down *folklore*.

Second, the "curriculum," such as it is, is dominated by *problems* rather than specific mathematical topics. A problem, in contrast to an exercise, is a mathematical question that one doesn't know how, at least initially, to approach. For example, "What is 3 times 5?" is an exercise for most people but a problem for a very young child. Computing 5^{34} is also an exercise, conceptually very much like the first example, certainly harder, but only in a "technical" sense. And a question like "Evaluate $\int_2^7 e^{5x} \sin 3x \, dx$" is also an exercise—for calculus students—a matter of "merely" knowing the right algorithm and how to apply it.

Problems, by contrast, do not come with algorithms attached. By their very nature, they require *investigation*, which is both an art and a science, demanding technical skill along with focus, tenacity, and inventiveness. Math circles teach students these skills, not with formal instruction, but by having them *do math* and observe others doing math. Students learn that a problem worth solving may require not minutes but possibly hours, days, or even years of effort. They work on some of the classic folklore problems and discover how these problems can help them investigate other problems. They learn how not to give up and how to turn errors or failures into opportunities for more investigation. A child in a math circle learns to do exactly what a research mathematician does; indeed, he or she does independent research, albeit at a lower level, and often—although not always—on problems that others have already solved.

Finally, many math circles have a culture similar to that of a sports team, with intense camaraderie, respect for the "coach," and healthy competitiveness (managed wisely, ideally, by the leader/facilitator). The math circle culture is often complemented by a variety of problem solving contests, often called *olympiads*. A mathematical olympiad problem is, first of all, a genuine problem (at least for the contestant), and usually requires an answer which is, ideally, a well-written argument (a "proof").

Why this book, and how to use it

The Mathematical Circles Library editorial board chose to translate this work from Russian into English because this book has an audacious goal—promised by its title—to develop mathematics through problems. This is not an original idea, nor just a Russian one. American universities have experimented for years with IBL (inquiry-based learning) and Moore-method courses, structured methods for teaching advanced mathematics through open-ended problem solving.[1]

But the authors' massive work is an attempt to curate sequences of problems for secondary students (the stated focus is on high school students, but that can be broadly interpreted) that allow them to discover and recreate much of "elementary" mathematics (number theory, polynomials, inequalities, calculus, geometry, combinatorics, game theory, probability) and start edging into the sophisticated world of group theory, Galois theory, etc.

The book is impossible to read from cover to cover—nor should it be. Instead, the reader is invited to start working on problems that he or she finds appealing and challenging. Many of the problems have hints and solution sketches, but not all. No reader will solve all the problems. That's not the point—it is not a contest. Furthermore, some of the problems are not supposed to be solved, but should rather be pondered. For example, as soon as it is "technically" possible to solve it, the book introduces Apollonius's problem (Problem 3.10.6 in Chapter 3), one of the deepest and most famous challenges of classical geometry, and the text provides references for learning more about it. Just because it is "too advanced" doesn't mean that it shouldn't be thought about!

Indeed, this is the philosophy of the book: mathematics is not a sequential discipline, where one is presented with a definition that leads to a lemma which leads to a theorem which leads to a proof. Instead it is an adventure, filled with exciting side trips as well as wild goose chases. The adventure is its own reward, but it also, fortuitously, leads to a deep understanding

[1] See, for example, https://en.wikipedia.org/wiki/Moore_method and http://www.jiblm.org.

and appreciation of mathematical ideas that cannot be achieved by passive reading.

English-language references

Most of the references cited in this book are in Russian. However, there are a few excellent books in English. Our two favorites are the classic *Geometry Revisited* [**CG67**] and [**Che16**], a recent and very comprehensive guide to "olympiad geometry."

Paul Zeitz
June 2020

Introduction

What this book is about and who it is for

A deep understanding of mathematics is useful both for mathematicians and for high-tech professionals. In particular, the "profession" in the title of this book does not necessarily mean the profession of mathematics.

This book is intended for high school students and undergraduates (in particular, those interested in olympiads). For more details, see "Olympiads and mathematics" on p. xv. The book can be used both for self-study and for teaching.

This book attempts to build a bridge (by showing that there is no gap) between ordinary high school exercises and the more sophisticated, intricate, and abstract concepts in mathematics. The focus is on engaging a wide audience of students to think creatively in applying techniques and strategies to problems motivated by "real world or real work" [**Mey**]. Students are encouraged to express their ideas, conjectures, and conclusions in writing. Our goal is to help students develop a host of new mathematical tools and strategies that will be useful beyond the classroom and in a number of disciplines [**IBL**, **Mey**, **RMP**].

The book contains the most standard "base" material (although we expect that at least some of this material will be review—that not all is being learned for the first time). But the main content of the book is more complex material. Some topics are not well known in the traditions of mathematical circles, but are useful both for mathematical education and for preparation for olympiads.

The book is based on classes taught by the authors at different times at the Independent University of Moscow, at the National Research University Higher School of Economics, at various Moscow schools, in preparing the Russian team for the International Mathematical Olympiad, in the "Contemporary Mathematics" summer school, in the Kirov and Kostroma Summer Mathematical Schools, in the Moscow visiting Olympiad School, in the "Mathematical Seminar" and "Olympiad and Mathematics" circles, and also at the summer Conference of the Tournament of Towns.

Much of this book is accessible to high school students with a strong interest in mathematics.[2] We provide definitions of concepts that are not standard in the school curriculum, or provide references. However, many topics are difficult if you study them "from scratch." Thus, the ordering of the problems helps to provide "scaffolding." At the same time, many topics are *independent* of each other. For more details, see p. xvi, "How this book is organized".

Learning by solving problems

We subscribe to the tradition of studying mathematics by solving and discussing problems. These problems are selected so that in the process of solving them the reader (more precisely, the solver) masters the fundamentals of important ideas, both classical and modern. The main ideas are developed incrementally with olympiad-style examples—in other words, by the simplest special cases, free from technical details. In this way, we show *how you can explore and discover these ideas on your own.*

Learning by solving problems is not just a serious approach to mathematics but also continues a venerable cultural tradition. For example, the novices in Zen monasteries study by reflecting on riddles ("koans") given to them by their mentors. (However, these riddles are rather more like paradoxes than what we consider to be problems.) See, for example, [**Su**]; compare with [**Pl**, pp. 26–33]. "Math" examples can be found in [**Ar04**, **BSh**, **GDI**, **KK08**, **Pr07-1**, **PoSe**, **SCY**, **Sk09**, **Va87-1**, **Zv**], which sometimes describe not only problems but also the principles of selecting appropriate problems. For the American tradition, see [**IBL**, **Mey**, **RMP**].

Learning by solving problems is difficult, in part, because it generally does not create the *illusion* of understanding. However, one's efforts are fully rewarded by a deep understanding of the material, at first, with the ability to carry out similar (and sometimes rather different) reasoning. Eventually, while working on fascinating problems, readers will be following the thought processes of the great mathematicians and may see how important concepts and theories naturally evolve. Hopefully this will help them make their own equally useful discoveries (not necessarily in math)!

Solving a problem, theoretically, requires only understanding its statement. Other facts and concepts are not needed. (However, useful facts and ideas will be developed when solving selected problems.) And you may need to know things from other parts of the book, as indicated in the instructions and hints. For the most important problems we provide hints, instructions, solutions, and answers, located at the end of each section. However, these should be referred to only after attempting the problems.

[2]Some of the material is studied in some circles and summer schools by those who are just getting acquainted with mathematics (for example, 6th graders). However, the presentation is intended for a reader who already has at least a minimal mathematical background. Younger students need a different approach; see, for example, [**GIF**].

As a rule, we present the *formulation* of a beautiful or important result (in the form of problems) before its *proof*. In such cases, one may need to solve later problems in order to fully work out the proof. This is always explicitly mentioned in the text. Consequently, if you fail to solve a problem, please read on. This guideline is helpful because it simulates the typical research situation (see [**ZSS**, Ch. 28]).

This book "is an attempt to demonstrate learning as *dialogue* based on solving and discussing problems" (see [**KK15**]).

Parting words *By A. Ya. Kanel-Belov*

To solve difficult olympiad problems, at the very least one must have a robust knowledge of algebra (particularly algebraic transformations) and geometry. Most olympiad problems (except for the easiest ones) require "mixed" approaches; rarely is a problem resolved by applying a method or idea in its pure form. Approaching such mixed problems involves combining several "crux" problems, each of which may involve single ideas in a "pure" form.

Learning to manipulate algebraic expressions is essential. The lack of this skill among olympians often leads to ridiculous and annoying mistakes.

Olympiads and mathematics

> To him a thinking man's job was not to deny one reality
> at the expense of the other, but to include and to connect.
>
> U. K. Le Guin. *The Dispossessed.*

Here are three common misconceptions: the best way to prepare for a math olympiad is by solving last year's problems; the best way to learn "serious" mathematics is by reading university textbooks; the best way to master any other skill is with no math at all. A further misconception is that it difficult to achieve any two of these three goals simultaneously, because they are so divergent. The authors share the belief that these three approaches miss the point and lead to harmful side effects: students either become too keen on emulation, or they study the *language* of mathematics rather than its *substance*, or they underestimate the value of robust math knowledge in other disciplines.

We believe that these three goals are not as divergent as they might seem. The foundation of mathematical education should be the *solution and discussion of problems interesting to the student, during which a student learns important mathematical facts and concepts.* This simultaneously prepares the student for math olympiads and the "serious" study of mathematics, and is good for his or her general development. Moreover, it is more effective for achieving success in any one of the three goals above.

Research problems for high school students

Many talented high school or university students are interested in solving research problems. Such problems are usually formulated as complex questions broken into incremental steps; see, e.g., [**LKTG**]. The final result may even be unknown initially, appearing naturally only in the course of thinking about the problem. Working on such questions is useful in itself and is a good approximation to scientific research. Therefore it is useful if a teacher or a book can support and develop this interest.

For a description of successful examples of this activity, see, for example, projects in the Moscow Mathematical Conference of High School Students [**M**]. While most of these projects are not completely original, sometimes they can lead to new results.

How this book is organized

You should not read each page in this book, one after the other. You can choose a sequence of study that is convenient for you (or omit some topics altogether). Any section (or subsection) of the book can be used for a math circle session.

The book is divided into chapters and sections (some sections are divided into subsections), with a plan of organization outlined at the start of each section. If the material of another section is needed in a problem, you can either ignore it or look up the reference. This allows greater freedom when studying the book but at the same time it may require careful attentiveness.

The topics of each chapter are arranged approximately in order of increasing complexity. The numbers in parentheses after a topic name indicate its "relative level": 1 is the simplest, and 4 is the most difficult. The first topics (not marked with an asterisk) are basic; unless indicated otherwise, you should begin your study with them. The remaining ones (marked with an asterisk) can be returned to later; unless otherwise stated, they are independent of each other. For problems marked with the degree sign (°), it suffices to give just an answer, not necessarily a proof. As you read, try to *return* to old material, but at a new level. Thus you should end up studying different levels of a topic *not sequentially* but as part of a mixture of topics.

The notation used throughout the book is given on p. xvii. Notation and conventions particular to a specific section are introduced at the beginning of that section. The book concludes with a subject index. The numbers in bold are the pages on which *formal definitions* of concepts are given.

Resources and literature

Besides sources for specialized material, we have also tried to include the very best popular writing on the topics studied. We hope that the bibliography, at least as a first approximation, can guide readers through the sea of popular scientific literature in mathematics. However, the great size of this genre guarantees that many notable works had to be omitted. Please note that sources in the bibliography are not necessary for solving the problems in this book, unless explicitly stated otherwise.

Many of the problems are not original, but the source (even if it is known) is usually not specified. When a reference is provided, it comes after the statement of the problem, so that readers can compare their solutions with the one given there. When we know that many problems in a section come from one source, we mention this.

We do not provide links to online versions of articles in the popular magazines *Quantum* and *Mathematical Enlightenment*; they can be found at the websites http://kvant.ras.ru, http://kvant.mccme.ru, and http://www.mccme.ru/free-books/matpros.html.

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Numbering and notation

The topics in each chapter are arranged approximately in order of increasing complexity of the material. The numbers in parentheses after the topic name indicate its “relative level”: 1 is the simplest, and 4 is the most difficult.

The first topics (not marked with an asterisk) are basic; unless indicated otherwise, you can begin to study the chapter with them. The rest of the topics (marked with an asterisk) can be returned to later; unless otherwise stated, they are independent of each other.

Each paragraph starting with a number (such as "1.1") is a problem. If the statement of the problem is in the form of an assertion, then the problem is to prove this assertion. More open-ended questions are called *challenges*; here one must come up with a precise statement and a proof; cf., for example, [**VIN**].

The most difficult problems are marked with asterisks (*). If the statement of the problem asks you to "find" something, then you need to give a "closed form" answer (as opposed to, say, an unevaluated sum of many terms). Again, if you are unable to solve a problem, read on; later problems may turn out to be hints.

Notation

- $\lfloor x \rfloor = [x]$ — (lower) integer part of the number x ("floor"); that is, the largest integer not exceeding x.
- $\lceil x \rceil$ — upper integer part of the number x ("ceiling"); that is, the smallest integer not less than x.
- $\{x\}$ — fractional part of the number x; equal to $x - \lfloor x \rfloor$.
- $d \mid n$ or $n \,\vdots\, d$ — d *divides* n; that is; $d \neq 0$ and there exists an integer k such that $n = kd$ (the number d is called a *divisor* of the number n).
- $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ — the sets of all real, rational, and integer numbers, respectively.
- $\mathbb{Z}_2$ — the set $\{0, 1\}$ of remainders upon division by 2 with the operations of addition and multiplication modulo 2.
- $\mathbb{Z}_m$ — the set $\{0, 1, \ldots, m-1\}$ of remainders upon division by m with the operations of addition and multiplication modulo m. (Specialists in algebra often write this set as $\mathbb{Z}/m\mathbb{Z}$ and use $\mathbb{Z}_m$ for the set of m*-adic integers* for the prime m.)
- $\binom{n}{k}$ — the number of k-element subsets of an n-element set (also denoted by C_n^k).
- $|X|$ — number of elements in set X.
- $A - B = \{x \mid x \in A \text{ and } x \notin B\}$ — the difference of the sets A and B.
- $A \subset B$ — means the set A is contained in the set B. In some books this is denoted by $A \subseteq B$, and $A \subset B$ means "the set A is in the set B and is not equal to B."
- We abbreviate the phrase "Define x by a" to $x := a$.

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Chapter 1

Triangle

The notation for this chapter is summarized in Fig. 1.

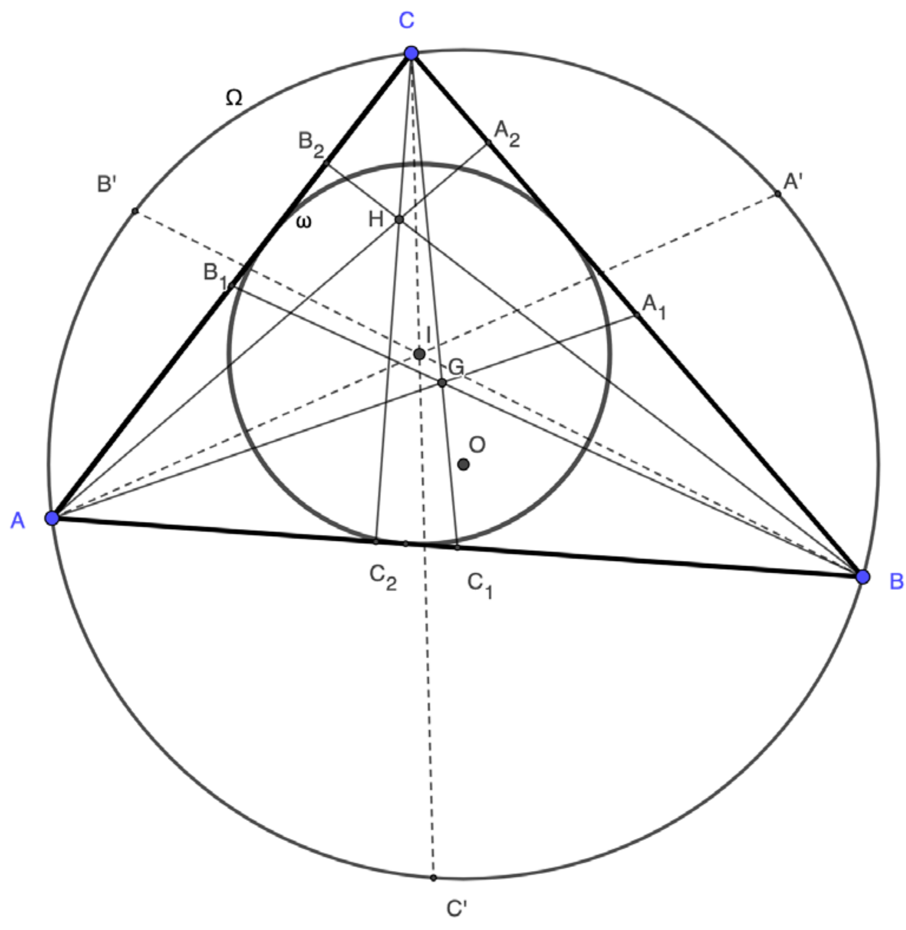

FIGURE 1

- Let ABC be a given triangle. Conventionally, A_i, B_i, and $C_i, i = 1, 2, \ldots$, denote points on the sides BC, CA, and AB respectively (or on extensions of these sides, if relevant).

- Let ω denote the inscribed circle (also called the "incircle"), with center I and radius r (also called the "inradius").
- Let Ω denote the circumscribed circle (also called the "circumcircle"), with center O and radius R (also called the "circumradius").
- Define the *medians* to be the segments joining the vertices with the midpoints of the opposite sides of the triangle. Let G denote the point of intersection of the medians (also known as the *center of gravity*, or *centroid*).
- Let H denote the intersection point of the heights of the triangle (also known as the *orthocenter*).
- Extend the angle bisectors AI, BI, CI so that they intersect with Ω at A', B', C' respectively. Thus, A', B', C' are the midpoints of the arcs AB, BC, CA respectively.
- The *orthotriangle* of ABC is the triangle whose vertices are the feet of the perpendiculars from A, B, and C to lines BC, AC, and AB (i.e., the heights).
- The *medial triangle* of ABC is the triangle whose vertices are the midpoints of the sides. The sides of the medial triangle are the *midlines* of the triangle ABC.
- By the *circle* ABC we mean the circumscribed circle of the triangle ABC.
- By a *circular segment* we mean a figure bounded by a circluar arc and a chord.
- An *external bisector* is the bisector of one of the external angles of a triangle.
- Denote by $h(A, BC)$ the perpendicular dropped from the point A to the line BC.
- Three or more lines are *concurrent* if they have a unique common point. Three or more points are *collinear* if they belong to one line.

  Some notable topics in the geometry of a triangle not covered by the content of this chapter can be found in [**EM90**, **Sh89**, **Ku92**].

1. Carnot's principle (1)
By V. Yu. Protasov and A. A. Gavrilyuk

1.1.1. Carnot's Theorem. Let A_1, B_1, C_1 be points lying on the sides or extensions of the sides of triangle ABC. Draw lines through these points perpendicular to the sides they are on. Prove that the lines intersect at a single point if and only if

$$|C_1A|^2 - |C_1B|^2 + |A_1B|^2 - |A_1C|^2 + |B_1C|^2 - |B_1A|^2 = 0.$$

1.1.2. Formulate and prove a generalized Carnot's theorem for arbitrary points in the plane A_1, B_1, C_1, not necessarily lying on lines containing the sides of a triangle ABC.

1.1.3.° For which of the following cases is it possible that the perpendiculars drawn to the sides (or extensions of sides) of the triangle through the specified points may not be concurrent?

1) A_1, B_1, C_1 are the points of tangency of the sides of triangle ABC to the inscribed circle.

2) A_2, B_2, C_2 are the points of tangency of the sides of triangle ABC to the corresponding excircles.

3) A_3, B_3, C_3 are the intersections of the angle bisectors with the corresponding sides of the triangle.

1.1.4. Consider three intersecting circles in the plane. Prove that their three pairwise common chords are concurrent.

Note: This statement is usually proven using the notion of the power of a point (see Problem 2.4.1). However, it is easy to derive it from the generalized Carnot's theorem.

1.1.5. Use Carnot's principle to give yet another proof that the heights of a triangle are concurrent.

1.1.6. On each side of a triangle, consider the point of intersection of the angle bisector of the angle opposite this side. Then draw the perpendicular to this side through this intersection point. Describe all triangles for which these three perpendiculars are concurrent.

1.1.7. On the sides of triangle ABC, construct the rectangles ABB_1A_1, BCC_2B_2, and CAA_2C_1. Prove that the perpendicular bisectors to segments A_1A_2, B_1B_2, and C_1C_2 are either concurrent or parallel.

1.1.8. Let A_1, B_1, C_1 be the midpoints of sides BC, AC, AB of triangle ABC, respectively. Let B_2 be the base of the perpendicular dropped from B to A_1C_1. Define A_2 and C_2 similarly. Prove that $h(C_1, A_2B_2)$, $h(B_1, A_2C_2)$, and $h(A_1, B_2C_2)$ are concurrent.

1.1.9. Prove that $h(A, B_1C_1)$, $h(B, A_1C_1)$, and $h(C, A_1B_1)$ are concurrent if and only if $h(A_1, BC)$, $h(B_1, AC)$, and $h(C_1, AB)$ are concurrent.

1.1.10. Given equilateral triangle ABC and a point D that does not lie on lines AB, AC, or BC, let A_1 be the incenter of triangle BCD, with B_1 and C_1 defined similarly. Prove that $h(A, B_1C_1)$, $h(B, A_1C_1)$, and $h(C, A_1B_1)$ are concurrent.

Suggestions, solutions, and answers

1.1.1. Let the perpendiculars drawn through points A_1 and B_1 intersect at point M. Applying the Pythagorean Theorem to the right triangles AMB_1 and CMB_1 yields $|B_1A|^2 - |B_1C|^2 = |MA|^2 - |MC|^2$. (This technique, where we replace the difference of the squares of the hypotenuses by the difference of the squares of their legs, is called *Carnot's principle.*)

Now let the perpendiculars to the sides of the triangle drawn through points A_1, B_1, C_1 intersect at point M. Applying Carnot's principle, we obtain the required equality.

Conversely, let points A_1, B_1, C_1 be such that

$$|C_1A|^2 - |C_1B|^2 + |A_1B|^2 - |A_1C|^2 + |B_1C|^2 - |B_1A|^2 = 0.$$

Let M denote the point of intersection of the perpendiculars drawn through A_1 and B_1 to the corresponding sides, and consider the perpendicular MC' dropped from M to AB. As above, we have

$$|C'A|^2 - |C'B|^2 + |A_1B|^2 - |A_1C|^2 + |B_1C|^2 - |B_1A|^2 = 0,$$

and thus C' coincides with C_1, which is what should be proved.

1.1.5. Apply the statement of the previous problem to three circles whose diameters are the sides of the triangle, or directly apply Carnot's Theorem to the feet of the heights.

1.1.7. To begin with, note that the perpendiculars dropped from A, B, C to lines A_1A_2, B_1B_2, C_1C_2, respectively, are concurrent. Indeed, for example, a perpendicular dropped from C to C_1C_2 divides the angle C into two angles equal to the angles of triangle CC_1C_2, whose sines are in the ratio $|CC_1|/|CC_2|$. Using similar relations for the two other vertices and Ceva's Theorem, we obtain the required statement.

Now suppose that lines A_1A_2, B_1B_2, and C_1C_2 form a triangle $A'B'C'$, and lines parallel to them passing through A, B, and C form the triangle $A''B''C''$ (the case where two of the lines A_1A_2, B_1B_2, and C_1C_2 are parallel is easier). Then, for example, triangles A_1B_1C' and ABC'' are congruent by SAA, and thus if B_0 is the midpoint of B_1B_2, then

$$\begin{aligned}|B_0C'|^2 - |B_0A'|^2 &= |A'C'|(|B_0C'| - |B_0A'|)\\ &= |A'C'|(|BC''| - |BA''|) = (|C''B|^2 - |A''B|^2) \cdot \frac{|A'C'|}{|A''C''|}.\end{aligned}$$

Applying Carnot's principle to the similar triangles $A'B'C'$ and $A''B''C''$ yields the statement of the problem.

2. The center of the inscribed circle (2)
By V. Yu. Protasov

The problems of this section are similar in content to problems from sections 3, 5, and 7 of this chapter ("The Euler line", "The orthocenter, orthotriangle, and nine-point circle", and "Bisectors, heights, and circumcircles", respectively).

1.2.1.° Choose the correct answer: Angle AIB is equal to
1) $\pi - \angle C$; 2) $(\pi + \angle C)/2$; 3) $2\angle C$.

1.2.2. Let $ABCD$ be a cyclic quadrilateral. Prove that the incenters of triangles ABC, BCD, CDA, DAB are the vertices of a rectangle.

1.2.3. Let $ABCD$ be a cyclic quadrilateral. Prove that the sum of the radii of the inscribed circles of triangles ABC and CDA is equal to the sum of the radii of the inscribed circles of triangles BCD and DAB.

1.2.4. Through point M inside a given triangle, three congruent circles are drawn, each of which is tangent to two sides of the triangle.

(a) Prove that M lies on the line connecting the centers of the inscribed and circumscribed circles of the triangle.

(b) Describe a method for constructing such a point M, given the triangle.

(c) Let x be the radius of the congruent circles. Prove that $\frac{2r}{3} \leq x \leq \frac{R}{3}$. Is it true that if one of these inequalities is an equality, then the triangle is equilateral?

(d) Prove *Euler's inequality*: $R \geq 2r$. For which triangles is it an equality?

1.2.5. Each of three congruent circles is tangent to two sides of a triangle, and a fourth circle, with the same radius, is tangent to these three circles.

(a) Prove that the center of the fourth circle lies on a line connecting the centers of the inscribed and circumscribed circles of the triangle.

(b) Describe how to construct such circles for the given triangle.

(c) Express the radius of the congruent circles in terms of r and R.

1.2.6. Let a triangle ABC be scalene, i.e., all three sides have different lengths. Denote by A_0, B_0, C_0 the midpoints of its sides. The inscribed circle is tangent to side BC at point A_1, and point A_2 is symmetric to A_1 with respect to the bisector of angle A. Define points B_2 and C_2 similarly. Prove that A_0A_2, B_0B_2, and C_0C_2 are concurrent.

Suggestions, solutions, and answers

1.2.2. By the trident lemma (Problem 1.7.3) we have $|C'A| = |C'I| = |C'B|$. Now let I_a, I_b, I_c, I_d be the centers of the inscribed circles of triangles BCD, CDA, DAB, ABC respectively. Then points A, B, I_c, I_d lie on one circle; therefore $\angle BI_dI_c = \pi - \angle BAI_c = \pi - \angle BAD/2$. Similarly, $\angle BI_dI_a = \pi - \angle BCD/2$. So $\angle I_aI_dI_c = (\angle BAD + \angle BCD)/2 = \pi/2$.

1.2.4. (a) *Suggestion.* Consider a homothety (see the definition in section 4 of Chapter 3) with a center at point I.

Path to solution (N. Medved). We have an arbitrary triangle ABC. Let O_a, O_b, and O_c be the centers of the congruent circles a, b, and c, and let I and O be the centers of the inscribed and circumscribed circles of triangle ABC, respectively. Draw the radii from the centers of a, b, c to the points of tangency with triangle ABC. These radii are equal (the circles are congruent), and when dropped to the same side will be parallel (the sides of the triangle are tangent to circles). So by connecting points O_a, O_b, O_c and drawing from them radii to points of tangency with the sides of the triangle, we get three rectangles; i.e., triangle ABC will be similar to triangle $O_aO_bO_c$, with the sides of triangle ABC being parallel to the corresponding sides of triangle $O_aO_bO_c$. Furthermore, note that lines AO_a, BO_b, and CO_c intersect at I, since they coincide with the angle bisectors of triangle ABC. Consequently, there exists a homothety with center I that transforms triangle ABC into triangle $O_aO_bO_c$. This homothety moves O to the center of the circumcircle of triangle $O_aO_bO_c$, that is, to M (since O_aM, O_bM, and O_cM are the radii of equal circles a, b, and c). Therefore, points I, O, and M are collinear.

(b) (N. Medved) From the solution of (a), we know that the required point M is the center of the circumscribed circle of triangle $O_aO_bO_c$; i.e., to solve the problem it suffices to construct triangle $O_aO_bO_c$, and then it is easy to find the center of its circumscribed circle, which is point M.

Start by noting that we can construct the radii of the inscribed and the circumscribed circles of triangle ABC. Simply construct the angle bisectors and connect their point of intersection with a tangent point in the triangle; this segment will be the radius of the inscribed circle of triangle ABC. For the circumcircle, construct the perpendicular bisectors of the sides and connect their intersection point with one of the vertices of the triangle; this segment will be the radius of circumcircle ABC.

From (a), we see that $\frac{x}{R} = \frac{r-x}{r}$, where x is the radius of the circumcircle of triangle $O_aO_bO_c$. This implies $x = \frac{Rr}{R+r}$, whence $\frac{x}{r} = \frac{R}{R+r}$. To construct a segment of length x, draw an arbitrary angle A_1, on one side of which we mark point B_1 such that $|A_1B_1| = R$; on the other side we mark points C_1 and D so that $|A_1C_1| = r$ and $|C_1D| = R$. Next, connect points B_1 and D to get segment B_1D. Draw a parallel segment from C_1 that intersects the side of angle A_1 at point F. The length of segment A_1F will equal x.

Draw three lines parallel to the sides of the triangle, at a distance of x from each side. The points of their intersection will be the vertices of the desired triangle $O_aO_bO_c$. Finally, construct the perpendicular bisectors of the sides of this triangle; their point of intersection will be the desired point M.

(c) Consider homothetic images of the given triangle with ratio $\frac{2}{3}$, with centers at the vertices of the given triangle. These three triangles have a single common point. From this, it follows that x cannot be less than $\frac{2r}{3}$. Further, from the similarity of the original triangle and the triangle with vertices at the centers of the given circles, it follows that $\frac{x}{R} = \frac{r-x}{r} = 1 - \frac{x}{r} \leq 1 - \frac{2}{3} = \frac{1}{3}$.

3. The Euler line
By V. Yu. Protasov

The problems of this section cover the same topics as problems in sections 2, 5, and 7 of this chapter.

1.3.1. In any triangle, points O, G, and H lie on one line (called the *Euler line*), and $|GH| = 2 \cdot |GO|$.

1.3.2.° The Euler line of a non-isosceles triangle passes through one of its vertices. What is the angle at this vertex?
1) $90°$ 2) $120°$ 3) $60°$ 4) There are no such triangles.

1.3.3. Prove that the Euler line is parallel to side AB if and only if $\tan A \cdot \tan B = 3$.

1.3.4. The Euler line of a triangle is parallel to one of its angle bisectors. Prove that either the triangle is isosceles, or one of its angles is $120°$.

1.3.5. Let $\angle A = 120°$. Prove that $|OH| = |AB| + |AC|$.

1.3.6. For each vertex of a triangle, construct the circle that goes through this vertex and the foot of the height from this vertex and which is tangent to the radius of the circumcircle drawn to this vertex. Prove that these three circles intersect at two points which lie on the Euler line of the triangle.

1.3.7. Assume that all angles of triangle ABC are less than $120°$. Define the *Torricelli point* of ABC to be the point T satisfying $\angle ATB = \angle BTC = \angle CTA = 120°$.

(a) Prove that the Euler line of triangle ATB is parallel to line CT.

Suggestion. Use Problem 1.3.3.

(b) Prove that the Euler lines of triangles ATB, BTC, and CTA are concurrent.

1.3.8. At the vertices of an acute triangle, draw tangents to its circumcircle. Prove that the center of the circumscribed circle of the triangle formed by these three tangents lies on the Euler line of the original triangle.

Suggestions, solutions, and answers

1.3.3. Angle C must be acute, since otherwise points O and H lie on opposite sides of AB. Since the distance from O to AB is $R\cos C$ and the height drawn from vertex C is $|AC|\sin A = 2R\sin A\sin B$, we see that the parallelism of the Euler line and line AB is equivalent to the equality $3\cos C = 2\sin A\sin B$. The result follows from $\cos C = -\cos(A + B) = \sin A\sin B - \cos A\cos B$.

1.3.6. It follows from the statement of the problem that the power of the point O relative to these circles is equal to R^2. In addition, if AA' and BB' are heights of the triangle, then the quadrilateral $ABA'B'$ is cyclic, so that $|HA| \cdot |HA'| = |HB| \cdot |HB'|$. Therefore, the powers of the point H relative to all three circles are also equal, i.e., line OH is their common radical axis.

4. Carnot's formula (2^*)
By A. D. Blinkov

Carnot's formula (named after the French mathematician, physicist, and politician Lazare Carnot, 1753–1823) states that in an acute triangle, the sum of the distances from the center of the circumscribed circle to the sides of the triangle is equal to the sum of the radii of the circumscribed and inscribed circles: $|OM_1| + |OM_2| + |OM_3| = R + r$, where M_1, M_2, M_3 are the midpoints of BC, CA, AB, respectively. A proof using Ptolemy's Theorem is given in section 6 of Chapter 2. Here we look at its applications and an alternative proof; these investigations will yield other important facts.

1.4.1. Let the bisector of angle A intersect the circumcircle of triangle ABC at point W, and let D be diametrically opposite to point W. Prove that

(a) $|M_1W| = (r_a - r)/2$;

(b) $|M_1D| = (r_b + r_c)/2$, where r is the radius of inscribed circle and r_a, r_b, r_c are the radii of the excircles.

1.4.2. Prove Carnot's formula.

For the following problems that explore applications of Carnot's formula, assume unless otherwise stated that the triangles are acute.

1.4.3. Prove that the sum of the distances from the vertices of a triangle to the orthocenter is equal to the sum of the diameters of its inscribed and circumscribed circles.

1.4.4. Prove that in triangle ABC the following inequalities hold:

(a) $|AH| + |BH| + |CH| \le 3R$;

(b) $3|OH| \ge R - 2r$.

1.4.5. (a) Prove that $m_a + m_b + m_c \le \frac{9}{2}R$, where m_a, m_b, and m_c are the lengths of the medians of the triangle.

(b) Let the bisectors of angles A, B, and C in triangle ABC intersect the circumcircle at points W_1, W_2, and W_3 respectively. Prove that $|AW_1| + |BW_2| + |CW_3| \le 6.5R - r$.

1.4.6. (a) Prove that the angles of a triangle ABC satisfy the inequality

$$\frac{3r}{R} \le \cos A + \cos B + \cos C \le \frac{3}{2}.$$

(b) Let AH_1, BH_2, and CH_3 be the heights of triangle ABC. Express the sum of the diameters of the circles circumscribed around triangles AH_2H_3, BH_1H_3, and CH_1H_2 in terms of R and r.

1.4.7. A triangle is inscribed in a circle of radius R. In each segment of the circle bounded by a side of the triangle and the lesser of the circular arcs, inscribe a circle of maximum radius. Find the sum of the diameters of these three circles and the radius of the incircle of the triangle.

1.4.8. (a) Prove that in triangle ABC, the following equality holds:

$$a(|OM_2| + |OM_3|) + b(|OM_1| + |OM_3|) + c(|OM_1| + |OM_2|) = 2pR$$

where $2p$ is the perimeter of ABC.

(b) **Erdős's inequality.** Let h_a be the greatest height of triangle ABC. Prove that $h_a \geq R + r$.

1.4.9. (a) Derive analogues of Carnot's formula for right and obtuse triangles.

(b) Quadrilateral $ABCD$ is cyclic. Let r_1 and r_2 be the radii of circles inscribed in triangles ABC and ADC, and let r_3 and r_4 be the radii of the circles inscribed in triangles ABD and CBD. Prove that $r_1 + r_2 = r_3 + r_4$.

1.4.10. Let d, d_1, d_2, and d_3 be the distances from the center O of the circumcircle of a triangle to the centers of its inscribed and exscribed circles. Prove that

$$R^2 = \frac{d^2 + d_1^2 + d_2^2 + d_3^2}{12}.$$

1.4.11. (a) Prove that if a point belongs to the segment connecting the bases of two angle bisectors of a triangle, then the sum of the distances from this point to two sides of the triangle is equal to the distance from it to the third side.

(b) Suppose the circumcenter of a triangle lies on the segment connecting the bases of two angle bisectors. Prove that the distance from the orthocenter of this triangle to one of its vertices is equal to $R + r$.

Suggestions, solutions, and answers

1.4.1. (a) Let points I and I_a, respectively, be the centers of the inscribed circle and the excircle tangent to side BC. Let K and P be the points of tangency of these circles with BC, and let L be the intersection point of I_aP with the line that passes through I which is parallel to BC. Let Q be the midpoint of IL. Since W is the midpoint of segment II_a (a consequence of the trident lemma; see Problem 1.7.3 in section 7) and $WM_1 \parallel IK \parallel LI_a$, it follows that segment WQ is a midline of triangle ILI_a, and M_1 lies on WQ. Therefore, $|WQ| = |I_aL|/2 = (r_a + r)/2$, and then $|M_1W| = |WQ| - r = (r_a - r)/2$.

(b) If I_b and I_c are the centers of the excircles and B_0 and C_0 are their respective points of tangency with BC, then, similarly to (a), we obtain that DM_1 is the midline of the trapezoid $I_bB_0C_0I_c$, i.e., the segment joining the midpoints of I_bB_0 and I_cC_0.

1.4.2. From the previous problem it follows that $r_a+r_b+r_c = r+4R$. Then

$$|OM_1| + |OM_2| + |OM_3| = 3R - (|W_1M_1| + |W_2M_2| + |W_3M_3|) = R + r.$$

1.4.3. Under the homothety that transforms the middle triangle to the original one, O is sent to H.

1.4.4. (a) We have $|AH| + |BH| + |CH| = 2R + 2r \leq 3R$, since $R \geq 2r$.

(b) By the triangle inequality, $|AH| + |OH| \geq R$. Similarly, $|BH| + |OH| \geq R$ and $|CH|+|OH| \geq R$. Therefore, $|AH|+|BH|+|CH|+3|OH| \geq 3R$; i.e., $2R + 2r + 3|OH| \geq 3R$.

1.4.5. (a) By the triangle inequality, considering triangle AOM_1, we have $m_a \leq |OM_1| + R$. Similarly, $m_b \leq |OM_2| + R$ and $m_c \leq |OM_3| + R$. Therefore,

$$m_a + m_b + m_c \leq |OM_1| + |OM_2| + |OM_3| + 3R = 4R + r \leq \frac{9}{2}R,$$

since $R \geq 2r$.

(b) By the triangle inequality, $|AW_1| \leq |AM_1| + |W_1M_1|$. Similarly, $|BW| \leq |BM_2| + |W_2M_2|$ and $|CW_3| \leq |AM_3| + |W_3M_3|$. Therefore

$$\begin{aligned} |AW_1| + |BW_2| + |CW_3| &\leq m_a + m_b + m_c + |W_1M_1| + |W_2M_2| + |W_3M_3| \\ &= m_a + m_b + m_c + 2R - r \\ &\leq 4.5R + 2R - r = 6.5R - r. \end{aligned}$$

1.4.6. (a) Carnot's formula implies $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$; then use inequality $R \geq 2r$.

(b) The diameters of the indicated circles are segments AH, BH, and CH; therefore the required expression follows from part (a) and from Problem 1.4.3.

1.4.7. *Answer*: $2R$. The circle of largest radius inscribed in the segment is tangent to the arc of the segment in its middle.

1.4.8. (a) We have

$$\begin{aligned} a(|OM_2| + |OM_3|) + b(|OM_1| + |OM_3|) + c(|OM_1| + |OM_2|) \\ &= (a+b+c)(|OM_1| + |OM_2| + |OM_3|) \\ &\quad - a|OM_1| - b|OM_2| - c|OM_3| \\ &= 2pR + 2S - 2S = 2pR. \end{aligned}$$

(b) The original inequality is equivalent to the inequality $2S \geq a(R+r)$. Since a is the smallest side of a triangle, we have

$$a(R+r) = a(|OM_1| + |OM_2| + |OM_3|) \leq a|OM_1| + b|OM_2| + c|OM_3| = 2S,$$

as required.

1.4.9. (a) In a right triangle, the distance from O to the larger side equals zero, and in an obtuse triangle it is negative.

(b) Consider the case where the circumcenter is an interior point of the quadrilateral; the other cases are dealt with similarly.

Since the quadrilateral $ABCD$ is cyclic, for any pair of opposite angles, one of them is not obtuse and the other is not acute. Suppose, for example, $\angle B \geq 90°$ and $\angle D \leq 90°$; then O lies inside triangle ADC and outside triangle ABC. Let M_1, M_2, M_3, and M_4 be the midpoints of sides AB, BC, CD, and DA respectively, and let M_5 be the midpoint of AC. By Carnot's formula applied to triangles ADC and ABC, we have $|OM_3| + |OM_4| + |OM_5| = R + r_2$ and $|OM_1| + |OM_2| - |OM_5| = R + r_1$. Therefore, $r_1 + r_2 = |OM_1| + |OM_2| + |OM_3| + |OM_4| - 2R$.

Similarly, considering triangles ABD and CBD yields

$$r_3 + r_4 = |OM_1| + |OM_2| + |OM_3| + |OM_4| - 2R.$$

1.4.10. Use Euler's formulas $|OI|^2 = R^2 - 2Rr$ and $|OI_i|^2 = R^2 + 2Rr_i$ $(i = a, b, c)$.

1.4.11. (a) If we consider the distances to the sides to be oriented, then the condition of the problem defines a line. Obviously, the bases of the bisectors lie on this line.

(b) Let $|OM_2| = |OM_1| + |OM_3|$. Then

$$|BH| = 2|OM_2| = |OM_1| + |OM_2| + |OM_3| = R + r.$$

Note: You can also show that the condition of the problem is equivalent to the equality $R = r_b$.

5. The orthocenter, orthotriangle, and nine-point circle (2) *By V. Yu. Protasov*

The problems of this section are close in subject to the problems in sections 2, 3, and 7 of this chapter.

1.5.1. Let ABC be an equilateral triangle. Describe the locus of points M inside ABC satisfying $\angle MAB + \angle MBC + \angle MCA = 90°$.

1.5.2. Let a, b, c be the lengths of sides of an acute triangle, and let u, v, w be the distances from the corresponding vertices to the orthocenter. Prove that $avw + bwu + cuv = abc$.

1.5.3.° Let AA', BB', CC' be the heights of triangle ABC. Then the orthocenter H of ABC is the (choose the correct answer)
1) orthocenter
2) center of gravity
3) center of the circumscribed circle
4) center of the inscribed circle
of triangle $A'B'C'$.

1.5.4. For a given acute triangle, find all of its *triangular billiards*, i.e., all triangles inscribed in the given triangle with the following property: two sides emerging from any vertex of the inscribed triangle form equal angles with the corresponding side of the given triangle.

1.5.5. Let $A_1B_1C_1$ be the orthotriangle of triangle ABC, and let A_2, B_2, and C_2 be the projections of A, B, and C to the lines B_1C_1, C_1A_1, and A_1B_1 respectively. Prove that the perpendiculars dropped from A_2, B_2, and C_2 onto lines BC, CA, and AB, respectively, are concurrent.

Suggestion. Apply Carnot's principle.

1.5.6. Prove that the points symmetric to the orthocenter with respect to the sides of the triangle and the points symmetric to the orthocenter with respect to the midpoints of the sides of the triangle all lie on the circumcircle of the triangle.

1.5.7. Prove that the midpoints of the sides of a triangle, the feet of its heights, and the midpoints of the segments connecting the vertices with the orthocenter all lie on the same circle (called the *nine-point circle*). The radius of this circle is $\frac{R}{2}$, and its center is the midpoint of segment OH.

1.5.8. The lengths of the sides of an acute triangle are multiplied by the cosines of the opposite angles. Prove that there exists a triangle with sides of these three lengths. What is the radius of its circumscribed circle if the radius of the circumscribed circle of the original triangle is R?

1.5.9. Thébault's Theorem. Let ABC be a given triangle with orthotriangle $A_1B_1C_1$. Prove that the Euler lines of triangles AB_1C_1, BC_1A_1, and CB_1A_1 are concurrent, with intersection point on the nine-point circle of triangle ABC.

1.5.10. For a given quadrilateral $ABCD$, prove that the nine-point circles of triangles ABC, BCD, CDA, DAB have a common point.

1.5.11. Brahmagupta's Theorem. If a cyclic quadrilateral has perpendicular diagonals, then the line passing through the intersection point of the diagonals and perpendicular to one of the sides bisects the opposite side.

1.5.12. If a quadrilateral has perpendicular diagonals, prove that the following eight points lie on one circle (the *eight-point circle*): the midpoints of the sides and the projections of the midpoints of the sides to opposite sides of the quadrilateral.

Suggestions, solutions, and answers

1.5.2. Since $\angle AHB = \pi - \angle ACB$, the radius of the circumscribed circle of triangle AHB is equal to R. Similarly, the radii of the circumscribed circles of triangles BHC and CHA are equal to R. Therefore, the statement of the problem follows from the fact that $S_{ABC} = S_{AHB} + S_{BHC} + S_{CHA}$ and the formula $S = \frac{abc}{4R}$.

1.5.6. Let points H_1 and H_2 be symmetric to point H with respect to line AB and the midpoint of segment AB. Then the radii of the circles circumscribed around the congruent triangles AHB, AH_1B, and BH_2A are equal. Since points H_1 and H_2 do not lie on circle ABH, they lie on the congruent circle ABC.

1.5.7. Apply the result of Problem 1.5.6 and apply a homothety with ratio $\frac{1}{2}$.

1.5.9. Triangle CB_1A_1 is the image of triangle CAB under the composition of a homothety with center C and a symmetry with respect to the bisector of angle C. Therefore, the angle between the Euler lines of triangles AB_1C_1 and BC_1A_1 is equal to angle C. Furthermore, the centers of the circles circumscribed around these triangles are the midpoints of segments HA and HB. Thus, the segment between these centers is seen from the intersection point of two Euler lines at angle C; hence this point lies on the nine-point circle (also called the Euler circle). Then the third line intersects the nine-point circle at the same point.

6. Inequalities involving triangles (3^*)
By V. Yu. Protasov

1.6.1. (a) Is it true that the area of the orthotriangle does not exceed the area of the medial triangle?

(b) Answer the same question, if it is known that the triangle is acute.

1.6.2. The bisectors of the angles of triangle ABC intersect the circumcircle at points A', B', C'. Prove that $S_{AC'BA'CB'} \geq 2S_{ABC}$.

1.6.3. Find the smallest α for which the following statement holds:

Let angle A have measure α. Inscribe a circle in this angle, tangent at points B and C. Another line tangent to the circle at a point M intersects segments AB and AC at points P and Q, respectively. Then $S_{PAQ} < S_{BMC}$ for any choice of points B, C, P, Q.

In Problems 1.6.4–1.6.7, let a, b, c denote the lengths of the sides of a given triangle, let x, y, z denote the distances from an arbitrary point M inside the triangle to its sides, and let u, v, w denote the distances from M to the vertices of the triangle.

1.6.4. Prove that for an arbitrary point M lying inside a triangle, the following inequalities hold:
(a) $u \geq \frac{b}{a}y + \frac{c}{a}z$; (b) $u \geq \frac{c}{a}y + \frac{b}{a}z$.

1.6.5. Show that for an acute triangle, there is a unique point M for which all three inequalities in Problem 1.6.4 (a) (for the three vertices of the triangle) become equalities. Describe this point. Answer the same question about the three inequalities from (b).

1.6.6. Prove that for arbitrary point M lying inside a triangle, the inequality $u+v+w \geq 2(x+y+z)$ holds (*Erdős's inequality*). For which triangles and for which points M does this inequality become an equality?

1.6.7. Prove that for an arbitrary point M lying inside a triangle, the following inequality holds:

$$2\Big(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\Big) \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

1.6.8. An arbitrary point M is selected inside triangle ABC. Prove that one of the angles MAB, MBC, MCA does not exceed $30°$. Formulate and prove a similar statement for a quadrilateral.

Suggestions, solutions, and answers

1.6.1. (b) *Answer*: True.

Let us prove a more general statement: if points A', B', C' lie on sides BC, CA, AB and lines AA', BB', CC' are concurrent, then

$$S_{A'B'C'} \leq \frac{S_{ABC}}{4}.$$

First, note that for any points A', B', C' lying on the sides of triangle ABC, the following equality holds:

$$S_{A'B'C'} = \frac{|AB'| \cdot |BC'| \cdot |CA'| + |BA'| \cdot |CB'| \cdot |AC'|}{4R}.$$

Indeed, setting $|AC'|/|AB| = \gamma$, $|CA'|/|CA| = \beta$, and $|BA'|/|BC| = \alpha$, we see that $S_{A'B'C} = \beta(1-\alpha)S_{ABC}$. This and two similar equalities yield

$$S_{A'B'C'} = S_{ABC} - S_{A'B'C} - S_{A'BC'} - S_{AB'C'} = S_{ABC}(\alpha\beta\gamma + (1-\alpha)(1-\beta)(1-\gamma)).$$

Replacing S_{ABC} with $\frac{abc}{4R}$, we obtain the required equality.

If segments AA', BB', CC' are concurrent, then Ceva's Theorem implies that $|AB'|\cdot|BC'|\cdot|CA'| = |BA'|\cdot|CB'|\cdot|AC'|$ and

$$|AB'|\cdot|BC'|\cdot|CA'| + |BA'|\cdot|CB'|\cdot|AC'|$$
$$= 2\sqrt{(|AB'|\cdot|B'C|)\cdot(|BC'|\cdot|C'A|)\cdot(|CA'|\cdot|A'B|)} \leq \frac{|AB|\cdot|BC|\cdot|CA|}{4},$$

and the required inequality immediately follows.

1.6.3. Prove first that triangle BMC is similar to triangle QIP, where I is the center of the inscribed circle of triangle PAQ. Besides, $S_{QIP}/S_{PAQ} = |PQ|/p$, where p is the perimeter of triangle PAQ. The fact that $p = 2|AB|$ will also be useful.

1.6.4. (a) Express the area of the non-convex quadrilateral with sides b, c, w, v in two ways.

(b) Consider a point symmetric to M with respect to the corresponding angle bisector.

1.6.6. Add all six inequalities from Problem 1.6.4 for the three vertices of the triangle and use the inequality $t + \frac{1}{t} \geq 2$ for $t > 0$.

7. Bisectors, heights, and circumcircles (2) *By P. A. Kozhevnikov*

These problems tie the classical elements of a triangle into a single structure. They are close in subject to problems from sections 2, 3, and 5 of this chapter.

1.7.1. Prove that triangles $AB'C'$ and $IB'C'$ are symmetric with respect to line $B'C'$ (and, similarly, for the pairs of triangles $\{CA'B', IA'B'\}$ and $\{BC'A', IC'A'\}$).

1.7.2. Prove that AA', BB', CC' are the heights of triangle $A'B'C'$.

1.7.3. The trident lemma. Prove that $|A'I| = |A'B| = |A'C|$ (and, similarly, $|B'I| = |B'C| = |B'A|$ and $|C'I| = |C'A| = |C'B|$).

1.7.4. (a) Prove that the main diagonals of the hexagon formed by the intersection of triangles ABC and $A'B'C'$ intersect at I and are parallel to the sides of triangle ABC.

(b) Let γ_c be the circle with center C' tangent to line AB. Similarly, define circle γ_b. Prove that the line passing through I parallel to BC is a common tangent to circles γ_b and γ_c. (M. Sonkin, *All-Russian Mathematical Olympiad*, 1999.)

1.7.5. Prove that $\frac{S_{ABC}}{S_{A'B'C'}} = \frac{2r}{R}$. (R. Mazov, *Quant*, 1982:10.)

1.7.6. Let a homothety with center I and ratio 2 transform triangle $A'B'C'$ into triangle $I_aI_bI_c$.

Prove that I_a, I_b, I_c are the centers of the excircles of triangle ABC.

1.7.7. Let A'', B'', C'' be the midpoints of arcs BAC, CBA, ACB of circle Ω respectively.

Prove that A'', B'', C'' are the midpoints of segments I_bI_c, I_cI_a, I_aI_b.

1.7.8. Let circle ω be tangent to sides BC, CA, AB at points A_1, B_1, C_1 respectively.

Prove that the following three lines are concurrent:
(a) A_1A', B_1B', C_1C';
(b) A_1A'', B_1B'', C_1C''.
(M. Sonkin, *All-Russian Mathematical Olympiad*, 1998.)

1.7.9. Prove that line IO passes through the orthocenter of triangle $A_1B_1C_1$.

Suggestions, solutions, and answers

1.7.1. We have (see Fig. 2) $\angle AB'C' = \angle ACC' = \angle BCC' = \angle BB'C'$; i.e., $\angle AB'C' = \angle IB'C'$. Similarly, $\angle AC'B' = \angle IC'B'$.

1.7.2. From Problem 1.7.1 it follows that points A and I are symmetric with respect to $B'C'$. Hence $AI \perp B'C'$.

Thus, we can simultaneously study the bisectors (of triangle ABC) and the heights (of triangle $A'B'C'$) in one figure. Triangle $A'B'C'$ can be an arbitrary acute triangle. (How can you choose the corresponding triangle ABC?) The symmetry of points A and I with respect to $B'C'$ implies the following important property of the orthocenter: a point symmetric to the orthocenter of a triangle with respect to one of its sides lies on the circumscribed circle (see Problem 1.5.6).

1.7.3. From Problem 1.7.1 it follows that $B'A = B'I$. Similarly, $B'C = B'I$.

1.7.4. (a) Let $K = AB \cap A'C'$, $L = BC \cap A'C'$, and $T = BI \cap KL$ (see Fig. 3). Since $BI \perp KL$, points K and L are symmetric with respect to T. And, since points B and I are symmetric with respect to T (Problem 1.7.1), $KBLI$ is a rhombus and $KI \parallel BC$. Similarly, $LI \parallel AB$. (Note the general fact that for any inscribed hexagon $ABCDEF$, the main diagonals of the hexagon formed by the intersection of triangles ACE and BDF are concurrent.)

(b) Lines BK and IK are symmetric with respect to line $A'C'$. Since BK is tangent to circle γ_c, the same is true for IK.

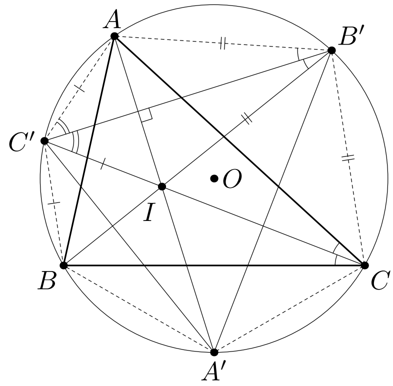

FIGURE 2

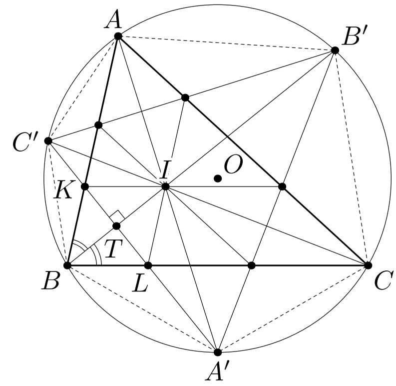

FIGURE 3

1.7.5. We calculate the area S of the hexagon $AB'CA'BC'$ in two ways. First,

$$\begin{aligned} S &= S_{AB'IC'} + S_{BC'IA'} + S_{CA'IB'} \\ &= 2(S_{B'IC'} + S_{C'IA'} + S_{A'IB'}) = 2S_{A'B'C'}. \end{aligned}$$

On the other hand,

$$\begin{aligned} S &= S_{OAB'C} + S_{OBC'A} + S_{OCA'B} \\ &= \frac{1}{2}(|OB|' \cdot |AC| + |OC'| \cdot |BA| + |OA'| \cdot |CB|) \\ &= \frac{R(|AC| + |BA| + |CB|)}{2} = \frac{R \cdot S_{ABC}}{r}. \end{aligned}$$

Equating these expressions yields the required equality.

1.7.6. Since $I_bI_c \parallel B'C'$ (see Fig. 4), line I_bI_c is perpendicular to the angle bisector AA'. Points A and I are symmetric with respect to $B'C'$; therefore I_bI_c passes through A, so I_bI_c is an external bisector[1] of triangle ABC. Similarly, I_aI_b and I_cI_a are external bisectors.

Note: It follows from the construction that I_a lies on line AI (this is clear, since the center of the excircle lies on the bisector) and $A'I_a = A'I$.

1.7.7. Points A'' and A' are diametrically opposite points on circle Ω; therefore $\angle A'AA'' = 90°$, so A'' lies on the external bisector I_bI_c of angle BAC. We have $AA'' \parallel B'C'$, so $|A''B'| = |AC'|$. But $|AC'| = |C'I|$ (Problem 1.7.3), so $|A''B'| = |C'I|$. Similarly, $|A''C'| = |B'I|$, implying that $A''B'IC'$ is a parallelogram. Thus $A''I$ bisects $B'C'$, and, consequently (from a homothety with center I), it bisects I_bI_c.

[1] *Editor's note:* Recall that an *external bisector* of a triangle is a line that bisects an exterior angle of the triangle. In Fig. 4, I_bI_c bisects the angle formed by AC and the extension of side BC. It also bisects the angle formed by BC and the extension of CA, since these angles are obviously equal.

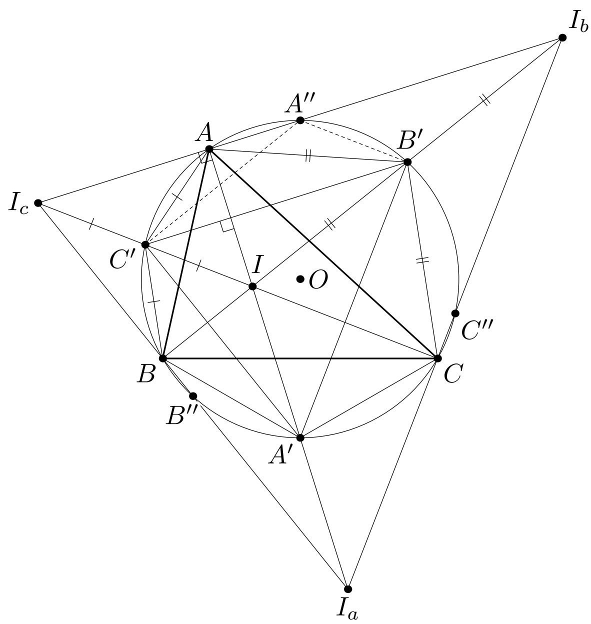

FIGURE 4

Note: For $I_aI_bI_c$, the circle Ω is the nine-point circle (with points A, B, C, A', B', C', A'', B'', C''), and triangle ABC is the orthotriangle (its bisectors coincide with the heights of the original triangle).

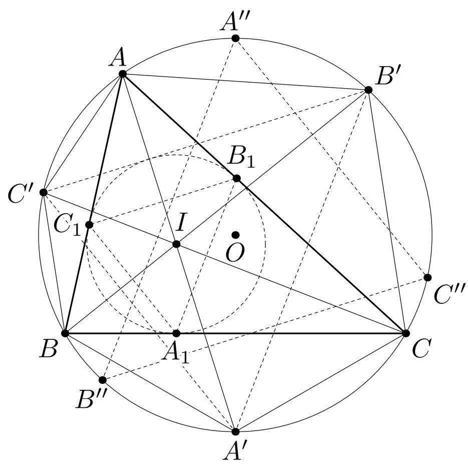

FIGURE 5

1.7.8. Triangles $A'B'C'$ and $A''B''C''$ are symmetric with respect to point O (see Fig. 5). The corresponding sides of triangles $A'B'C'$, $A''B''C''$, and $A_1B_1C_1$ are parallel (they are perpendicular to bisectors AA', BB', and

CC' of triangle ABC). Thus, triangles $A'B'C'$, $A''B''C''$, and $A_1B_1C_1$ are pairwise homothetic.

1.7.9. Triangles $A'B'C'$ and $A_1B_1C_1$ are homothetic, so their Euler lines (lines connecting the orthocenter and the center of the circumscribed circle) are parallel or coincide. But IO is the Euler line of triangle $A'B'C'$, and I is the center of the circumscribed circle of triangle $A_1B_1C_1$. Therefore their Euler lines must coincide.

8. "Semi-inscribed" circle (3^*)
By P. A. Kozhevnikov

Let triangle ABC have circumcircle Ω. Define A' and A'' to be the midpoints of the two arcs BC of Ω, with A' on the arc not containing A and A'' on the arc that contains A. Define B', B'' and C', C'' similarly.

Define the *semi-inscribed* circle S_A to be the circle tangent to sides AB and AC and circle Ω (internally). The key goals of this section are to prove the following facts.

- The line passing through the tangency points of the semi-inscribed circle on the sides of the triangle contains I.
- The point of tangency of the semi-inscribed circle with Ω lies on the line $A''I$.

Main series of problems—1

Prove the following statements:

1.8.1. A line perpendicular to angle bisector AI passing through point I intersects AB and AC at K and L, respectively. Then circles BKI, CLI, and Ω intersect at a single point T.

1.8.2. Points T, I, A'' are collinear.

1.8.3. Points T, K, C' are collinear.

1.8.4. The points K, L, and T are the points of tangency of the circle S_A with lines AB and AC and circle Ω.

1.8.5. (a) Line CC' is tangent to circle $TBKI$.

(b) Point T is the center of a rotational homothety (see the definition before Problem 3.5.13 in Chapter 3)[2] which transforms triangle BKI into triangle ILC.

[2] *Editor's note:* A *rotational homothety* (also called a *spiral similarity*) is a composition of a rotation and a homothety with the same center point.

Main series of problems—2

1.8.6. Line AT passes through the center of a homothety with positive[3] coeffcient (see the definition in section 4 of Chapter 3) that transforms circle ω (the incircle) into circle Ω (the circumcircle).

1.8.7. Let A_1 and A_2 be the tangent points of the inscribed and excribed circles with side BC, respectively. Then

(a) AA' bisects angle TAA_2;

(b) $\angle BTA_1 = \angle ABC$. (Problem 4.7.7 from [**Ako17**].)

1.8.8. Let AT intersect KL at Z. Then $\angle BZK = \angle CZL$. (Problem 4.7.5 from [**Ako17**].)

1.8.9. Lines KL, TA', and BC are either concurrent or parallel. (I. Sharygin.)

1.8.10. The point of intersection Y_A of the lines from the previous problem and the points Y_B and Y_C, defined in a similar way, are collinear.

Supplementary problems—1

1.8.11. Let P be an arbitrary point on arc $BA'C$.

(a) Let $P_b = BB' \cap PC'$ and $P_c = CC' \cap PB'$. Then circle PP_bP_c passes through T. ([**Zas14**], Problem 8.8, 2013.)

(b) Let J_b and J_c be the centers of the inscribed circles of triangles PAB and PAC, respectively. Then circle PJ_bJ_c passes through T. (Problem 4.7.9 from [**Ako17**].)

(c) Let the tangents to ω from point P intersect BC at U_1 and U_2. Then circle PU_1U_2 passes through T. (Problem 4.7.10 from [**Ako17**].)

(d) Let the lines passing through I parallel to the bisectors of the angles made by the lines AP and BC intersect BC at V_1 and V_2. Then circle PV_1V_2 passes through T. (See a special case of Problem 4.7.18 from [**Ako17**].)

Supplementary problems—2

The following problems investigate "generalized semi-inscribed" circles, i.e., circles tangent to two lines and a circle.

1.8.12. Let D be a point on side AC of triangle ABC, and let S_1 be a circle internally tangent to circle Ω at point R and also tangent to segments BD and AD at points M and N, respectively.

(a) Prove that points B, M, I, R lie on a circle.

(b) **Savayama's lemma.** Line MN passes through the center I of the inscribed circle ω of triangle ABC.

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[3] *Editor's note:* A homothety with a negative coefficient is a rotational homothety with rotation angle 180 degrees.

1.8.13. Let D be a point on the side AC of triangle ABC, S_1 the circle tangent to segments BD and AD and also internally tangent to circle Ω, and S_2 a circle tangent to segments BD and CD and also internally tangent to circle Ω.

(a) **Thébault's Theorem.** The line joining the centers of circles S_1 and S_2 passes through I.

(b) Prove that circles S_1 and S_2 are congruent if and only if D is the tangency point of an inscribed circle with the side AC.

Deeper results in this direction can be found in [**Koz12**, **Pro92**, **Pro08**, **Gi90**].

1.8.14. Find analogues of the proposed problems for "semi-inscribed" and "generalized semi-inscribed" circles *externally* tangent to Ω.

Suggestions, solutions, and answers

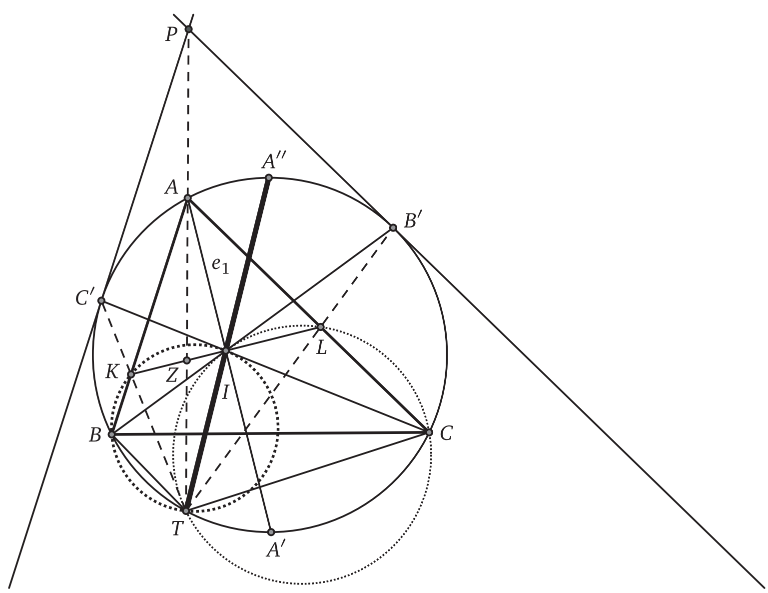

FIGURE 6

1.8.1. This is a special case of the following well-known fact: if points X', Y', Z' lie on lines YZ, ZX, XY respectively, then circles $XY'Z'$, $YZ'X'$, $ZX'Y'$ are concurrent. In this case, points B, C, I lie on the lines containing the sides of triangle AKL; therefore circles BKI, CLI, and ABC are concurrent.

1.8.2. Considering circles $TBKI$ and $TCLI$ we see that $\angle BTI = \angle AKI = \angle ALI = \angle CTI$ (see Fig. 6). Thus TI bisects angle BTC, so TI passes through the midpoint of arc A''.

1.8.3. Looking at the circle $TBKI$ we see that

$$\begin{aligned}\angle BTK &= \angle BIK = \angle BIA - \angle KIA \\ &= (90^\circ + \angle C/2) = \angle BCC' = \angle BTC'.\end{aligned}$$

1.8.4. The previous problem, together with the fact that $KL \parallel B'C'$, implies that the triangles TLK and $TC'B'$ are homothetic, and hence the circles TKL and Ω are tangent at the point T. Next, under the homothety the line AB maps to the line that is parallel to AB and passes through C', i.e., to the tangent to Ω at the point C'. Therefore, AB is tangent to TKL at the point K.

1.8.5. (a) Calculating angles yields $\angle KBI = \angle KIC'$.

(b) The rotational homothety with center T that transforms circle $TBKI$ into circle $TCLI$ transforms B into I, K into L, and I into C.

1.8.6. Considering the homothety mapping triangle TKL into $TC'B'$, it is clear that TA passes through point P of the intersection of the tangents to Ω at B' and C'. But a homothety with a positive coefficient that transforms ω into Ω must transform point A into point P.

Another solution to this problem can be obtained by considering the homotheties with positive coefficients that transform ω into S_A, S_A into Ω, and Ω into ω and applying the theorem of three homotheties (see also Problem 2.5.3 in Chapter 2).[4]

1.8.7. (a) Perform the inversion with center A and radius $\sqrt{|AB| \cdot |AC|}$, and then a reflection with respect to the angle bisector BAC. The composition of these transformations swiches points B and C, and maps line BC and circle Ω into each other (see Fig. 7); therefore circle S_A maps to the excircle, and consequently point T maps to point A_2.

Note: A consequence of the previous problem is that a homothety center with a positive coefficient that transforms ω into Ω and the Nagel point[5] are isogonal conjugates (see section 10, "Isogonal conjugation and the Simson line").[6]

(b) Denote the other point of intersection of AA_2 with Ω by X (see Fig. 7). It follows from (a) that arcs BX and CT are congruent. Reflection about the perpendicular bisector of BC transforms X and A_2 into T and A_1, respectively; thus the other point of intersection of TA_1 with Ω is symmetric to A with respect to the perpendicular bisector of BC.

[4] The centers of three homotheties whose composition is the identity transformation are collinear.

[5] The *Nagel point* is the point of intersection of lines passing through vertices of a triangle and tangency points of its excircles.

[6] *Editor's note:* The *isogonal conjugate* of a point P with respect to triangle ABC is the point of concurrence of the reflections of PA, PB, and PC about the angle bisectors of A, B, and C, respectively.

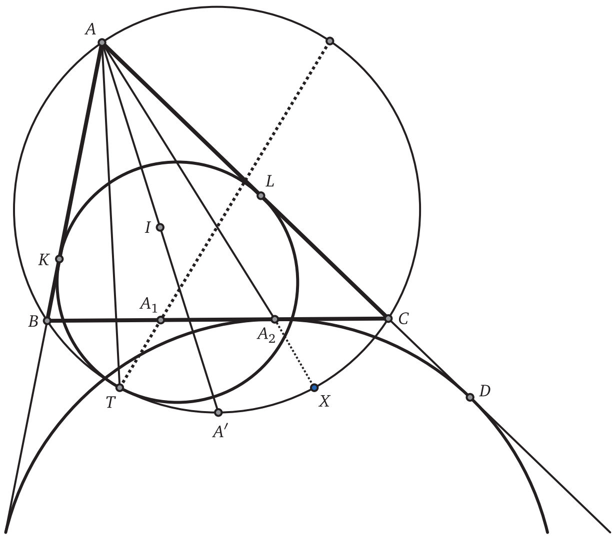

FIGURE 7

1.8.8. It suffices to prove that $BKZ \sim CLZ$ (see Fig. 6). Using rotational homothety (see Problem 1.8.5) we get $|BK|/|KT| = |IL|/|LT|$ and $|IK|/|KT| = |CL|/|LT|$. The equality $|IK| = |IL|$ implies $|BK|/|CL| = |KT|^2/|LT|^2$. Since TA is a symmedian[7] of TKL, we have $ZK/ZL = KT|^2/LT|^2$. The equality $\angle BKZ = \angle CLZ$ implies the desired similarity.

1.8.9. For example, the required intersection point Y_A is the radical center (the intersection point of radical axes) of circles Ω, IBC, and ITA' (calculating angles shows that the last two circles are tangent to line KL).

Another solution can be obtained by considering the intersection point W of $A'T$ and $A''A$. The point I is the orthocenter of triangle $A'A''W$, so $WI \parallel BC$. Further, it is sufficient to show that a homothety with center A' which transforms A to I will map line WI to line BC.

Note: The statement of the problem can be generalized to the case of an arbitrary circle Ω passing through points B and C (see [**Koz12**]).

1.8.10. The point Y_A is the radical center of the circles Ω and IBC and the point I (here a point is viewed as a circle of zero radius). Therefore Y_A lies on the radical axis of circles Ω and I. The points Y_B and Y_C lie on the same radical axis.

1.8.11. (a, b) The rotational homothety from Problem 1.8.5 maps P_b and J_b to P_c and J_c, respectively.

Geometric solutions of parts (c) and (d) can be found, for example, in [**DLC15**].

[7]A *symmedian* is the line that is symmetric to the median of a triangle with respect to the corresponding angle bisector.

1.8.12. The measures of arcs RN and RB' of circles S_1 and Ω are equal (using a homothety with center R); therefore $\angle ARB' = \angle ACB' = \angle CAB'$ (see Fig. 8). Triangles $B'AR$ and $B'NA$ are similar, implying $|B'N| \cdot |B'R| = |B'A|^2$. Since $|B'I| = |B'A|$ (the trident lemma), we have $|B'N| \cdot |B'R| = |B'I|^2$, so triangles $B'IR$ and $B'NI$ are similar; therefore $\angle RNI = \angle RIB$.

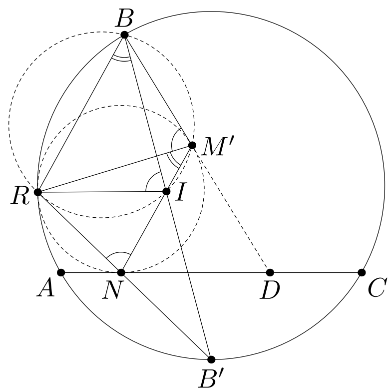

FIGURE 8

Let M' be the second intersection point of NI with S_1. Again using the equality of arcs RN and RB' of circles S_1 and Ω, we get $\angle RM'N = \angle RBB'$. From this equality it follows that points B, M', I, R lie on a circle; therefore $\angle RIB = \angle RM'B$.

Thus, $\angle RNM' = \angle RM'B$, so BM' is tangent to S_1; i.e., $M' = M$.

1.8.13. (a) Let I_1 and I_2 be the centers of circles S_1 and S_2; let S_1 be tangent to BD and AD at points M_1 and N_1, and let S_2 be tangent to BD and CD at points M_2 and N_2 (see Fig. 9). From the previous problem it follows that $I = M_1N_1 \cap M_2N_2$.

Points N_1, N_2, D are collinear, and points I_1, I_2, I are such that $I_1N_1 \parallel I_2N_2$ (perpendicular to AC), $I_1D \parallel IN_2$, and $I_2D \parallel IN_1$ (I_1D and I_2D are the bisectors of angles ADB and CDB). Then it is not difficult to establish that points I_1, I_2, I are collinear. (See Problem 1.9.2 in section 9.)

(b) Let S_1 and S_2 be congruent circles. Then $|AN_1| = |CN_2|$ and $|I_1N_1| = |IB_1| = |I_2N_2|$. Thus triangles IN_1B_1 and I_2DN_2 are congruent, so $|N_1B_1| = |DN_2|$. It follows that B_1 and D are symmetric with respect to the midpoint of AC as required.

1.8.14. *Note:* There are algebraic reasons why, for example, the inscribed circle and excircles have similar properties. The equation of a circle tangent to these three straight lines is the equation of either the inscribed circle or one of the excircles; the difference is only in the geometric arrangement. Therefore, statements about the inscribed circle often have “twin” counterparts for the excircles.

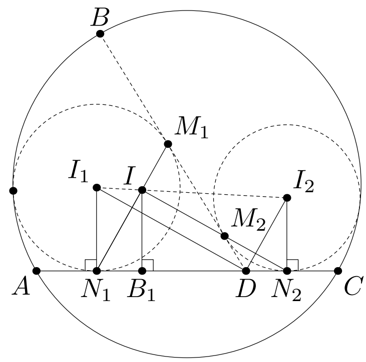

FIGURE 9

9. The generalized Napoleon's theorem (2*)
By P. A. Kozhevnikov

The classical version of Napoleon's Theorem states that the centers of equilateral triangles constructed on the sides of an arbitrary triangle (and exterior to it) are the vertices of an equilateral triangle.

Napoleon's Theorem is a special case of the statement of Problem 5.1.7 in Chapter 5. In this section we will prove another generalization of Napoleon's Theorem.

Below we invite you to solve a series of problems that apparently have no connection with Napoleon's Theorem. You can solve these problems using any means, and then see how they can be solved using the generalized Napoleon's theorem.

Introductory problems

1.9.1. Prove that the centers of the squares erected externally on the sides of a parallelogram are the vertices of a square.

1.9.2. On the legs of trapezoid $ABCD$, construct triangles ABE and CDF such that $AE \parallel CF$ and $BE \parallel DF$. Prove that if E lies on side CD, then F lies on side AB.

1.9.3. (a) Two circles intersect at points A and B. Through point A, draw a line intersecting the first circle at point C and the second circle at point D (assume that points C and D lie on opposite sides of point A). Let M and N be the midpoints of arcs BC and BD that do not contain the point A, and let K be the midpoint of segment CD. Prove that MKN is a right angle. (D. Tereshin, *All-Russian Mathematical Olympiad*, 1997.)

(b) Chord AB divides a circle into two circular segments, and one of them is rotated around A by some angle. Let D be the image of B under this rotation. Prove that the segments that connect the midpoints of the arcs of the circular segments with the midpoint of segment BD are perpendicular to each other. (W. Nasyrov, [**Kva92**].)

1.9.4. Draw lines l_1 and l_2 through vertex A of triangle ABC that are symmetric with respect to the bisector of angle A. Prove that the following points lie on a circle: the projections of points B and C onto l_1 and l_2 respectively; the midpoint of side BC; and the foot of the height dropped from vertex A.

1.9.5. In cyclic quadrilateral $ABCD$, let the diagonals intersect at point E, let points K and M be the midpoints of sides AB and CD, and let points L and N be the projections of E onto BC and AD respectively. Prove that $KM \perp LN$.

1.9.6. In two circles intersecting at points P and Q, two cyclists A and B simultaneously started moving from point P with equal angular speed, one moving clockwise and the other counterclockwise. Prove that the two cyclists are always equidistant from a fixed point. (Problem about cyclists, the case of movement in different directions, N. Vasilyev and I. Sharygin, [**Kva79**].)

1.9.7. Let K be the midpoint of side AC of acute triangle ABC. On sides AB and BC, construct (interior to triangle ABC) isosceles triangles ABM and BCN such that $|AM| = |BM|$, $\angle AMB = \angle AKB$, $|BN| = |CN|$, and $\angle BNC = \angle BKC$. Prove that the circumcircle of triangle MNK is tangent to AC. (A. Antropov and M. Uryev, [**Kva15**].)

1.9.8. On the circumcircle of triangle ABC, place points A_1, B_1, C_1 so that AA_1, BB_1, and CC_1 are concurrent. Denote the reflection of points A_1, B_1, C_1 across sides BC, CA, AB by A_2, B_2, C_2 respectively. Prove that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar. (A. Zaslavsky, [**Zas09b**].)

Formulation and proof of the generalized Napoleon's theorem

Let $\angle(\vec{a}, \vec{b})$ denote the rotational angle from vector $\vec{a} \neq \overrightarrow{0}$ to vector $\vec{b} \neq \overrightarrow{0}$, measured counterclockwise. This angle is determined up to integer multiples of 2π. For example, the equality $\angle(\vec{a}, \vec{b}) \equiv 0 \pmod{2\pi}$ means that $\angle(\vec{a}, \vec{b}) = 2k\pi$ for some $k \in \mathbb{Z}$.

1.9.9.* Generalized Napoleon's theorem. On the sides of triangle ABC, construct (possibly degenerate) triangles BCA_1, CAB_1, ABC_1 so that the points A, B, C, A_1, B_1, C_1 are pairwise distinct and the following two conditions hold:

1) $\angle(\overrightarrow{A_1B}, \overrightarrow{A_1C}) + \angle(\overrightarrow{B_1C}, \overrightarrow{B_1A}) + \angle(\overrightarrow{C_1A}, \overrightarrow{C_1B}) \equiv 0 \pmod{2\pi}$;

2) $|AB_1| \cdot |BC_1| \cdot |CA_1| = |BA_1| \cdot |CB_1| \cdot |AC_1|$.

Then the angles of triangle $A_1B_1C_1$ are given by the equalities

$$\angle(\overrightarrow{A_1C_1}, \overrightarrow{A_1B_1}) \equiv \angle(\overrightarrow{BC_1}, \overrightarrow{BA}) + \angle(\overrightarrow{CA}, \overrightarrow{CB_1}) \pmod{2\pi};$$
$$\angle(\overrightarrow{B_1A_1}, \overrightarrow{B_1C_1}) \equiv \angle(\overrightarrow{CA_1}, \overrightarrow{CB}) + \angle(\overrightarrow{AB}, \overrightarrow{AC_1}) \pmod{2\pi};$$
$$\angle(\overrightarrow{C_1B_1}, \overrightarrow{C_1A_1}) \equiv \angle(\overrightarrow{AB_1}, \overrightarrow{AC}) + \angle(\overrightarrow{BC}, \overrightarrow{BA_1}) \pmod{2\pi}.$$

Note: The theorem assumes that point A_1 is different from B, C, B_1, C_1, etc. However, it may be the case that the vertices of some of the triangles BCA_1, CAB_1, ABC_1, and $A_1B_1C_1$ are collinear. In this case, the triangle in question is degenerate, and its angles are considered equal (up to 2π) to 0, 0, and π.

Thus, the theorem states that under conditions 1 and 2, the angles of triangle $A_1B_1C_1$ depend only on the angles of the triangles built on the sides of triangle ABC, and not on the angles of triangle ABC. The condition of the theorem can also be described in the following elegant way (see [**Bel10**]): given points M, N, P, T, construct on the sides of triangle ABC triangles ABC_1, BCA_1, CAB_1 that are similar (preserving orientation) to triangles MNT, NPT, PMT respectively. Indeed, the fulfillment of conditions 1 and 2 in this case can be verified directly. On the other hand, if triangles ABC_1, BCA_1, CAB_1 satisfy the conditions of the theorem, and triangles MNT and NPT are similar to triangles ABC_1 and BCA_1, respectively, then PMT is similar to CAB_1.

The construction behind the generalized Napoleon's theorem is interesting, in that you can find even more beautiful facts around it; for example, circles ABC_1, BCA_1, CAB_1, and $A_1B_1C_1$ have a common point. (From this we can see that, in fact, in this construction we could have started with each of the triangles ABC_1, BCA_1, CAB_1, and $A_1B_1C_1$ and obtained the other three.)

This theorem, in a somewhat weakened form (with $A_1B = A_1C$, $B_1C = B_1A$, and $C_1A = C_1B$), was proposed by I. F. Sharygin in the early 1990s as a problem in the journal *Mathematics in School*.

If $\angle(\overrightarrow{A_1B}, \overrightarrow{A_1C}) = \angle(\overrightarrow{B_1C}, \overrightarrow{B_1A}) = \angle(\overrightarrow{C_1A}, \overrightarrow{C_1B}) = 2\pi/3$ and if $A_1B = A_1C$, $B_1C = B_1A$, and $C_1A = C_1B$, then the theorem becomes the classical Napoleon's Theorem. If each of angles $\angle(\overrightarrow{A_1B}, \overrightarrow{A_1C})$, $\angle(\overrightarrow{B_1C}, \overrightarrow{B_1A})$, $\angle(\overrightarrow{C_1A}, \overrightarrow{C_1B})$ is equal (up to 2π) to 0 or π, then the generalized Napoleon's theorem turns into Menelaus's Theorem.

Suggestions, solutions, and answers

Let us show that all the problems in this section can be treated as special cases of the generalized Napoleon's theorem. For some problems, we also outline alternative solutions. In the solutions below we allow triangles to be degenerate.

1.9.1. Let K, L, M, N be the centers of squares constructed respectively on sides AB BC, CD, DA of parallelogram $ABCD$; O is the center of the parallelogram. Applying the theorem to triangles ABK, BCL, CAO constructed on the sides of triangle ABC, we obtain that triangle KOL is an isosceles right triangle with right angle O. Similarly, triangles LOM, MON, NOK are isosceles right triangles with right angle O.

An alternative solution follows by noting that KAN and KBL are congruent triangles, obtained from each other by a rotation of 90°.

1.9.2. Let F' be a point on AB such that $\frac{|AF'|}{|BF'|} = \frac{|AD|}{|ED|} \cdot \frac{|EC|}{|BC|}$. The theorem can be applied to triangles ABF', BEC, EAD constructed on the sides of triangle ABE. Applying the theorem, we get $\angle F'CD = \angle AED$ and $\angle F'DC = \angle BEC$, which implies $F'C \parallel AE$ and $F'D \parallel BE$; i.e., $F' = F$.

Sketch of a second solution. Let $O = AB \cap CD$ and $F_1 = AB \cap CF$. Then $BC \parallel AD$ and $F_1C \parallel AE$ imply $\frac{|OB|}{|OA|} = \frac{|OC|}{|OD|}$ and $\frac{|OA|}{|OF_1|} = \frac{|OE|}{|OC|}$. Multiplying the equalities, we get $\frac{|OB|}{|OF_1|} = \frac{|OE|}{|OD|}$, and thus $F_1D \parallel BE$; i.e., $F_1 = F$.

Sketch of a third solution. From Pappus's Theorem (cf. Problem 3.9.3 (b) in Chapter 3) it follows that points $X = BC \cap AD$, $Y = FC \cap AE$, and $Z = FD \cap BE$ are on the line at infinity; therefore points $A = EY \cap DX$, $B = EZ \cap CX$, and $F = CY \cap DZ$ must be collinear.

1.9.3. (a) Construct isosceles triangles BMC and BND on the sides of triangle BCD such that $\angle BMC + \angle BND = 180°$, and consider the degenerate isosceles triangle CKD. The statement of the problem then follows from the generalized Napoleon's theorem.

(b) This is a special case of part (a) where $BC = BD$.

1.9.4. Let K and L be the projections of points B and C onto l_1 and l_2, respectively, let M be the midpoint of BC, and let AH be a height. The right triangles ABK and ACL are similar; therefore the theorem can be applied to triangles BAK, ACL, CBM constructed on the sides of triangle ABC. This yields

$$\angle(MK, ML) \equiv \angle(CA, CL) + \angle(BK, BA) \pmod{2\pi}.$$

Since points A, B, K, H are concyclic, and also points A, C, L, H are concyclic, it follows that

$$\begin{aligned}\angle(HK, HL) &= \angle(HK, HA) + \angle(HA, HL) \\ &\equiv \angle(BK, BA) + \angle(CA, CL) \pmod{2\pi}.\end{aligned}$$

Consequently, $\angle(ML, MK) \equiv \angle(HL, HK) \pmod{2\pi}$; i.e., points K, L, M, H are concyclic.

1.9.5. Because the quadrilateral is cyclic, we have $\angle EAN = \angle EBL$. The right triangles ANE and BLE are similar, so the theorem can be applied to triangles ANE, BLE, ABK constructed on the sides of triangle ABE. This yields

$$\angle(\overrightarrow{LK}, \overrightarrow{LN}) \equiv \angle(\overrightarrow{EA}, \overrightarrow{EN}) \equiv \angle(\overrightarrow{EL}, \overrightarrow{EB}) \equiv \angle(\overrightarrow{NL}, \overrightarrow{NK}) \pmod{2\pi};$$

hence, triangle KLN is isosceles ($KL = KN$). Similarly, $ML = MN$. It follows that the quadrilateral $KLMN$ is symmetric with respect to its diagonal KM.

1.9.6. Let A and B be arbitrary positions of the cyclists, and let A_0 and B_0 be the positions of the cyclists when they have traversed half a circle, so PA_0 and PB_0 are the diameters of the circles. Consider triangle PA_0B_0 and construct right triangles PA_0A and PB_0B on its sides along with the degenerate triangle A_0MB_0, where M is the midpoint of A_0B_0. This construction satisfies the generalized Napoleon's theorem, implying that $|MA| = |MB|$.

1.9.7. Point M is the midpoint of arc AKB; therefore KM is an external bisector of triangle AKB, so $\angle CKM = \angle CKB/2$. Consider triangle ABC and construct triangles ABM and BCN on its sides along with the degenerate triangle AKC. This construction satisfies the generalized Napoleon's theorem, from which it follows that $\angle MNK = \angle ABM = (180° - \angle AMB)/2 = (180° - \angle AKB)/2 = \angle CKB/2 = \angle CKM$. The resulting equality of angles $\angle MNK = \angle CKM$ solves the problem.

Note: This problem is connected with another theme. For triangle AKB, the point M is the midpoint of arc AKB of the circumscribed circle. If we place equal-length segments AX and BY on rays AK and BK, then points X, Y, K, M will be concyclic (see, for example, [**Pol12**]). Construct the point Y on the median BK such that $|BY| = |AK| = |CK|$; then in our case $X = K$, which corresponds to a circle tangent to the line BC.

1.9.8. It is easy to see that the conditions of the generalized Napoleon's theorem are fulfilled by triangles ABC_1, BCA_1, CAB_1 constructed on the sides of triangle ABC (condition 2 follows from the fact that the main diagonals of the cyclic hexagon $AC_1BA_1CB_1$ intersect at one point). Therefore, the conditions of the theorem are satisfied also for triangles ABC_2, BCA_2, CAB_2. From the theorem it follows that

$$\begin{aligned}\angle(\overrightarrow{A_2C_2}, \overrightarrow{A_2B_2}) &\equiv \angle(\overrightarrow{BC_2}, \overrightarrow{BA}) + \angle(\overrightarrow{CA}, \overrightarrow{CB_2})\\ &\equiv \angle(\overrightarrow{BA}, \overrightarrow{BC_1}) + \angle(\overrightarrow{CB_1}, \overrightarrow{CA}) \equiv \angle(\overrightarrow{A_1B_1}, \overrightarrow{A_1C_1}) \pmod{2\pi}.\end{aligned}$$

Similarly,

$$\begin{aligned}\angle(B_2A_2, B_2C_2) &\equiv \angle(B_1C_1, B_1A_1),\\ \angle(C_2B_2, C_2A_2) &\equiv \angle(C_1A_1, C_1B_1) \pmod{2\pi}.\end{aligned}$$

We conclude that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar (and are oppositely oriented).

1.9.9. Consider the rotational homothety H_a with center A_1 transforming B to C, the rotational homothety H_b with center B_1 transforming C to A, and the rotational homothety H_c with center C_1 transforming A to B. When performing H_a, the arbitrary vector $\vec{a}$ is rotated by $\angle(\overrightarrow{A_1B}, \overrightarrow{A_1C})$ and its length is multiplied by $\frac{|A_1C|}{|A_1B|}$. Similar statements hold for H_b and H_c. Thus, when performing the composition $H_c \circ H_b \circ H_a$, the vector $\vec{a}$ rotates by angle $\angle(\overrightarrow{A_1B}, \overrightarrow{A_1C}) + \angle(\overrightarrow{B_1C}, \overrightarrow{B_1A}) + \angle(\overrightarrow{C_1A}, \overrightarrow{C_1B}) \equiv 0 \pmod{2\pi}$ (i.e., does not change direction), and its length is multiplied by $\frac{|A_1C|}{|A_1B|} \cdot \frac{|B_1A|}{|B_1C|} \cdot \frac{|C_1B|}{|C_1A|} = 1$ (i.e., the length does not change). Given that $H_c(H_b(H_a(B))) = H_c(H_b(C)) = H_c(A) = B$ (i.e., the composition $H_c \circ H_b \circ H_a$ has a fixed point B), we see that $H_c \circ H_b \circ H_a$ is the identity transformation.

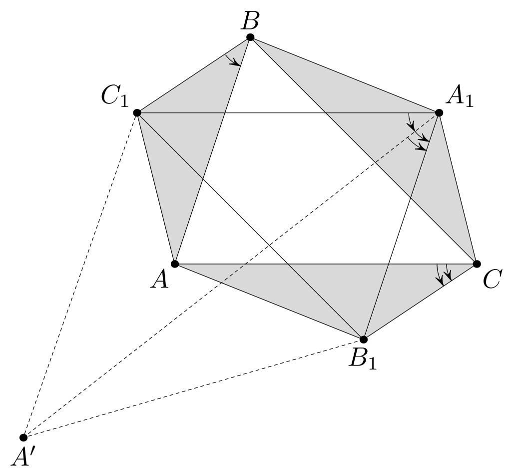

FIGURE 10

Write $H_b(A_1) = A'$. Then

$$H_c(A') = H_c(H_b(A_1)) = H_c(H_b(H_a(A_1))) = A_1.$$

Triangles $B_1XH_b(X)$ are similar with the same orientation for all points $X \neq B_1$; hence triangles B_1A_1A' and B_1CA are similar with the same orientation so that $\angle(\overrightarrow{A_1A'}, \overrightarrow{A_1B_1}) \equiv \angle(\overrightarrow{CA}, \overrightarrow{CB_1}) \pmod{2\pi}$. Similarly, triangles C_1A_1A' and C_1BA are similar with the same orientation, implying $\angle(\overrightarrow{A_1C_1}, \overrightarrow{A_1A'}) = \angle(\overrightarrow{BC_1}, \overrightarrow{BA}) \pmod{2\pi}$. Consequently,

$$\begin{aligned}\angle(\overrightarrow{A_1C_1}, \overrightarrow{A_1B_1}) &= \angle(\overrightarrow{A_1C_1}, \overrightarrow{A_1A'}) + \angle(\overrightarrow{A_1A'}, \overrightarrow{A_1B_1}) \\ &\equiv \angle(\overrightarrow{BC_1}, \overrightarrow{BA}) + \angle(\overrightarrow{CA}, \overrightarrow{CB_1}) \pmod{2\pi}.\end{aligned}$$

The remaining equalities of the angles are proved similarly.

Note: Compare this argument with the construction of the center of the composition of two rotational homotheties.

10. Isogonal conjugation and the Simson line (3^*)
By A. V. Akopyan

Before attempting the problems of this section, we recommend that you work on problems from sections 2, 3, 5, and 7 of this chapter.

Let P be a point that does not lie on the circumcircle of triangle ABC. The triangle whose vertices are the projections of point P onto the sides of triangle ABC or their extensions is called the *pedal triangle* of point P relative to triangle ABC. The circumcircle of the pedal triangle is called the *pedal circle* of point P with respect to triangle ABC.

1.10.1.° Which of the following triangles is not the pedal triangle for any point relative to a non-isosceles triangle? The triangle whose vertices are
(a) the midpoints of the sides;
(b) the intersections of the angle bisectors with the opposite sides;
(c) the tangent points of the inscribed circle;
(d) the feet of the heights.

1.10.2.° Let the sides of triangle ABC be tangent to the corresponding excircles at points A', B', C'. For which of the following points will triangle $A'B'C'$ be the pedal triangle?
(a) The centroid.
(b) The incenter.
(c) A point symmetric to the incenter with respect to the circumcenter.
(d) Such a point does not exist.

1.10.3. Prove that the projections of point P onto the sides of triangle ABC or their extensions are collinear if and only if P lies on the circumscribed circle of triangle ABC.

The straight line thus obtained is called the *Simson line* of P with respect to triangle ABC.

1.10.4. Let P lie on the circumcircle of triangle ABC, and suppose that P is not one of the vertices of this triangle. Let B' be chosen on the circumcircle so that line $PB' \perp AC$. Prove that chord BB' (or the tangent to the circle at point B, if $B = B'$) is parallel to the Simson line of P.

1.10.5. Prove that when point P moves along the circumcircle, the Simson line rotates in the opposite direction and its rotation speed is half the rate of change of arc PA.

1.10.6. Let ABC be a triangle that has no right angle, with orthocenter H. Prove that the Simson line of point P with respect to ABC bisects PH.

The *circumcevian triangle* of P with respect to triangle ABC is the triangle whose vertices are the second intersection points of lines AP, BP, CP with the circumcircle ABC (if any of the lines AP, BP, CP are tangent, then that vertex of the circumcevian triangle coincides with the corresponding vertex of ABC).

1.10.7. Prove that the pedal and circumcevian triangles of a point P with respect to triangle ABC are similar with the same orientation.

Let ABC be an arbitrary triangle. For any point P that is not a vertex of the triangle, reflect lines AP, BP, and CP about the bisectors of the corresponding angles of the triangle.

1.10.8. Prove that the resulting three lines are either concurrent or parallel.

Denote this intersection point by P' if it exists and is unique. We say that P' is *isogonally conjugate* to P with respect to triangle ABC, and we call the transformation sending P to P' the *isogonal conjugation.* (See also footnote 6 on p. 22 for a definition of isogonal conjugate.)

1.10.9.° How many fixed points does the isogonal conjugation have?
(a) 0; (b) 1; (c) 2; (d) 4.

1.10.10. Reflect P about the sides of a triangle. Prove that if the three reflection points are different and not collinear, then the center of the circle passing through them is isogonally conjugate to P. (This, of course, is also a proof that isogonal conjugation is well-defined.)

Prove the following elementary properties of isogonal conjugation.

1.10.11. If point P does not lie on the lines containing the sides of a triangle, then P' exists and is unique, and P is isogonally conjugate to P'.
These two points are called *isogonally conjugate.*

1.10.12. A point isogonally conjugate to a point lying on a line containing a side of the triangle is the vertex of the triangle opposite to that side.

1.10.13. Let P lie on the circumcircle of triangle ABC and be different from A, B, and C. Then the reflections of lines AP, BP, CP about the corresponding angle bisectors are perpendicular to the Simson line of P with respect to ABC.

1.10.14. Consider two points that do not lie on the lines containing the sides of a triangle or on its circumcircle. Then the pedal circles of these two points coincide if and only if the points are isogonally conjugate.

1.10.15. The orthocenter and the circumcenter of a triangle are isogonally conjugate.

Note: Problems 1.10.14 and 1.10.15 imply that the midpoints of the sides and the feet of the heights of the triangle lie on the same circle (for another proof, see section 5 of this chapter).

1.10.16.* Let the tangents to the circumcircle of a triangle ABC at points B and C intersect at A_1. Prove that the line AA_1 is the reflection of the median drawn to side BC about the angle bisector of angle A.

1.10.17. Pascal's Theorem. Let points A, B, C, D, E, and F lie on the same circle. Let AB and DE intersect at X, BC and EF intersect at Y, and AF and CD intersect at Z. Then X, Y, Z are collinear.

Deeper results in this direction can be found in [**AkZa07a**, **AkZa07b**].

Suggestions, solutions, and answers

We consider only one of several possible cases in the problems below; the other cases are dealt with in similar ways.

1.10.3. Let P_a, P_b, and P_c be the projections of point P onto sides BC, CA, and AB, respectively. We consider the case shown in Fig. 11.

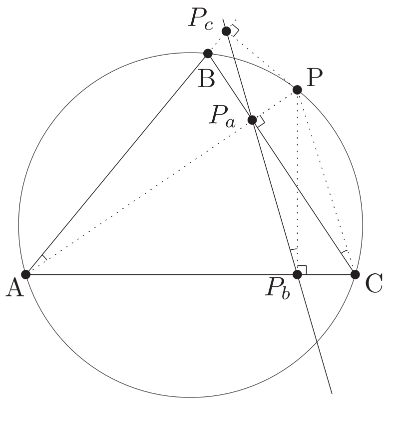

FIGURE 11

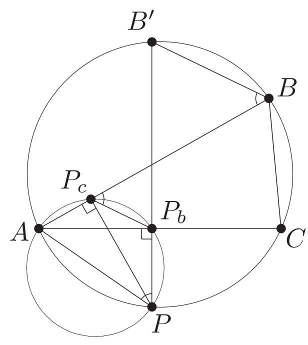

FIGURE 12

The quadrilateral PCP_bP_a is cyclic, so $\angle PP_bP_a = \angle PCP_a$. Likewise, $\angle PP_bP_c = \angle PAP_c$. Points P_a, P_b, P_c are collinear if and only if $\angle PP_bP_c = \angle PP_bP_a$ or, equivalently, $\angle PAP_c = \angle PCP_a$. But this means that P lies on the circumcircle of triangle ABC.

1.10.4. Consider the case shown in Fig. 12. The projections of point P onto sides AB and AC are denoted by P_c and P_b respectively. Then $\angle ABB' = \angle APB'$ as angles subtending arc AB'. Since the quadrilateral AP_cP_bP is cyclic (AP is the diameter of its circumscribed circle, all the vertices are different, and P does not coincide with A, B, or C) and the sum of opposite angles of a cyclic quadrilateral is $180°$, we get $\angle APB' = \angle APP_b = 180° - \angle AP_cP_b = \angle BP_cP_b$. Therefore, $P_bP_c \parallel BB'$.

1.10.5. This follows easily from the previous problem.

1.10.6. Consider the case shown in Fig. 13.

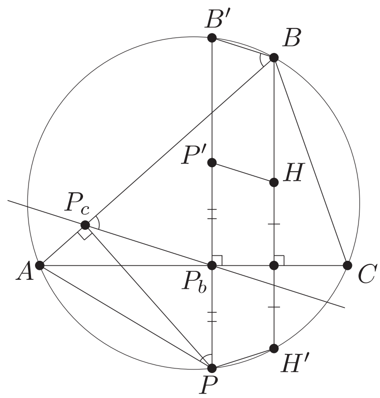

FIGURE 13

It is easy to see that $\angle AHC = 180° - \angle ABC$, which means that point H', which is symmetric to point H with respect to AC, lies on the circumcircle of ABC (see Fig. 13). Since PB' and BH' are perpendicular to AC, the quadrilateral $PB'BH'$ is a trapezoid (we assume that $P \neq H'$; otherwise the statement of the problem is obvious), which is isosceles since it is cyclic. This means that the line symmetric to PH' with respect to AC (i.e., parallel to the axis of symmetry of the trapezoid) will be parallel to BB'. Consequently, line $P'H$ is parallel to BB', and therefore, according to Problem 1.10.3, it is parallel to the Simson line of point P (here P' is the image of point P under symmetry with respect to AC). Since P_b (the projection of point P to the side AC) is the midpoint of PP', the Simson line is the midline of triangle HPP', and consequently it bisects PH.

1.10.7. Consider the case shown in Fig. 14.

Points P_a, P_b, P_c are the vertices of the pedal triangle, points A', B', C' are the vertices of the circumcevian triangle, and $\angle AA'C' = \angle ACC' = \angle P_bP_aP$. The last equality is true since the quadrilateral PP_aCP_b is cyclic. Similarly, $\angle AA'B' = \angle P_cP_aP$, which implies that $\angle C'A'B' = \angle P_bP_aP_c$. Likewise, $\angle A'B'C' = \angle P_aP_bP_c$ and $\angle A'C'B' = \angle P_aP_cP_b$. Consequently, triangles $A'B'C'$ and $P_aP_bP_c$ are similar.

1.10.8. Since triangles $A'B'C'$ and $P_aP_bP_c$ are similar, the point P in triangle $A'B'C'$ has a corresponding point P' in triangle $P_aP_bP_c$. This point will be isogonally conjugate to P with respect to $P_aP_bP_c$.

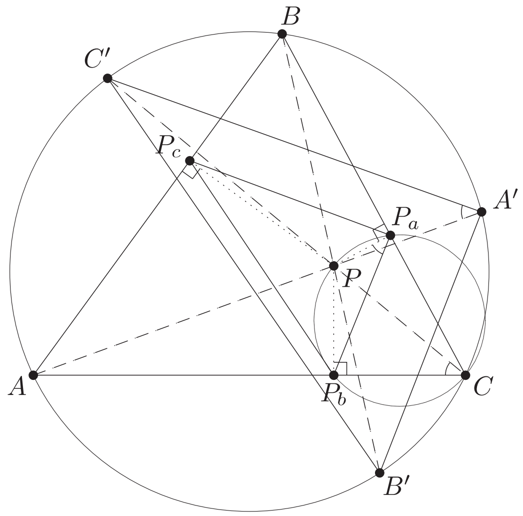

FIGURE 14

1.10.10. Consider the case shown in Fig. 15.

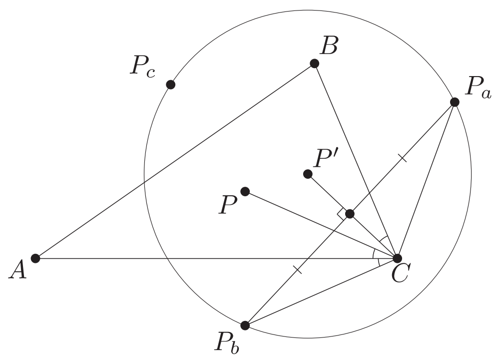

FIGURE 15

Let P_a be symmetric to P with respect to side BC, with P_b and P_c defined similarly (see Fig. 15). Let P' be the circumcenter of triangle $P_aP_bP_c$. As C is equidistant from P_a and P_b, CP' is the perpendicular bisector of segment P_aP_b. This means that $\angle P_aCP' = \frac{1}{2}\angle P_aCP_b = \angle C$. But then

$$\angle BCP' = \angle P_aCP' - \angle BCP_a = \angle C - \angle BCP = \angle ACP.$$

Similarly, we have $\angle ABP' = \angle CBP$ and $\angle BAP' = \angle CAP$. Consequently, P' is isogonally conjugate to P with respect to triangle ABC.

1.10.13. Consider the case shown in Fig. 16; other cases are dealt with similarly.

Let P lie on the circumcircle and let P_b and P_c be the projections of point P onto sides AC and AB, respectively. Let X be the point of intersection

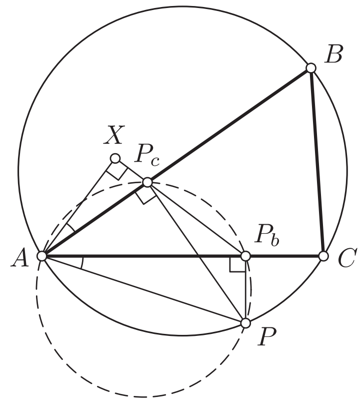

FIGURE 16

of the Simson line of point P with the line a that is symmetric to AP with respect to the angle bisector at A. The quadrilateral APP_bP_c is cyclic, so

$$\begin{aligned}\angle AP_cP_b &= 180^\circ - \angle APP_b = 180^\circ - (90^\circ - \angle PAP_b)\\ &= 90^\circ + \angle PAP_b = 90^\circ + \angle XAP_c.\end{aligned}$$

But since an exterior angle of a triangle is equal to the sum of the opposite interior angles, we have $\angle AXP_c = 90^\circ$. Similarly, the lines symmetric to PB and PC with respect to the bisectors of the corresponding angles are perpendicular to P_cP_b.

1.10.14. Indeed, if points P and P' are isogonally conjugate, then their pedal circle is a circle whose center is the midpoint of PP', with radius $|P'P_a|/2 = |PP'_a|/2$, where P_a and P'_a are symmetric to P and P' with respect to the side BC (see Fig. 15).

Conversely, if the pedal circles of points P and Q coincide, then, as proved above, they coincide with the pedal circle of the point P' that is isogonally conjugate to point P. By the pigeonhole principle, two of the three vertices of the pedal triangle of point Q coincide with the vertices of the pedal triangle either of point P or of point P'. Consequently, point Q coincides with one of these points, because the projections of a point onto two lines completely determine the position of this point.

1.10.15. (a) Consider the case where triangle ABC is acute. Let H and O be the orthocenter and circumcenter, respectively. Then $\angle BAH = 90^\circ - \angle B$; but $\angle AOC = 2\angle B$, so $\angle OAC = \frac{1}{2}(180^\circ - 2\angle B) = 90^\circ - \angle B$, whence it follows that $\angle BAH = \angle OAC$, and this means that AH and AO are symmetric with respect to the angle bisector at A. We argue similarly for the other two angles.

1.10.16. The point A', symmetric to A with respect to M_a, obviously lies on the median AM_a (see Fig. 17).

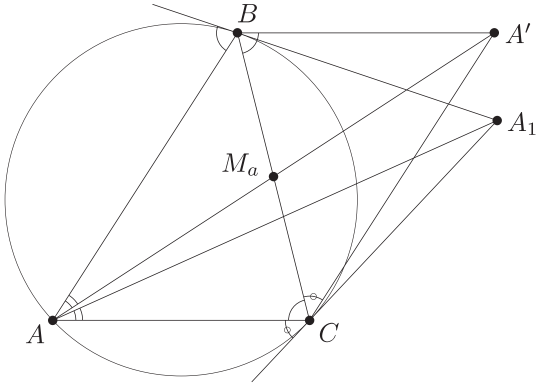

FIGURE 17

It is easy to check that lines BA' and CA' are symmetric to the tangents of the circumcircle with respect to the bisectors of the corresponding angles. Therefore, A' and A_1 are isogonal conjugates.

1.10.17. We will consider only one case of the configuration of points on the circle (see Fig. 18).

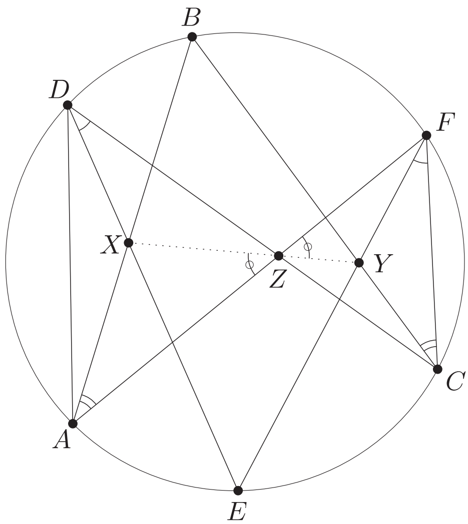

FIGURE 18

Angles BAF and BCF are equal because they are subtended by the same arc. Similarly, angles CDE and CFE are equal. Furthermore, triangles AZD and CZF are similar. Consider the similarity that transforms triangle AZD to triangle CZF. Under this transformation, X goes to X', which is the isogonal conjugate of Y with respect to triangle CZF (because of the equal angles mentioned above). Therefore, $\angle AZX = \angle CZX' = \angle FZY$, implying that X, Z, and Y are collinear.

Note: Another proof of Pascal's Theorem can be obtained with the help of projective transformations (see section 9 in Chapter 3).

Additional reading

For more information about the geometry of triangles see [**EM90**, **Sh89**, **Ku92**]; about the "semi-inscribed" circle see [**Koz12**, **Pro92**, **Pro08**, **Gi90**]; about the generalized Napoleon's theorem see [**Bel10**], [**Pol12**], [**Kva15**], [**Kva79**]; and about isogonal conjugation and the Simson line see [**AkZa07a**, **AkZa07b**].

Chapter 2

Circle

1. The simplest properties of a circle (1)
By A. D. Blinkov

2.1.1.$^\circ$ Lines a, b, and c intersect pairwise and are not concurrent. How many different circles can be simultaneously tangent to each of these lines?

(a) One;
(b) two;
(c) three;
(d) four;
(e) infinitely many;
(f) impossible to determine.

2.1.2. A cue ball is launched from an arbitrary point of a circular billiard table. Prove that there exists a circle on this table such that the trajectory of the ball never crosses it. (See [**GT86**].)

2.1.3. Three non-coinciding circles with equal radius pass through the point H. Let A, B, and C be their pairwise points of intersections other than H. Prove that H is the orthocenter of triangle ABC. (See [**Pra19**].)

2.1.4. The extensions of sides AB and CD of cyclic quadrilateral $ABCD$ intersect at P, and the extensions of sides BC and AD intersect at Q. Prove that the intersection points of the bisectors of angles AQB and BPC with the sides of the quadrilateral are the vertices of a rhombus. (See [**Pra19**].)

2.1.5. The centers of three pairwise externally tangent circles lie at the vertices of a right triangle with perimeter P. Find the radius of a fourth circle, which is tangent to the three given circles and contains them in its interior.

2.1.6. In convex quadrilateral $ABCD$, let $\angle A = \alpha$ and $\angle B = \beta$. The bisectors of the exterior angles at C and D intersect at point P. Find angle CPD.

2.1.7. A circle with radius R is tangent to the hypotenuse AB of right triangle ABC and to extensions of its legs. Find the perimeter of the triangle.

Suggestions, solutions, and answers

2.1.2. Let the ball be launched along line AB, not passing through the center of the table. Then the trajectory of the ball is the piecewise-linear path $ABCD\ldots$. When the ball is reflected, the angle ABO of incidence (i.e., the angle between the path of the ball and the radius of the circular table drawn to the point of impact of the ball with the wall) is equal to the angle of reflection CBO (i.e., the angle between the next segment of the path and the same radius; see Fig. 1). The measures of these angles will not change with successive reflections of the ball from the wall, which follows from the congruence of the isosceles triangles whose bases are the segments of the trajectory of the ball (i.e., congruent to chord BC). Therefore the distance from the center of the circle to any segment of the path is constant. So any circle whose center coincides with the center of the table and whose radius is less than this distance suffices.

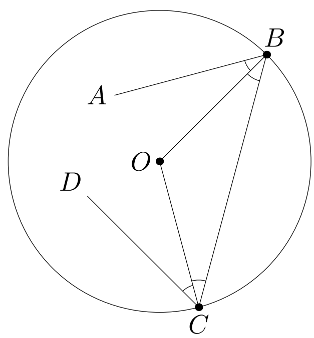

FIGURE 1

If the ball is launched along a straight line passing through the center of the table, the trajectory of its movement will be a "degenerate piecewise-linear path" coinciding with a diameter of the circle. This path will partition the circle into two semicircles. Then any circle inside one of these semicircles suffices.

2.1.3. Let O_1, O_2, and O_3 be the centers of the given circles (see Fig. 2). Since all sides of the quadrilateral BO_1HO_2 are equal, it is a rhombus, so $BO_2 \parallel HO_1$. Similarly, AO_1HO_3 is a rhombus, so $AO_3 \parallel HO_1$. Therefore, $BO_2 \parallel AO_3$. Furthermore, $|BO_2| = |AO_3|$, so ABO_2O_3 is a parallelogram; i.e., $AB \parallel O_2O_3$. Since $CH \perp O_2O_3$, we see that $CH \perp AB$. Likewise, it can be shown that $AH \perp BC$; therefore H is the orthocenter of triangle ABC.

2.1.4. Let the bisectors of angles AQB and BPC intersect at R, and denote the other intersections by $PR \cap BC = N$, $PR \cap AD = E$, $QR \cap AB = M$, and $QR \cap CD = F$ (see Fig. 3). Let $\angle BQM = \alpha$, $\angle BPN = \beta$, and $\angle PBC = \varphi$.

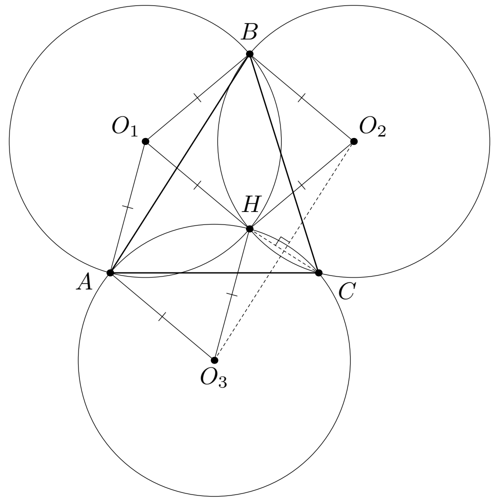

FIGURE 2

Consider quadrilateral $MBNR$:

$$\begin{aligned}\angle MRN &= 360^\circ - (\angle B + \angle M + \angle N)\\ &= 360^\circ - (180^\circ - \varphi) - (\varphi + \alpha) - (\varphi + \beta) = 180^\circ - (\varphi + \alpha + \beta).\end{aligned}$$

We will calculate $\varphi + \alpha + \beta$. In quadrilateral $ABCD$, we have $\angle A = \varphi + 2\alpha$, $\angle C = \varphi + 2\beta$, and $\angle A + \angle C = 2\varphi + 2\alpha + 2\beta = 180^\circ$, since this quadrilateral is cyclic. Therefore $\varphi + \alpha + \beta = 90^\circ$, so $\angle MRN = 90^\circ$. Thus, the angle bisectors QR and PR in triangles QNE and PMF, respectively, are also heights, so these triangles are equilateral. Thus R is the midpoint of segments MF and NE. Consequently, the diagonals of quadrilateral $MNEF$ are bisected by their point of intersection; i.e., this quadrilateral is a parallelogram. Moreover, these diagonals are perpendicular, so $MNFE$ is rhombus.

2.1.5. Let A, B, and C be the centers of the given circles; $\triangle ABC$ has a right angle at C. Let O be the point in the plane such that $ACBO$ is rectangle. Let A_1 denote the intersection point furthest from O of the ray OC with the circle centered at C; define B_1 and C_1 similarly (for the ray OA and the circle centered at A and for the ray OB and the circle centered at B respectively), as shown in Fig. 4. Let us prove that O is the center of the fourth circle. To do this, we denote the radii of the given circles by x, y, and z. Then $|OC_1| = |OB| + |BC_1| = y + z + x$, $|OA_1| = |OC| + |CA_1| = x + z + y$, and $|OB_1| = |OA| + |AB_1| = x + y + z$. Thus O is equidistant from the three points A_1, B_1, and C_1; i.e., it is the center of the circle passing through these points. It's clear that this circle is tangent to the first three, since points A_1, B_1, and C_1 lie on the center lines. The radius of this circle is $R = x + y + z = P/2$.

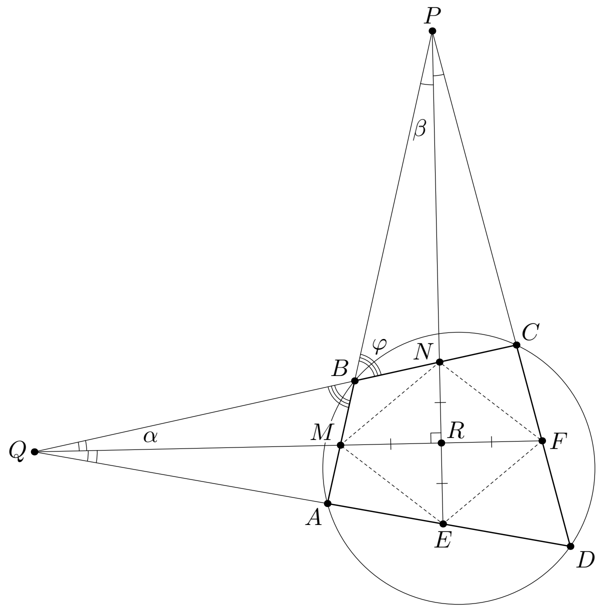

FIGURE 3

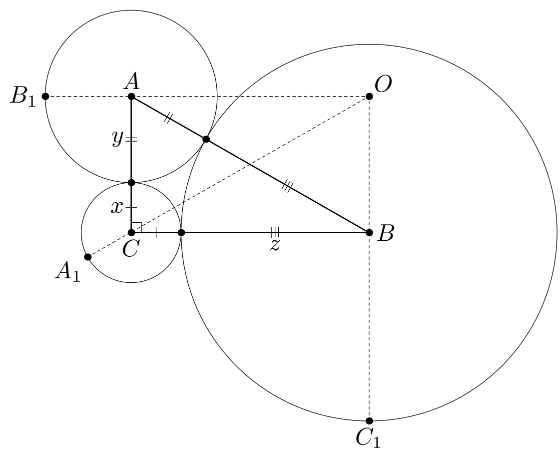

FIGURE 4

2. Inscribed angles (1)
By A. D. Blinkov and D. A. Permyakov

2.2.1.° Choose three conditions, each of which is equivalent to the condition that the convex quadrilateral $ABCD$ is cyclic.

(a) $|AB| + |CD| = |BC| + |AD|$;
(b) $\angle BAD + \angle BCD = 180°$;
(c) $ABCD$ is a rectangle;
(d) $ABCD$ is a rhombus;

(e) $\angle BAC = \angle BDC$;
(f) angle ABC is equal to the exterior angle at vertex D.

2.2.2.° Let ABC be an equilateral triangle with side length a. Point D is located at a distance a from vertex A. What are the possible values of angle BDC?

2.2.3. In triangle ABC, the inequality $|AB| > |BC|$ holds. A point P is chosen on side AB so that $|BP| = |BC|$. The angle bisector BM intersects the circumcircle at N. Prove that quadrilateral $APMN$ is cyclic. (See [**GT86**].)

2.2.4. From an arbitrary point M lying inside a given angle with vertex A, perpendiculars MP and MQ are dropped to the sides of the angle or their extensions. From point A, the perpendicular AK is dropped onto line PQ. Prove that $\angle PAK = \angle MAQ$.

2.2.5. In triangle ABC, construct medians AA_1 and BB_1. Prove that $\angle CAA_1 = \angle CBB_1$ if and only if $|AC| = |BC|$. (See [**Pra19**].)

2.2.6. Points A, B, C, D lie, in order, on a circle. Let A_1, B_1, C_1, and D_1 be the midpoints of arcs AB, BC, CD, and DA respectively. Prove that $A_1C_1 \perp B_1D_1$.

2.2.7. Circles ω_1 and ω_2 are externally tangent at A. Through this point, draw lines B_1B_2 and C_1C_2, where B_1 and C_1 lie on ω_1 and B_2 and C_2 lie on ω_2. Prove that $B_1C_1 \parallel B_2C_2$.

2.2.8. Let O be the center of the excircle of triangle ABC that is tangent to side BC. A circle with center D passes through A, B, and O. Prove that A, B, C, and D are concyclic. (See [**GT86**].)

2.2.9. Point O in the interior of triangle ABC has the following property: lines AO, BO, and CO pass through the circumcenters of triangles BCO, ACO, and ABO, respectively. Prove that O is the incenter of ABC. (See [**Pra19**].)

2.2.10. Rectangle $ABCD$ is circumscribed about right triangle APQ so that points P and Q lie on sides BC and CD, respectively. Let P' and Q' be the midpoints of AP and AQ. Prove that $BQ'C$ and $CP'D$ are right triangles.

2.2.11. Points A, B, C, D lie, in order, on a circle. Let M be the midpoint of arc AB. Let the intersection points of chords MC and MD with chord AB be E and F, respectively. Prove that $FECD$ is a cyclic quadrilateral.

2.2.12. Circles S_1 and S_2 intersect at point A. Through point A, draw a line intersecting S_1 at point B and S_2 at point C, so that points B, A, and C lie on the line in this order. At points C and B, draw tangents to the circles, intersecting at point D. Prove that the measure of angle BDC does not depend on the choice of the line passing through point A.

2.2.13. Let C be a point on chord AB of circle S with center O. The circumcircle of AOC intersects circle S at point $D \neq A$. Prove that $|BC| = |CD|$.

2.2.14. Let $ABCDE$ be a cyclic pentagon. Let the distances from E to the lines AB, BC, and CD be a, b, and c, respectively. Find the distance from E to AD.

2.2.15. Let AA_1, BB_1, and CC_1 be the heights of triangle ABC. Let B_2 and C_2 be the midpoints of BB_1 and CC_1. Prove that triangles $A_1B_2C_2$ and ABC are similar.

2.2.16. A circle is divided into equal arcs by n diameters. Prove that the feet of the perpendiculars dropped from an arbitrary point M inside the circle onto these diameters

(a) lie on one circle;

(b)* are vertices of a regular polygon.

2.2.17.* Let ACA_1A_2 and BCB_1B_2 be squares constructed externally to triangle ABC with bases on the triangle. Prove that lines A_1B, A_2B_2, and AB_1 are concurrent.

Suggestions, solutions, and answers

2.2.3. Since angles ANB and ACB are inscribed and are subtended by the same arc, they are equal (see Fig. 5). It also follows that triangles BMP and BMC are congruent (by SAS), so $\angle BPM = \angle BCA$. Thus, $\angle APM + \angle ANM = 180°$; i.e., quadrilateral $APMN$ is cyclic.

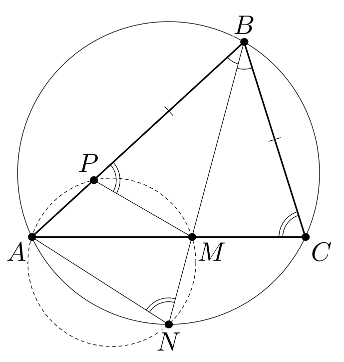

FIGURE 5

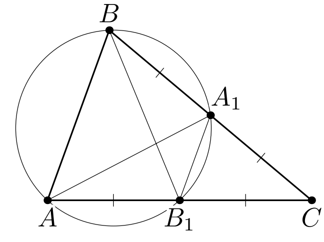

FIGURE 6

2.2.5. Segment A_1B_1 is a midline of triangle ABC, so AB_1A_1B is a trapezoid. Consider the circle circumscribed around $\triangle ABB_1$ (see Fig. 6).

1. If $|AC| = |BC|$, then $|AB_1| = |BA_1|$; i.e., trapezoid AB_1A_1B is isosceles, and point A_1 lies on the same circle. Therefore, $\angle A_1AB_1 = \angle B_1BA_1$, since these angles are inscribed in a circle and subtended by the same arc B_1A_1.

2. If $\angle CAA_1 = \angle CBB_1$, then again A_1 lies on the same circle, so trapezoid AB_1A_1B is isosceles. Therefore, $|AB_1| = |BA_1|$; i.e., $|AC| = |BC|$.

Note: Part 1 can also be proved using the fact that $\triangle AA_1B_1 \cong \triangle BB_1A_1$ (by SSS).

2.2.8. Since O is the center of an excircle of triangle ABC, ray AO is an angle bisector of BAC, and BO is an angle bisector of B_1BC, where B_1 is a point on the extension of side AB (see Fig. 7). Clearly D, C, and O lie on one side of line AB. Connecting D with A and B, we get $\angle AOB = \frac{1}{2}\angle ADB$. Since $\angle B_1BO$ is an external angle of $\triangle ABO$, we get

$$\angle B_1BO = \angle AOB + \angle BAO = \frac{1}{2}(\angle ADB + \angle BAC),$$

and thus $\angle B_1BC = \angle ADB + \angle BAC$.

On the other hand, since B_1BC is an exterior angle of $\triangle ABC$, we have $\angle B_1BC = \angle BCA + \angle BAC$. Hence, $\angle ADB = \angle BCA$; i.e., points A, B, C, and D lie on the same circle.

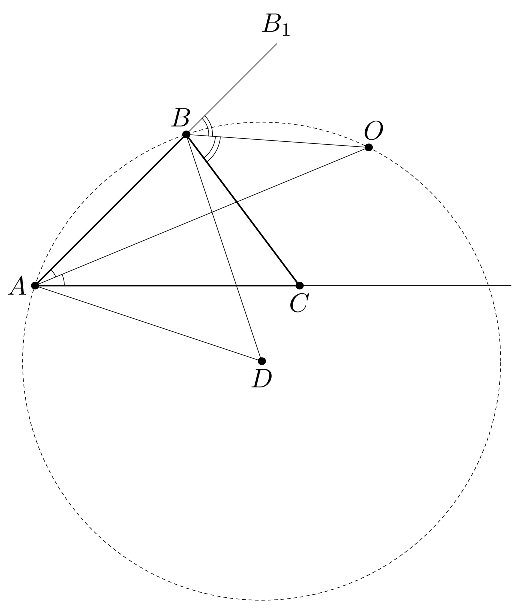

FIGURE 7

2.2.9. Consider the circle with center P circumscribed around $\triangle AOC$ (see Fig. 8). Since OPC is a central angle and OAC is an inscribed angle sub-

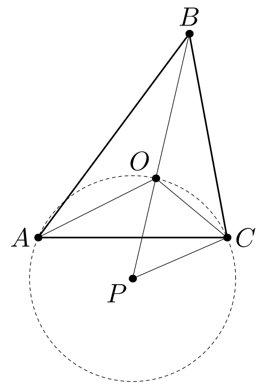

FIGURE 8

tended by the same arc, we have

$$\angle COP = \frac{1}{2}(180^\circ - \angle OPC) = 90^\circ - \angle OAC.$$

Then

$$\angle BOC = 180^\circ - \angle COP = 90^\circ + \angle OAC.$$

Similarly, considering the circumcircle of AOB, we have $\angle BOC = 90^\circ + \angle OAB$. Thus $\angle OAB = \angle OAC$; i.e., AO bisects angle BAC. Likewise, CO bisects BCA. Therefore, O is the incenter of triangle ABC.

2.2.14. *Answer*: ac/b.

3. Inscribed and circumscribed circles (2)
By A. A. Gavrilyuk

2.3.1. Let $ABCD$ be a convex quadrilateral. Through the intersection points of the extensions of its sides, two lines are drawn which divide $ABCD$ into four quadrilaterals. Prove that if the small quadrilaterals containing B and D are tangential, then $ABCD$ is also tangential.

2.3.2. Let $ABCD$ be a convex quadrilateral with A_1 and A_2, B_1 and B_2, C_1 and C_2, and D_1 and D_2 lying on sides AB, BC, CD, and DA, respectively, and arranged in that order. Then A_1C_2, A_2C_1, B_1D_2, B_2D_1 divide $ABCD$ into 9 quadrilaterals. Let the central quadrilateral and the small quadrilaterals containing the vertices of $ABCD$ be tangential. Prove that $ABCD$ is also tangential.

2.3.3. Let B_1 and B_2 be the points of tangency of side AC of triangle ABC with the inscribed circle and escribed circles (tangent to AC), respectively. Let A_2 and C_2 be the points of tangency of this escribed circle with the extensions of sides BC and BA. Prove that $|CB_1| = |AB_2| = |AC_2| = \frac{|AC|+|BC|-|AB|}{2}$.

2.3.4. Let $ABCD$ be a cyclic quadrilateral, and let H_C and H_D be the orthocenters of triangles ABD and ABC, respectively. Prove that CDH_CH_D is a parallelogram.

2.3.5. Let $ABCD$ be a tangential quadrilateral.[1] Prove that the incenters of triangles ABC, BCD, CDA, and DAB are vertices of a rectangle.

2.3.6. Let ABC be an acute triangle. Construct a circle whose diameter is AC, and let the second intersection points of this circle with sides AB and BC be K and L, respectively. Let M be the intersection point of the tangents to the circle at these points. Prove that $BM \perp AC$.

2.3.7. Let $ABCD$ be a tangential trapezoid with bases AD and BC. Let O be the intersection point of the diagonals of this trapezoid. Let r_1, r_2, r_3, and r_4 denote the radii of the incircles of AOD, AOB, BOC, and COD, respectively. Prove that $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$.

2.3.8.* Points F_1 and F_2 are foci of an ellipse. Points A and C are the points of intersection of segments BF_1 and BF_2 with this ellipse, respectively. Let D be the intersection point of segments F_1C and F_2A. Prove that $ABCD$ is a tangential quadrilateral.

Suggestions, solutions, and answers

2.3.6. Point M is the center of the circle circumscribed around the triangle BKL.

4. The radical axis (2)
By I. N. Shnurnikov and A. I. Zasorin

2.4.1. Let S be a circle and P be a point. A line through P intersects S at A and B. Prove that the product $|PA| \cdot |PB|$ does not depend on the choice of the line. (See [**Pra19**].)

This product (taken with positive sign if P lies outside the circle and with negative sign if P lies inside) is called the *power of point* P relative to S. (See [**Pra19**].)

[1] *Editor's note:* A convex quadrilateral is called *tangential* if it has an inscribed circle, i.e., a circle that is tangent to all four sides.

2.4.2. Prove that for a point P lying outside circle S, its power with respect to S is equal to the square of the length of the tangent to the circle from this point. (See [**Pra19**].)

2.4.3. Prove that the power of the point P with respect to circle S is equal to $d^2 - R^2$, where R is the radius of circle S and d is the distance from P to the center of the circle. (See [**Pra19**].)

2.4.4. Radical axis. Let S_1 and S_2 be nonconcentric circles. Prove that the geometric locus of points for which the power with respect to S_1 is equal to the power with respect to S_2 is a line. (See [**Pra19**].)

This line is called the *radical axis* of the circles S_1 and S_2.

2.4.5. Prove that the radical axis of two intersecting circles passes through their intersection points. (See [**Pra19**].)

2.4.6. Consider three circles whose centers are not collinear. Prove that the three radical axes of each pair of circles are concurrent. (See [**Pra19**].)

This common point is called the *radical center* of the three circles.

2.4.7. Consider three pairwise intersecting circles. Consider the three lines drawn through the points of intersection of any two of the circles. Prove that these three lines are concurrent or are parallel. (See [**Pra19**].)

2.4.8. Let S_1 and S_2 be nonconcentric circles. Prove that the geometric locus of the centers of circles intersecting both these circles at a right angle consists of the radical axis but omitting their common chord (assuming that S_1 and S_2 intersect).

2.4.9. (a) Prove that the midpoints of the four common tangents to two non-intersecting circles are collinear.

(b) Let AB and CD be common external tangents to two circles (A and D lie on one circle; B and C lie on the other). Prove that the chords formed by the intersection of AC with these two circles have equal length.

2.4.10. Let A_1 and B_1 lie on sides BC and AC of triangle ABC, respectively. Line L passes through the intersection points of the two circles with diameters AA_1 and BB_1. Prove that L

(a) passes through the orthocenter of triangle ABC;

(b) passes through C if and only if $|AB_1| : |AC| = |BA_1| : |BC|$.

2.4.11. The extensions of sides AB and CD of quadrilateral $ABCD$ intersect at point F, and the extensions of sides BC and AD intersect at point E. Prove that the circles with diameters AC, BD, and EF have the same radical axis, and that the orthocenters of triangles ABE, CDE, ADF, and BCF lie on it.

2.4.12.* Three circles intersect pairwise at points A_1 and A_2, B_1 and B_2, and C_1 and C_2. Prove that $|A_1B_2| \cdot |B_1C_2| \cdot |C_1A_2| = |A_2B_1| \cdot |B_2C_1| \cdot |C_2A_1|$.

5. Tangency (2)
By I. N. Shnurnikov and A. I. Zasorin

Problems 2.5.1–2.5.5 are interesting in their own right, but they also serve as helpful lemmas for other problems.

2.5.1. Circle ω_2 is internally tangent to circle ω_1 at D, and is tangent to chord AB of ω_1 at C. Let E be the midpoint of the arc AB that does not contain point D. Prove that C, D, and E are collinear and that $|BE|^2 = |CE| \cdot |DE|$.

2.5.2. Chords AB and CD of circle ω_1 intersect at point P. Circle ω_2 is internally tangent to circle ω_1 at point S and is tangent to segments PB and PD. Let A_0 and C_0 be the midpoints of arcs CD and AB, respectively (the arcs do not contain point S). Let lines l and m be tangent to ω_1 at A_0 and C_0, respectively, and let $Q = l \cap m$. Prove that S, P, and Q are collinear.

2.5.3. The "three hats" theorem. Consider three circles with distinct radii, none lying inside any other. Prove that the intersection points of the three pairs of common external[2] tangents are collinear.[3]

2.5.4. Let ω_1 and ω_2 be non-intersecting circles with centers at points O_1 and O_2, respectively. Let AB and CD be common external and internal tangents, respectively, with A and D lying on ω_1 and B and C lying on ω_2. Let $P = AD \cap BC$. Prove that P lies on the segment O_1O_2.

2.5.5.* Three chords of circle ω intersect pairwise at points A, B, and C. Let ω_A be tangent to rays AC and AB and also internally tangent to ω at A'. Likewise, define circles ω_B and ω_C and points B' and C', respectively. Prove that AA', BB', and CC' are concurrent lines.

2.5.6.* Inscribe non-intersecting circles ω_1 and ω_2 in angle POQ, with ω_1 closer to O. Draw ray OR internally to the angle formed by rays OP and OQ, and let A, B, C, and D be the intersection points of this ray with circles ω_1 and ω_2, in order; i.e., A is closest to O, then B, etc. Draw tangents l_A and m_A to ω_2 from A. Let τ_1 be a circle that is tangent to l_A, m_A, and ω_1. Likewise, draw tangents l_D and m_D from D to ω_1, and let τ_2 be tangent to these two lines and to ω_2. Prove that τ_1 and τ_2 have radii of equal length.

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[2] *Editor's note:* An external tangent of two circles is a common tangent such that the circles lie on the same side of the tangent.

[3] *Editor's note:* This is also known as *Monge's Theorem*.

6. Ptolemy's and Casey's Theorems (3*)
By A. D. Blinkov and A. A. Zaslavsky

6.A. Ptolemy's Theorem

2.6.1. (a) Prove *Ptolemy's inequality*: for any four distinct points A, B, C, and D,

$$|AB| \cdot |CD| + |AD| \cdot |BC| \geq |AC| \cdot |BD|.$$

b) **Ptolemy's Theorem.** This inequality becomes an equality if and only if $ABCD$ is a cyclic quadrilateral.

2.6.2. In an acute triangle ABC, let $|BC| = a$ and $|AC| = b$. Find $|AB|$ in terms of a, b, and the circumradius R.

2.6.3. In triangle ABC, the angle bisector of A intersects the circumcircle at point W. Let I be the incenter.

(a) Express $|AW|/|IW|$ in terms of the sides of the triangle.

(b) Prove that $|AW| > \frac{|AB|+|AC|}{2}$.

2.6.4. Erect a square externally on hypotenuse AB of the right triangle ABC. Let O be its center. Find $|OC|$ in terms of legs a and b of the triangle.

2.6.5. Let ABC be an equilateral triangle and consider a point P.

(a) Prove that if P lies on the circumcircle, then the distance from P to one of the vertices of the triangle is equal to the sum of the distances from P to the other two vertices.

(b) **Pompeiu's Theorem**. If P does not lie on the circumcircle, then a triangle can be constructed of segments of lengths $|PA|$, $|PB|$, $|PC|$.

2.6.6. Let m be the sum of the distances from point X, lying outside a square, to the two nearest neighboring vertices of the square. Find the largest value of the sum of the distances from X to the other two vertices of the square.

2.6.7. Let M and N be the midpoints of diagonals AC and BD of the cyclic quadrilateral $ABCD$. If $\angle ABD = \angle MBC$, prove that $\angle BCA = \angle NCD$. (*Kolmogorov Cup Competition*, 1999.)

2.6.8. Let A, B, C, and D be four consecutive vertices of a regular heptagon. Prove that

(a) $\frac{1}{|AB|} = \frac{1}{|AC|} + \frac{1}{|AD|}$;

(b) $\frac{1}{\sin(\pi/7)} = \frac{1}{\sin(2\pi/7)} + \frac{1}{\sin(3\pi/7)}$.

2.6.9. Let $ABCDEF$ be a convex hexagon with $|AB| = |BC| = a$, $|CD| = |DE| = b$, and $|EF| = |FA| = c$. Prove that $\frac{a}{|BE|} + \frac{b}{|AD|} + \frac{c}{|CF|} \geq \frac{3}{2}$.

2.6.10. Find the length of the diagonals of a cyclic quadrilateral with side lengths a, b, c, d.

2.6.11. Derive Carnot's formula from Ptolemy's Theorem (see section 4 of Chapter 1).

6.B. Casey's Theorem

2.6.12. The generalized Ptolemy's theorem (Casey's Theorem).

(a) Let α, β, γ, δ be boundary circles of non-intersecting discs. Prove that there exists a circle which is externally tangent to all four circles or a line tangent to them so that all four circles belong to the same half-plane with respect to the line if and only if

$$l_{\alpha\beta}l_{\gamma\delta} + l_{\alpha\delta}l_{\beta\gamma} = l_{\alpha\gamma}l_{\beta\delta},$$

where $l_{\alpha\beta}$ denotes the length of the common external tangent to circles α and β, etc.

(b) Formulate an analogue of Casey's Theorem for the case where the required circle is internally tangent to some of the given circles.

2.6.13. Formulate a similar statement for "degenerate" cases, such as when

(a) one of the given circles is a line;

(b) two of the circles are lines;

(c) one of the circles degenerates into a point.

2.6.14. On sides AC and BC of triangle ABC construct points X and Y, respectively, such that $XY \parallel AB$. Prove that there exists a circle passing through X and Y that is tangent to the escribed circles of the triangle that are inscribed in angles A and B, and that the tangency is of the same form (both internally tangent, or both externally tangent).

2.6.15. Prove **Feuerbach's Theorem**, which states that the circle passing through the midpoints of the sides of a triangle is tangent to its inscribed and escribed circles.

2.6.16. Consider three circles, each of which is internally tangent to one of the excircles of a triangle and externally tangent to the other two. Prove that these circles have a common point.

2.6.17. Consider two circles lying one outside the other. An arbitrary circle tangent to them in the same way (either both internally, or both externally) intersects one of their two common internal tangents at points A and A', and intersects the other one at points B and B'. Prove that among the lines AB, AB', $A'B$, and $A'B'$ two of them are parallel to the common external tangents to the given circles.

2.6.18. Let a_1 and a_2 be concentric circles. Circles b_1 and b_2 are externally tangent to a_1 and internally tangent to a_2, and circles c_1 and c_2 are both internally tangent to a_1 and a_2. Suppose that b_1 and b_2 intersect c_1 and c_2 at eight points. Prove that these points lie on two circles or straight lines distinct from b_1, b_2, c_1, c_2. (V. Protasov, *III Olympiad in Honor of I. F. Sharygin.*)

Suggestions, solutions, and answers

2.6.1. Apply the inversion with center A and use Problems 3.10.2 and 3.10.5 in Chapter 3. Note that Ptolemy's inequality holds even for points that are not in the same plane.

2.6.2. *Answer*: $\frac{a\sqrt{4R^2-b^2}+b\sqrt{4R^2-a^2}}{2R}$.

Draw diameter CD and apply Ptolemy's Theorem to the resulting quadrilateral.

Note that it is easy to derive trigonometric addition relations from this formula.

2.6.3. (a) Since W is the center of the circle circumscribed around triangle IBC, applying Ptolemy's Theorem to quadrilateral $ABWC$ gives

$$\frac{|AW|}{|IW|} = \frac{b+c}{a}.$$

(b) Use $|BW| + |CW| > |BC|$, $|BW| = |CW| = |IW| > |BC|/2$, and (a) above.

2.6.4. *Answer*: $\frac{a+b}{\sqrt{2}}$.

Since $\angle ABC = \angle AOC = 90°$, $AOBC$ is cyclic. Apply Ptolemy's Theorem.

2.6.5. Use Ptolemy's Theorem.

2.6.6. *Answer*: $m(\sqrt{2}+1)$.

Let $ABCD$ be a square with side length a and let X be a point outside this square such that $|XA|+|XB| = m$. Consider the diagonals of the square, and use Ptolemy's inequalities for the (possibly self-intersecting) quadrilaterals $XADB$ and $XBCA$ to get $|XC| + |XD| \leq m(\sqrt{2}+1)$, with equality achieved if and only if X lies on the arc AB of the circle circumscribed about the square.

2.6.7. Both conditions are equivalent to $|AB| \cdot |CD| = |AD| \cdot |BC|$.

2.6.8. (a) Circumscribe a circle around regular heptagon $ABCDEFG$ and apply Ptolemy's Theorem to quadrilateral $ACDE$.

(b) The statement follows from part (a) and the law of sines.

2.6.9. Let $|AC| = x$, $|CE| = y$, and $|AE| = z$. Applying Ptolemy's inequality to quadrilaterals $ABCE$, $ACDE$, and $ACEF$ and summing up the resulting inequalities, we get

$$\frac{a}{|BE|} + \frac{b}{|AD|} + \frac{c}{|CF|} \geq \frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y}.$$

Using the arithmetic-harmonic mean inequality for three positive numbers, we see that the right-hand side is not less than $\frac{3}{2}$. Equality is achieved only for a regular hexagon.

2.6.10. *Answer*: $\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$ and $\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$.

Apply Ptolemy's Theorem to the given cyclic quadrilateral and the two others derived from it by rearranging the sides.

2.6.11. Let M_1, M_2, M_3 be the midpoints of BC, CA, AB. Applying Ptolemy's Theorem to cyclic quadrilaterals OM_1CM_2, OM_2AM_3, OM_3BM_1 and summing up the obtained equalities, we get

$$Rp = d_1\frac{b+c}{2} + d_2\frac{c+a}{2} + d_3\frac{a+b}{2},$$

where p is the semiperimeter of the triangle. Since $\frac{ad_1+bd_2+cd_3}{2} = S_{ABC} = pr$, this equality is equivalent to the required one.

2.6.12. We prove that the equality is necessary.

Let O and R be the center and radius of the required circle, and let O_a and O_b be the centers of circles α and β with radii r_a and r_b, respectively. Let A, B, C, D be points of tangency of the circle with α, β, γ, δ, respectively. Applying the law of cosines to the (possibly degenerate) triangles OO_aO_b and OAB, we get

$$\begin{aligned}|O_aO_b|^2 &= (R+r_a)^2 + (R+r_b)^2 - 2(R+r_a)(R+r_b)\cos\angle AOB \\ &= (R+r_a)^2 + (R+r_b)^2 - 2(R+r_a)(R+r_b)\frac{2R^2 - |AB|^2}{2R^2}.\end{aligned}$$

Therefore, the length of the common external tangent to circles α and β is equal to

$$\sqrt{|O_aO_b|^2 - (r_a - r_b)^2} = \frac{|AB|}{R}\sqrt{(R+r_a)(R+r_b)}.$$

Similarly, we find the lengths of the common tangents to the remaining pairs of circles. Applying Ptolemy's Theorem to quadrilateral $ABCD$ yields the required equality.

Sufficiency is obtained with a routine proof by contradiction. Let circles α, β, γ, δ satisfy the equality in the theorem. Construct circle ω tangent to α, β, γ, and construct circle δ' also tangent to ω such that the lengths of its common tangents with circles α and β are the same as those of circle δ. Then, by the above, the common tangents with γ, δ, and δ' are also equal, from which it is easy to see that these circles coincide.

(b) If α and β lie on opposite sides of the required circle or line, then their common external tangent $l_{\alpha\beta}$ must be replaced with an internal one.

2.6.13. (a) If α degenerates into a line, then the common outer (inner) tangent $l_{\alpha\beta}$ must be replaced by $\sqrt{d}$, where d is the distance from the farthest (closest) point of circle β to the line α.

(b) If circles α and β degenerate into nonparallel lines, then the common outer (inner) tangent $l_{\alpha\beta}$ must be replaced by $\cos\frac{\varphi}{2}$, where φ is the acute (obtuse) angle between these lines. The details are left to the reader.

(c) If the circle α degenerates into a point, then $l_{\alpha\beta}$ must be replaced with a tangent from this point to circle β.

2.6.14. The statement directly follows from Casey's Theorem applied to points X and Y and the escribed circles.

2.6.15. Apply Casey's Theorem to the midpoints of the sides and the inscribed (escribed) circle.

2.6.16. Apply Casey's Theorem to the three escribed circles and their radical center.

2.6.17. Fix a point A on one of the internal tangents. Through A one can draw two circles that are tangent to the given ones in the same way (internal or external). On the other hand, if we draw a line through A parallel to one of external tangents and intersecting the internal tangent at point B, then by Casey's Theorem (see Problem 2.6.3) there exists a circle passing through A and B and tangent to the given ones.

2.6.18. Perform an inversion with center at the intersection point of c_1 and c_2 and use the previous problem.

Chapter 3

Geometric transformations

In this chapter we introduce the concept of geometric transformations by using them to solve interesting problems that seemingly have nothing to do with transformations. Only then, once the idea of transformations is motivated, do we investigate properties of transformations on their own.

1. Applications of transformations (1)
By A. D. Blinkov

A *rotation* around O by an angle φ is a transformation of the plane that leaves O in place and maps any point X other than O to the point X' such that $|OX| = |OX'|$ and the oriented angle between the vectors $\overrightarrow{OX}$ and $\overrightarrow{OX'}$ is equal to φ. This informal definition can be used in solutions of the problems; a rigorous definition is given in the solution of Problem 3.2.8 (a). A rotation by 180° is called a *central symmetry* (about O).

A *translation* by the vector $\vec{m}$ is a transformation (about O) mapping any point X to the point X' such that $\overrightarrow{XX'} = \vec{m}$.

An *axial symmetry* (also called a *reflection*) with respect to a line l is a transformation mapping any point X to the point X' such that $XX' \perp l$ and l bisects the segment XX'. A *glide-reflectional symmetry* is the composition of an axial symmetry and translation by a vector parallel to the axis of the symmetry.

3.1.1.° A parallelogram has exactly four axes of symmetry. Which of the following statements are correct?

(a) It is a rectangle other than a square;
(b) it is a rhombus but not a square;
(c) it is a square;
(d) there is no such parallelogram.

3.1.2.° A triangle has a center of symmetry. Which of the following statements is true?

(a) It is scalene;
(b) it is equilateral;

(c) it is isosceles;
(d) there is no such triangle.

3.1.3.$^\circ$ In a regular n-gon there are two axes of symmetry intersecting at an angle of $20°$. What is the smallest possible value for n?
(a) 6;
(b) 9;
(c) 12;
(d) 18.

3.1.4.$^\circ$ Given two points $A(1,0)$ and $B(-1,2)$, find the coordinates of
(a) the image of B under reflection about the x-axis;
(b) the image of A under reflection about point B;
(c) the image of A under the translation that takes B to A;
(d) the image of B under rotation with center A by $-90°$.

3.1.5.$^\circ$ Let ABC be an equilateral triangle with area S and center O. Find the area of the intersection of this triangle with its image
(a) under translation by $\overrightarrow{OC}$;
(b) under reflection about point O;
(c) under reflection about the line containing the triangle's midline;
(d) under rotation by $120°$ about O.

3.1.6.$^\circ$ Let ABC be an isosceles right triangle with equal sides AC and BC. Let K and M lie on AC and BC, respectively, such that $|AK| = |CM|$. Let D be the midpoint of AB. Find $\angle KDM$.

3.1.7. Prove that the lines drawn through the midpoints of the sides of a cyclic quadrilateral perpendicular to their opposite sides are concurrent. (See [**Pra95**].)

3.1.8. Let M lie on the diameter AB of a circle. Chord CD passes through M at an angle of $45°$ to AB. Prove that $|CM|^2 + |DM|^2$ does not depend on the choice of M. (See [**Pra95**].)

3.1.9. Let $ABCD$ be a square with side length a. Let M and N lie on BC and CD, respectively, such that the area of triangle AMN is equal to the sum of the areas of triangles ABM and ADN. Find $\angle MAN$ and the length of the height of triangle AMN dropped from vertex A. (See [**Got96a**].)

3.1.10. Let P and Q lie on legs a and b of a right triangle, and let perpendiculars PK and QH be dropped to the hypotenuse. Find the smallest possible value of $|KP| + |PQ| + |QH|$. (See [**VGRT87**].)

3.1.11. Let $ABCD$ be a trapezoid with bases AD and BC, and let E be the intersection of its diagonals. Prove that if the trapezoid is tangential, then $\angle AED$ is obtuse. (See [**MS17**].)

3.1.12. Let $ABCD$ be a parallelogram with heights BK and BL, and let $|KL| = a$ and $|BD| = b$. Find the distance from B to the orthocenter of triangle BKL. (See [**Pra95**].)

3.1.13. An equilateral triangle ABC is inscribed in a circle. Let M be a point on this circle, and let a and b be the distances from A and B to M, respectively. Find the length of CM. (See [**Pra95**].)

3.1.14. Let CM and CH be a median and a height, respectively, of triangle ABC. Consider lines drawn through an arbitrary point P in the plane of the triangle that are perpendicular to CA, CM, and CB. These lines intersect the line CH at A', M', and B'. Prove that $|A'M'| = |B'M'|$. (See [**Pra95**].)

Suggestions, solutions, and answers

3.1.6. *Answer*: $90°$.

Rotation by $90°$ around D sends A to C and C to B. Consequently, this rotation takes K to M.

3.1.7. Let $ABCD$ be a quadrilateral inscribed in a circle with center O, and let E, F, G, and H be the midpoints of its sides (see Fig. 1). Then $OH \perp AD$. Since $EFGH$ is a parallelogram, $M = EG \cap FH$ is the midpoint of each of its diagonals.

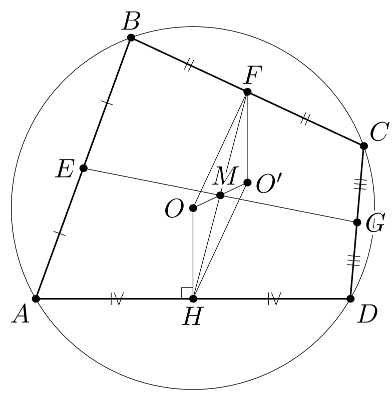


FIGURE 1

Let $O' = Z_M(O)$ (i.e., it is a point centrally symmetric to point O with respect to M). Then $F = Z_M(H)$, so $O'FOH$ is a parallelogram, which means that $O'F \perp AD$. Considering the perpendiculars OG, OF, and OE to the other sides of the quadrilateral $ABCD$ and their images under central symmetry with respect to M, we see that they also pass through point O'.

3.1.8. Let C' be the image of C under reflection about AB (see Fig. 2). Then $\angle CMC' = 90°$, so triangle DMC' is a right triangle, and we get $|CM|^2 + |DM|^2 = |C'M|^2 + |DM|^2 = |DC'|^2$.

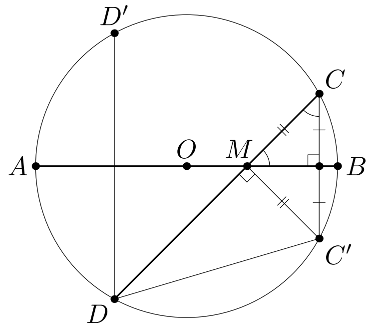


FIGURE 2

Regardless of the position of point M, arc $C'D$ corresponds to the inscribed angle $C'CD$ with measure $45°$, so $|DC'|^2$ does not depend on the choice of point M. Therefore, $|CM|^2 + |DM|^2$ also does not depend on the choice of point M.

Of course, the problem can be solved directly with the Pythagorean Theorem.

3.1.9. *Answer*: $\angle MAN = 45°$; length of the height is equal to a.

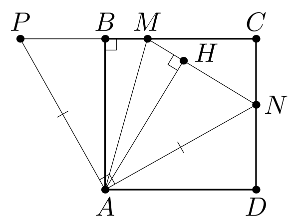


FIGURE 3

Consider the rotation with center A by $90°$: the image of N is the point P lying on ray CB (see Fig. 3). The statement of the problem implies $S_{MAN} = S_{ABM} + S_{ADN} = S_{ABM} + S_{ABP} = S_{MAP}$. Since triangles MAN and MAP share the common side AM and $|AN| = |AP|$, we get $\sin \angle PAM = \sin \angle MAN$.

Since these triangles are acute and $\angle PAN = 90°$, we have $\angle MAN = \angle PAM = 45°$. Thus $\triangle PAM = \triangle NAM$ (by SAS), and so their respective heights are equal: $|AH| = |AB| = a$.

3.1.10. *Answer*: $\frac{2ab}{\sqrt{a^2+b^2}}$.

Consider triangle ABC, in which $\angle C = 90°$; $|BC| = a$, $|AC| = b$; $P \in [BC]$, $PK \perp AB$, $K \in [AB]$; and $Q \in [AC]$, $QH \perp AB$, $H \in [AB]$ (see Fig. 4). Consider the symmetry with respect to BC. This sends A and K to A' and K', respectively. Similarly, symmetry with respect to AC

sends B and H respectively to B' and H'. Thus $AB'A'B$ is a rhombus and $|KP| + |PQ| + |QH| = |K'P| + |PQ| + |QH'|$.

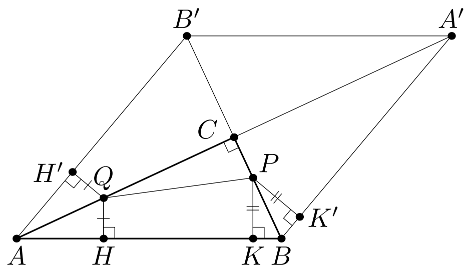


FIGURE 4

Since points K' and H' lie on parallel lines, and since $[PK']\perp[A'B]$ and $[QH']\perp[AB']$, the smallest value of the sum is reached if and only if K', P, Q, and H' belong to the common perpendicular to AB' and BA'. The requested value of the sum in this case is equal to the distance between these lines, i.e., the length of the height of the rhombus or the double height of the triangle ABC, and therefore is equal to $\frac{2ab}{\sqrt{a^2+b^2}}$.

3.1.11. Consider trapezoid $ABCD$. Under the translation by $\overrightarrow{BC}$, the image of the diagonal BD is CK, and $|BD| = |CK|$ and $\angle AED = \angle ACK$ (see Fig. 5). The image of side AB under this translation is CP; therefore $|AB| = |CP|$.

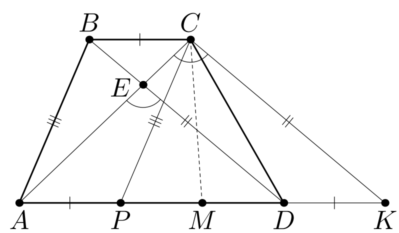


FIGURE 5

Draw CM, the median of triangle ACK. Since $|AP| = |BC| = |DK|$, CM is also the median of triangle PCD. Consequently,

$$|CM| < \frac{|CP| + |CD|}{2} = \frac{|AB| + |CD|}{2} = \frac{|AD| + |BC|}{2} = \frac{|AK|}{2},$$

since the trapezoid is tangential.

Consider a circle with diameter AK. The point M is its center and, since $|CM| < \frac{|AK|}{2}$, the point C lies inside this circle. Thus $\angle ACK$ is obtuse, so $\angle AED$ is obtuse.

Another solution can be obtained by noting that the circles with diameters AB and CD are tangent at the center of the inscribed circle.

3.1.12. *Answer*: $\sqrt{b^2 - a^2}$.

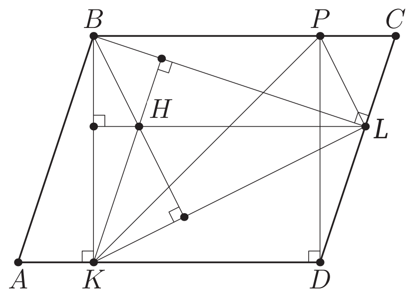


FIGURE 6

Let H be the orthocenter of triangle BKL (see Fig. 6). Since $HL \perp BK$ and $KH \perp BL$, we see that $HL \parallel AD$ and $KH \parallel CD$; i.e., $LHKD$ is a parallelogram.

Consider the translation by the vector $\overrightarrow{HL}$. This takes K to D and takes B to a point P lying on BC. Since $PD \parallel BK$, $BPDK$ is a rectangle, and $|PK| = |BD| = b$. Furthermore, $BH \perp KL$ implies $PL \perp KL$ and $|PL| = |BH|$ (by the definition of translation). Thus, $\triangle PKL$ is a right triangle, so $|BH| = |PL| = \sqrt{|KP|^2 - |KL|^2} = \sqrt{b^2 - a^2}$.

3.1.13. *Answer*: $a + b$ if M lies on the arc AB not containing C; otherwise $|a - b|$.

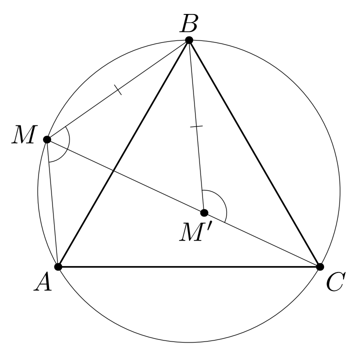


FIGURE 7

Let M lie on the arc from A to B that does not contain C (see Fig. 7). Consider the rotation with center B by $60°$, which takes A to C. If M' is the image of M under this rotation, then $\angle BM'C = \angle BMA = 120°$.

Since $\angle MBM' = 60°$ and $|BM'| = |BM| = b$, triangle MBM' is equilateral, and $\angle BM'M = 60°$. The fact that $\angle BM'C + \angle BM'M = 180°$ implies that M' lies on CM, which implies that

$$|CM| = |CM'| + |M'M| = |AM| + |BM| = a + b.$$

Similarly, if M lies on arc AC not containing B, then $b = a + |CM|$, and if M lies on arc BC not containing B, then $a = b + |CM|$. The first of these cases is possible if $b > a$, and the second is possible if $b < a$.

3.1.14. Consider the rotation $R_P^{90^\circ}$ with center P by 90° (see Fig. 8). This rotation takes lines PA', PB', and $A'B'$ to lines parallel to CA, CB, and AB, respectively. Therefore, the image of triangle $PA'B'$ is triangle $PA''B''$, the sides of which are respectively parallel to the sides of the given triangle. Thus $\triangle PA''B'' \sim \triangle CAB$, and if PM'' is the median of $PA''B''$, then $PM'' \parallel CM$, since the similarity transformation preserves the proportionality of segments and the magnitude of the angles. Since $PM'' = R_P^{90^\circ} PM'$, M' lies at the middle of segment $A'B'$.

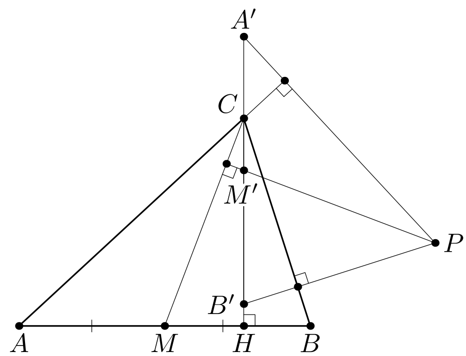


FIGURE 8

Note that the center of rotation can be taken to be any point in the plane, not necessarily P.

2. Classification of isometries of the plane (2) *By A. B. Skopenkov*

3.2.1. Let l and m be lines, $\vec{a}$ be a vector, and φ and θ be angles. Let S_l be symmetry (reflection) with respect to line l, let $T_{\vec{a}}$ be translation by vector $\vec{a}$, and let R_A^φ be rotation around A by the angle φ. Find

(a) $S_l \circ S_l$; (b) $S_l \circ S_m$ for $l \parallel m$, $l \neq m$;

(c) $S_l \circ S_m$ for $l \cap m \neq \emptyset$; (d) $S_l \circ T_{\vec{a}}$; (e) $S_l \circ R_A^\varphi$; (f) $R_A^\varphi \circ R_B^\theta$.

3.2.2. An *isometry* of a line is a mapping of the line into itself that preserves distances between points. A *motion* of a line is either a translation or a symmetry. (For the line, symmetry can only be defined with respect to a point. It is an analogue of both the central and the axial symmetries of the plane.)

Translations by different vectors and symmetries with different centers are pairwise different.

An *isometry* of a plane is a mapping of the plane into itself that preserves distances. Later in the book, isometries will be called *rigid motions*. Isometries were used to find the solutions of problems in section 1 of this chapter; other examples from elementary geometry can be found in [**Pra95**, Ch. 15–18].

3.2.3. In the plane, rigid motions preserve

(a) angles; (b) areas of polygons;

(c) straight lines (i.e., they transform straight lines into straight lines);

(d) parallelism of straight lines; (e) circles.

3.2.4. Let A, B, C be non-collinear points in a plane, and let f and g be rigid motions.

(a) If $f(A) = A, f(B) = B$, and $f(C) = C$, then $f = 1$ (the identity transformation).

(b) If $f(A) = g(A), f(B) = g(B)$, and $f(C) = g(C)$, then $f = g$.

3.2.5. (a) Any isometry of a plane can be decomposed into a composition of an axial symmetry and an isometry having a fixed point.

(b) Any isometry of a plane with a fixed point can be decomposed into a composition of an axial symmetry and an isometry having at least two fixed points.

(c) **Chasles' Theorem.** Any rigid motion of a plane is either a rotation, a translation, or a glide-reflectional symmetry.

A rigid motion of a plane is called *proper* if it can be decomposed into a composition of an even number of axial symmetries. Informally speaking, an isometry is proper if it "preserves orientation."

Caution: this informal interpretation cannot be used (without formalization) in a proof.

3.2.6. Which of the following transformations are proper isometries?

(a) Translation; (b) rotation; (c)* axial symmetry.

3.2.7. (a) The composition of two proper isometries is a proper isometry.

(b) The composition of a proper and an improper isometry (in any order) is an improper isometry.

(c) The composition of two improper isometries is a proper isometry.

3.2.8. (a) If an isometry is proper, then *any* of its decompositions into a composition of axial symmetries contains an even number of symmetries.

(b) Each proper isometry of the plane is either a rotation or a translation.

Hints

3.2.1. (d) Decompose the translation vector into the sum of vectors that are parallel and perpendicular to l: $\vec{a} = \vec{a}_{\|} + \vec{a}_{\perp}$. Represent the translation by $\vec{a}_{\perp}$ in the form $T_{\vec{a}_{\perp}} = S_l \circ S_{l'}$.

(e) Represent the rotation in the form $R_A^{\alpha} = S_{l'} \circ S_m$, where $l' \parallel l$.

(f) Represent the rotations in the forms $R_A^{\alpha} = S_n \circ S_l$ and $R_B^{\alpha} = S_l \circ S_m$.

3.2.6. (c) *Answer*: No. This is equivalent to the statement 3.2.8 (a).

3.2.8. (a) Let us define a *rotation* to be the composition of two axial symmetries with intersecting axes. (This definition is necessary because the usual definition employs a visual, non-rigorous representation of the direction of rotation and the orientation of angles via a "clock" metaphor.)

The composition of any two axial symmetries is a rotation or translation. It is clear that the composition of translations is a translation. Similar to 3.2.1 (f), the composition of two rotations is a rotation or a translation. Likewise, the composition of a rotation and a translation is a rotation. Similar to 3.2.1 (d, e), the composition of an axial symmetry with a rotation or with a translation is not the identity transformation.

Therefore, the composition of an *odd* number of axial symmetries with a rotation or a translation is not the identity transformation. This implies the required assertion.

3. Classification of isometries of space (3*)
By A. B. Skopenkov

Symmetry (or *reflection*) with respect to a plane is defined literally as symmetry with respect to a line with the word "line" replaced by the word "plane."

3.3.1. In addition to the notation introduced in Problem 3.2.1, we denote planes by α and β and let S_α denote the symmetry with respect to α. Find the compositions

(a) $S_\alpha \circ S_\alpha$; (b) $S_\alpha \circ S_\beta$ when $\alpha \parallel \beta$, $\alpha \neq \beta$;

(c) $S_\alpha \circ S_\beta$ when $\alpha \cap \beta \neq \emptyset$; (d) $S_l \circ S_\alpha$ when $l \perp \alpha$.

A *rotation* of space about the directed line l by the angle α is a map R_l^α of space that sends each point X to the point X' defined as follows. Take the projection O of a point X onto l (possibly $O = X$). Construct the plane π that passes through O and is orthogonal to the line l. Use a positive direction for angles in the plane π consistent with the direction of line l according to the *right-hand rule*. Take the unique point $X' \in \pi$ for which $|OX| = |OX'|$ and the oriented angle $X'OX$ in the π plane is α. Informally, we are rotating about the axis l by the angle α in the counterclockwise direction when viewed from "above" in the positive direction of l.

The composition of a rotation and a mirror symmetry with respect to a plane perpendicular to the axis of rotation (or, equivalently, the composition of a rotation and a central symmetry with respect to a point lying on the axis of rotation) is called a *rotational symmetry* of space.

3.3.2. Using the notation of Problem 3.2.1, represent the following transformations as rotations or rotational symmetries:

(a) $R_l^\alpha \circ R_m^\beta$ for $l \cap m \neq \emptyset$; (b) $R_l^\alpha \circ T_{\vec{a}}$;

(c)* $S_l \circ S_m$ for skew lines l and m;

(d) a composition of symmetries with respect to three planes having a common point.

An *isometry* of space is a mapping of space into itself that preserves distances.

3.3.3. Spatial isometries preserve

(a) angles; (b) area of polygons;

(c)* volumes of polyhedra; (d) planes; (e) straight lines;

(f) half-spaces; (g) spheres; (h) circles;

(i) parallelism.

An isometry of space is called *proper* if it can be decomposed into a composition of an even number of mirror symmetries.

3.3.4. Which of the following transformations are proper isometries?

(a) Mirror symmetry;

(b) axial symmetry of space;

(c) central symmetry of space.

(Compare with the transformations of lines and planes discussed above.)

3.3.5. If an isometry is proper, then any decomposition of it into a composition of mirror symmetries contains an even number of symmetries.

3.3.6. (a) **Theorem.** Any proper isometry of space with a fixed point is a rotation.

(b) (Challenge.) Any *proper isometry of a sphere* (determine by yourself what this means) is a rotation.

(c)* **Theorem.** Any proper isometry of space is a *screw motion*, i.e., a composition of rotation and translation such that the rotation axis is aligned with the vector of translation.

3.3.7. (a) **Theorem.** Any isometry of space with a fixed point is a rotation or a rotational symmetry.

(b)* (Challenge.) **Theorem.** Any isometry of space is a screw motion, rotational symmetry, or glide-reflectional symmetry (determine by yourself what these mean).

For more information about the classification of isometriess see also [**Sol80**]. A somewhat related topic is discused in [**SS85**].

Hints

3.3.6. (a) This isometry decomposes into a composition of two rotations with intersecting axes. In other words, if $f(O) = O$ and $f(A) = A'$ does not lie on the line AO, then $f = R_l^{\angle A'OA}$, where $O \in l \perp AOA'$.

(c) First prove that this isometry decomposes into a composition of a translation and a proper isometry with a fixed point.

4. An application of similarity and homothety (1)
By A. D. Blinkov

A transformation (of a line, plane, or space) that maps each point X to a point X' such that $\vec{OX'} = k\vec{OX}$ is called the *homothety H_O^k with center O and coefficient k.*

3.4.1.° Given two points $A(1, 0)$ and $B(-1, 2)$, find the coordinates of

(a) the image of point B under the homothety with center A and coefficient $k = -0.5$;

(b) the image of point A under a rotational homothety (i.e., the composition of a rotation and a homothety with common center) with center at the origin, coefficient $k = 2\sqrt{2}$, and an angle of 45°.

3.4.2.° The lengths of the bases of a trapezoid are 6 cm and 12 cm. Find the length of the segment with endpoints on its legs that is parallel to the bases of the trapezoid and passes through the intersection point of its diagonals.

3.4.3.° Let K be a point on side CD of square $ABCD$ such that $|CK| : |DK| = 1 : 2$. Erect a smaller square whose side is KD that is exterior to $ABCD$. Specify the center, coefficient, and angle of the rotational homothety (i.e., the composition of a rotation and a homothety with common center) which transforms the larger square into the smaller one.

3.4.4.° A rectangle has side lengths 2 and 5. A line parallel to a side of this rectangle divides it into two similar non-congruent rectangles. Find the dimensions of these two rectangles.

3.4.5.° Let ABC be a right triangle, with height CD drawn from the vertex of the right angle. The inradii of triangles ADC and BDC are r_1 and r_2, respectively. Find the inradius of triangle ABC.

3.4.6. Point K lies on diagonal BD of parallelogram $ABCD$. Line AK intersects lines BC and CD at L and M, respectively. Prove that $|AK|^2 = |LK| \cdot |KM|$. (See [**Pra95**].)

3.4.7. On side BC of triangle ABC, construct a point M such that the line passing through the feet of the perpendiculars dropped from M to AB and AC is parallel to BC. (See [**Pra95**].)

3.4.8. Two circles are tangent internally at point M. Chord AB of the larger circle is tangent to the smaller circle at point P. Prove that MP bisects angle AMB. (See [**Pra95**].)

3.4.9. In trapezoid $ABCD$ with bases AD and BC, angle A is a right angle, E is the intersection point of the diagonals, and F is the foot of the perpendicular dropped from E to side AB. Find $\angle CFE$ in terms of $\angle DFE = \alpha$.

3.4.10. Trapezoids $ABCD$ and $ADPQ$ lie in different half-planes with respect to their common base AD. Prove that the intersection points of lines AB and CD, AP and DQ, and BP and CQ are collinear. (See Problem 19.25 (a) from [**Pra95**].)

3.4.11. Three similar triangles exterior to triangle ABC are constructed: $\triangle A'BC \sim \triangle B'CA \sim \triangle C'AB$. Prove that the centroids of triangles ABC and $A'B'C'$ coincide. (See [**Pra95**].)

3.4.12. Let E lie inside square $ABCD$. Let ET be the height of triangle ABE, let K be the intersection point of DT and AE, and let M be the intersection point of CT and BE. Prove that segment KM is the side of a square inscribed in triangle ABE. (See [**FKBKY08**].)

3.4.13. Let ABC be an acute triangle. Find the locus of the centers of rectangles $PQRS$ such that P and Q lie on side AC and R and S lie on sides AB and BC, respectively. (See [**Pra95**].)

3.4.14. (a) The radius of the circumscribed sphere of a tetrahedron is not less than three times the radius of the inscribed sphere.

(b) Given two concentric spheres and a point X on the outer sphere, draw a line passing through X, such that the spheres cut three equal segments from this line.

(*Hint.* Start with the two-dimensional analogue.)

3.4.15. Find $H_A^k \circ H_B^l$ on (a) a line; (b) a plane.

3.4.16. Points A and B move along two intersecting lines with constant but unequal speeds V_A and V_B, respectively. Prove that there exists a point O such that at any instant of time $|AO| : |BO| = V_A : V_B$, and describe how to construct it (cf. Problem 3.5.2).

Suggestions, solutions, and answers

3.4.6. Since $\triangle ABK \sim \triangle MDK$ (equal angles), we have $\frac{|AK|}{|KM|} = \frac{|BK|}{|KD|}$ (see Fig. 9). Likewise, $\triangle ADK \sim \triangle LBK$, so $\frac{|AK|}{|KL|} = \frac{|KD|}{|BK|}$. Multiplying yields $\frac{|AK|^2}{|KM|\cdot|KL|} = 1 \iff |AK|^2 = |LK| \cdot |KM|$.

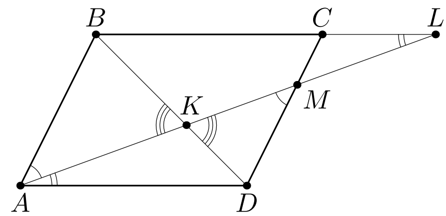


FIGURE 9

3.4.7. We have $MK \perp AB$, $K \in AB$ and $ML \perp AC$, $L \in AC$ (see Fig. 10).

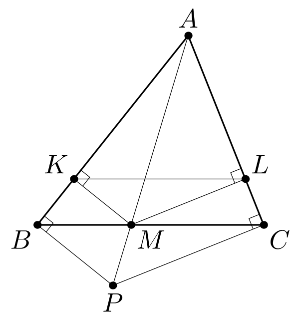


FIGURE 10

First method. Consider the homothety with center A that takes K to B. Since $KL \parallel BC$, this homothety takes L to C. Let $P = H_A(M)$. Then $PB \perp AB$ and $PC \perp AC$.

Thus, it suffices to construct perpendiculars to sides AB and AC at points B and C, respectively, and connect their intersection point P with A. Then $M = AP \cap BC$.

Second method. Draw perpendiculars to sides AB and AC from vertices B and C respectively, and let their intersection be P. Since $\angle ABP + \angle ACP = 180^\circ$, quadrilateral $BACP$ is cyclic, so $\angle APC = \angle ABC = \beta$ and $\angle APB = \angle ACB = \gamma$. Therefore, $\angle BAP = 90^\circ - \gamma$ and $\angle CAP = 90^\circ - \beta$.

Thus $M = AP \cap BC$ because the perpendiculars drawn from M to sides AB and AC form triangles AKM and ALM that are similar to ABP and ACP, respectively. The construction reduces to attaching one of the found angles (with the side AP) to an appropriate half-plane.

The problem has a solution if the point P lies between the sides of the angle BAC i.e., if the angles B and C of this triangle are acute.

3.4.8. Let the radii of the given circles be R and r. Then the image of the larger circle under the homothety H_M^k with center M and coefficient $k = \frac{r}{R}$ is the smaller circle. Let $H_M^k(A) = A'$ and $H_M^k(B) = B'$. Then $H_M^k(AB) = A'B'$, where $A'B'$ is a chord of the smaller circle parallel to AB. This implies (considering the inscribed angles and angles between the tangent and the chord) that $\angle AMP = \angle A'B'P = \angle B'PB = \angle BMP$. Therefore, MP bisects angle AMB.

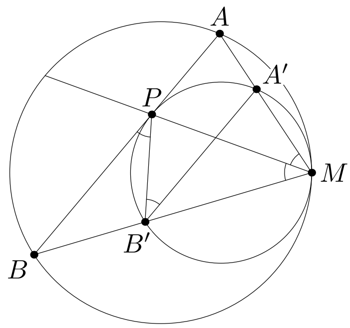


FIGURE 11

3.4.9. *Answer*: α.

Solution. First method. Construct trapezoid ABC_1D_1 symmetric to the given one with respect to line AB (see Fig. 12). Point $E_1 = [AC_1] \cap [BD_1]$ is symmetric to E. In any trapezoid the ratio of the distances from the intersection point of the diagonals to the bases is equal to the ratio of the lengths of these bases. The respective distances from points E and F to the bases BC and AD are equal, and $|BC| : |AD| = |C_1C| : |D_1D|$. Thus F is the intersection point of the diagonals of the trapezoid D_1DCC_1, so $\angle CFE = \angle D_1FE_1 = \angle DFE = \alpha$.

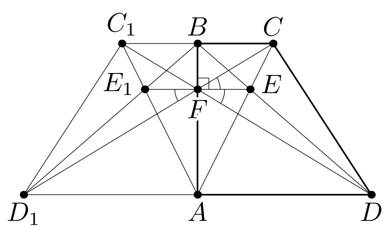


FIGURE 12

Second method. Since triangles EBC and EDA are similar and the respective distances from points E and F to the bases BC and AD are equal, we have $|BC| : |AD| = |BF| : |AF|$. Therefore, triangles CBF and DAF are also similar; so $\angle CFE = \angle BCF = \angle ADF = \angle DFE = \alpha$.

3.4.11. Let P be a rotational homothety (defined in Problem 3.4.1) that takes $\overrightarrow{CB}$ to $\overrightarrow{CA'}$ (see Fig. 13). Similar triangles yield $P(\overrightarrow{BA}) = \overrightarrow{BC'}$ and $P(\overrightarrow{AC}) = \overrightarrow{AB'}$.

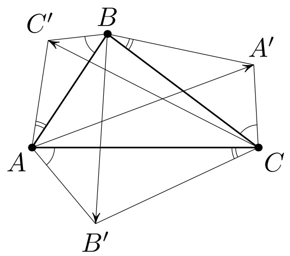


FIGURE 13

Therefore,

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{AC} + \overrightarrow{CA'} + \overrightarrow{BA} + \overrightarrow{AB'} + \overrightarrow{CB} + \overrightarrow{BC'}$$
$$= \big(\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BA}\big) + \big(P(\overrightarrow{CB}\,) + P(\overrightarrow{AC}) + P(\overrightarrow{BA})\big) = \vec{0}.$$

Let M be the centroid of triangle ABC. Then $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = \vec{0}$. Consequently,

$$\overrightarrow{MA'} + \overrightarrow{MB'} + \overrightarrow{MC'} = \overrightarrow{MA} + \overrightarrow{AA'} + \overrightarrow{MB} + \overrightarrow{BB'} + \overrightarrow{MC} + \overrightarrow{CC'} = \vec{0};$$

that is, M is the centroid of triangle $A'B'C'$.

3.4.12. Let $|ET| = h$, and let a be the side length of square $ABCD$. Since $ADK \sim ETK$ (see Fig. 14 a) we have $\frac{|AK|}{|EK|} = \frac{|DK|}{TK} = \frac{a}{h}$. Likewise, $BCM \sim ETM$ implies that $\frac{|BM|}{|EM|} = \frac{a}{h}$. From $\frac{|AK|}{|EK|} = \frac{|DK|}{|TK|} = \frac{|BM|}{|EM|} = \frac{a}{h}$ it follows that $\frac{|AE|}{|KE|} = \frac{|DT|}{|KT|} = \frac{|BE|}{|ME|} = \frac{a+h}{h}$. Since E is the common angle of triangles AEB and KEM and $\frac{|AE|}{|KE|} = \frac{|BE|}{|ME|} = \frac{a+h}{h}$, we have $AEB \sim KEM$. Therefore $\frac{|AB|}{|KM|} = \frac{a+h}{h}$ and $\angle BAE = \angle MKE$, so KM and AB are parallel (see Fig. 14 b); moreover, $|KM| = \frac{ah}{a+h}$. Drawing perpendiculars KK_1 and MM_1 to AB (see Fig. 14 c), we obtain rectangle KK_1M_1M, which is inscribed in triangle ABE. Since $DAT \sim KK_1T$, it follows that $\frac{|DA|}{|KK_1|} = \frac{|DT|}{|KT|}$, that is, $|KK_1| = \frac{ah}{a+h} = |KM|$. Therefore KK_1M_1M is a square.

3.4.13. *Answer*: Segment OH (excluding the endpoints O and H), where O is the midpoint of the height drawn from vertex B and H is the midpoint of AC.

Solution. Let O be the midpoint of height $[BM]$ and let H be the midpoint of $[AC]$. Let D and E be the midpoints of RQ and PS, respectively (see Fig. 15). Since $RQ \parallel BM$ and $A = BR \cap HQ$, there is a homothety

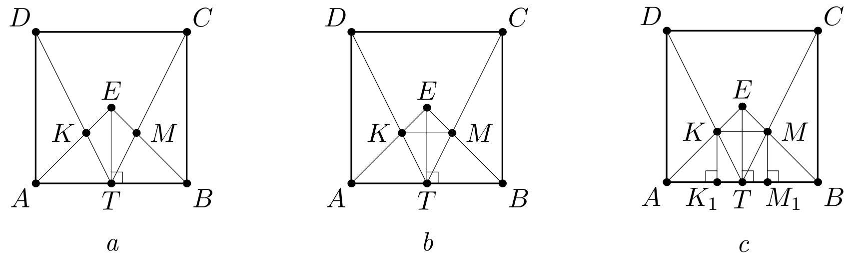


FIGURE 14

with center A which takes RQ to BM. Therefore this homothety takes D to O, so $D \in OA$. Similarly, we can show that $E \in OC$.

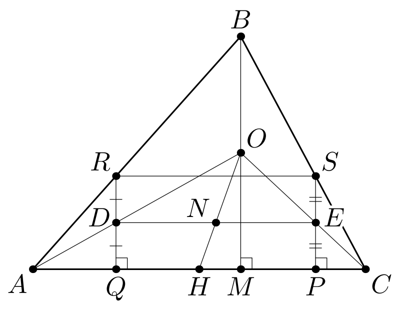


FIGURE 15

Let N be the center of the rectangle $PQRS$. It is also the midpoint of segment DE. Since there is a homothety with center O which takes DE to AC, we see that $N \in OH$.

Conversely, for any interior point of the segment OH there exists a rectangle satisfying the condition, for which it is the center.

3.4.16. *Path to solution.* Let A and B be the positions of the points at some moment, and let A' and B' be their positions after a time interval t (see Fig. 16). Then O must satisfy the condition

$$|AO| : |BO| = |A'O| : |B'O| = V_A t : V_B t = |AA'| : |BB'|.$$

Therefore, triangles AOA' and BOB' are similar, so $\angle AOA' = \angle BOB'$.

Thus there exists a rotational homothety with center O that takes segment AA' to BB'.

Let P be the intersection point of these straight lines, and let α be the angle between them. Since a rotational homothety with center O takes A to B, segment AB should be "visible" from point O at the angle α. Consequently, O lies on the arc of the circumcircle of triangle APB. Similarly, segment $A'B'$ must be "visible" from O at the angle α, which means that O

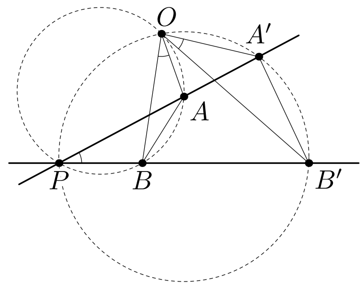


FIGURE 16

lies on an arc of the circumcircle of triangle $A'PB'$. Consequently, O is the second intersection point of these circles.

If the constructed circles are tangent, then both points pass through P simultaneously. In this case, the point P is what we seek.

5. Rotational homothety (2)
By P. A. Kozhevnikov

5.A. Introductory problems involving cyclists

3.5.1. Two circles intersect at points P and Q. Cyclists A and B simultaneously begin cycling clockwise from point P with equal angular speed, each traveling on a different circle.

(a) Prove that line AB always passes through Q.

(b) Prove that triangles PAB are always similar to each other and to triangle PO_1O_2, where O_1 and O_2 are the centers of the circles.

(c) Find the locus of the midpoints of segments AB, of the incenters of triangles PAB, and of any corresponding points of the similar triangles PAB.

(d) **The cyclist problem.** Prove that A and B are always equidistant from a fixed point. (See [**Kva79**].)

3.5.2. Consider three circles having a common point O and pairwise distinct intersection points P, Q, and R. Three cyclists A, B, and C simultaneously begin to travel clockwise on different circles, with equal angular velocity, starting at O.

(a) Prove that all triangles ABC are similar to each other and to triangle $O_1O_2O_3$, where O_1, O_2, and O_3 are the centers of the circles.

(b) What is the trajectory of the centroid of triangle ABC?

3.5.3. Two cyclists P and Q ride at constant (but not necessarily equal) speeds along two straight lines intersecting at point O.

(a) Find the trajectory of the midpoint of segment PQ.

(b) Prove that if the speeds of the cyclists are equal, then the midpoint of one of the arcs PQ of circle OPQ is fixed.

(c) Prove that if the cyclists pass O at different times, then circles OPQ have a second common point that is different from O.

3.5.4. Given a fixed triangle ABC, cyclists P, Q, R travel along lines BC, CA, AB, respectively, so that the angles between RP and PQ, PQ and QR, and QR and RP are fixed.

(a) Prove that the intersection point of the circles RAQ, RBP, PCQ is fixed.

(b) Find the locus of the incenters of the triangles PQR.

5.B. Main problems

3.5.5. Three cyclists P, Q, and R ride with fixed velocities along three straight lines. It is known that at two points in time, the triangle PQR is similar to a fixed triangle XYZ (preserving orientation). Prove that this condition will be satisfied at all times.

3.5.6. Triangle PQR is inscribed in a similar triangle ABC (so that $P \in BC$, $Q \in CA$, $R \in AB$, $\angle P = \angle A$, $\angle Q = \angle B$, and $\angle R = \angle C$).

(a) Prove that the circumcenter of triangle ABC coincides with the orthocenter of PQR.

(b) Find the maximum value of $\frac{S_{ABC}}{S_{PQR}}$.

(c) Prove that the circumcenter of triangle PQR is equidistant from the circumcenter and orthocenter of triangle ABC.

3.5.7. Take three arbitrary parallel lines d_a, d_b, d_c through the vertices of triangle ABC. Let lines d'_a, d'_b, d'_c be symmetric to d_a, d_b, d_c with respect to BC, CA, AB respectively. The intersections of these lines determine a triangle XYZ. Find the locus of the incenters of these triangles. (A. A. Zaslavsky, *Olympiad in Honor of I. F. Sharygin*, 2009.)

3.5.8. Consider a convex quadrilateral $ABCD$ whose sides BC and AD are equal in length but not parallel. Let E and F be interior points of segments BC and AD, respectively, satisfying the condition $|BE| = |DF|$. Lines AC and BD intersect at P, lines BD and EF intersect at Q, and lines EF and AC intersect at R. Prove that for all possible ways to choose points E and F, the circumcircles of triangles PQR have a common point other than P. (See **[IMO]**, 2005.)

3.5.9. Let O and I be the circumcenter and incenter of a non-equilateral triangle ABC. Points D, E, and F are chosen on sides BC, CA, and AB, respectively, in such a way that $|BD|+|BF| = |CA|$ and $|CD|+|CE| = |AB|$. The circumcircles of triangles BDF and CDE intersect at points D and P. Prove that $|OP| = |OI|$. (See **[IMO]**, 2012.)

5.C. Additional problems

3.5.10. Inscribe an equilateral triangle with minimal side length in a given acute triangle.

3.5.11. A right triangle ABC cut out of plywood is placed on the floor. Three nails are driven into the floor (each one close to a side of the triangle) so that the triangle cannot be rotated without taking it off the floor. The first nail is closer to A and divides side AB in the ratio $1 : 3$. The second is closer to C and divides side BC in the ratio $2 : 1$. How does the third nail divide side AC? (*Moscow Mathematical Olympiad*, 1998.)

3.5.12. A convex polygon M is placed in a triangle T. Prove that this can be done in such a way that one of the sides of M lies on a side of T.

A *rotational homothety* (or *spiral similarity*) $H_O^{k,\varphi} := H_O^k \circ R_O^\varphi$ is the composition of a homothety and a rotation with the same center.

3.5.13. (a) Circles α and β intersect at points A and B. Let H be a rotational homothety with center at point A mapping α to β. Prove that for any point $X \in \alpha$, its image $H(X)$ is obtained by intersecting BX with β. (See [**Pra95**, 19.27].)

(b) Circles $S_1, \dots, S_n$ pass through the point O. A grasshopper jumps from point $X_i \in S_i$ to point $X_{i+1} \in S_{i+1}$ so that the line X_iX_{i+1} passes through the second intersection point of circles S_i and S_{i+1}. Prove that after n jumps (from S_1 to $S_2, \dots,$ from S_n to S_1) the grasshopper returns to its starting point. (See [**Pra95**, 19.28].)

(c) Let the ends of segments AB and CD be pairwise distinct, and let P be the intersection point of lines AB and CD. Show that the center of the rotational homothety that takes AB to CD is an intersection point of the circumcircles of triangles ACP and BDP (and not equal to P). (See [**Pra95**, 19.41 (b)].)

More applications of rotational homothety can be found in [**Sp98**].

Suggestions, solutions, and answers

3.5.1. (a) The equality of the angular velocities implies that the oriented angles $\angle(PQ, QA)$ and $\angle(PQ, QB)$ are equal.

(b) Angle $\angle(BA, AP) = \angle(QA, AP)$ is constant and equals the angle $\angle(O_2O_1, O_1P)$.

(c) If M is the midpoint of AB, then $\angle(QM, MP)$ is constant, so M moves along circle Γ passing through P and Q.

Let N be any point of triangle PAB (at some fixed time). Consider a rotational homothety (see the definition before Problem 3.5.13) with center P that maps A into N. It maps the trajectory of A (a circle) into the trajectory of N.

(d) The perpendicular bisectors of AB pass through the point diametrically opposite to point Q in circle Γ (the locus of midpoints of AB from part (c)).

3.5.2. (a) Apply Problem 3.5.1 to pairs of circles.

(b) Triangles ABC at two different moments in time are related by a rotational homothety with center O. If N is any point of triangle ABC, then the rotational homothety with center O that takes A to N maps the trajectory of A (a circle) to the trajectory of N. The trajectory we seek is a circle passing through point O.

Comment. Compare Problem 3.5.1 to Problem 3.4.16 in the previous section. Here, in fact, we obtained a description of a family of similar triangles ABC circumscribed around a given triangle PQR.

3.5.3. (a) The midpoint of segment PQ moves in a straight line. If $\vec{p}$ and $\vec{q}$ denote the velocity vectors for the cyclists, then the midpoint moves at velocity $(\vec{p}+\vec{q})/2$.

(b) This is a special case of the solution of part (c).

(c) Let P and Q, P' and Q', and P'' and Q'' be the positions of the cyclists at three different points in time, and let Z be the second intersection point of circles POQ and $P'OQ'$. It is not difficult to prove that $\triangle ZPP' \sim \triangle ZQQ'$. This implies $\triangle ZPP'' \sim \triangle ZQQ''$, and from the equality of angles $\angle(OP'', P''Z) = \angle(OQ'', Q''Z)$ it follows that circle $P''OQ''$ also passes through Z.

Comment. Note that the point Z from the solution of part (c) is the center of the rotational homothety which maps PP' to QQ', and it is also a point from Problem 3.4.16. In addition, this point is the Miquel point (defined in Problem 8.3.11 in Chapter 8) for all quadruples of lines PP', QQ', PQ, $P'Q'$ and is the focus of the parabola tangent to all lines PQ.

3.5.4. (a) Let circles RAQ, RBP, PCQ intersect at point X. Then $\angle(PX, XQ)$ and $\angle(QX, XR)$ are fixed and they determine the position of the point X relative to triangle PQR. This means that $\angle(XP, PQ)$ and $\angle(XQ, QR)$ (and all similar angles) are fixed. Therefore $\angle(XC, CA) = \angle(XP, PQ)$ and $\angle(XA, AB) = \angle(XQ, QR)$ are fixed; that is, X is fixed.

Comment. Here, in fact, we obtained the description of a family of similar triangles PQR inscribed in the given triangle ABC. Everything is defined by the fixed point X. By specifying the point P, we can determine points Q and R based on the circles PXB and PXC.

(b) Let N be any point of the triangle PQR. Consider the rotational homothety with center X that takes P to N. It maps the trajectory of P (a straight line) to the trajectory of N.

3.5.5. Apply the methods of the solution to Problem 3.5.3 (c). You can also use vectors; for example, use a vector expression for $\overrightarrow{P''Q''}$ in terms of $\overrightarrow{PQ}$

and $\overrightarrow{P'Q'}$, where P and Q, P' and Q', and P'' and Q'' are the positions of the two cyclists at three different points in time.

3.5.6. (a) Consider the point Z mentioned in the suggestions for Problem 3.5.3 (c) above. It suffices to verify that Z coincides with O and with the orthocenter of triangle PQR in the special case where P, Q, R are the midpoints of BC, CA, AB, respectively.

(b) The area S_{PQR} is proportional to $|ZP|^2$. The segment ZP has minimal length when PQR is a pedal triangle of point Z, that is, when P, Q, and R are the midpoints of the sides.

(c) At the moment when P, Q, and R are the midpoints of the sides, the center W of circle PQR is the midpoint O_9 of segment OH (the center of the nine-point circle). At any other time $\triangle OO_9W \sim \triangle OA'P$, so $\angle OO_9W = 90°$, and thus W lies on the perpendicular bisector of OH.

3.5.7. *Answer*: Consider three directions of lines d_a, d_b, d_c, for example, perpendicular to the sides of the triangle, and construct the corresponding incenters. The desired locus is the circle through these three points.

Let A', B', C' be points symmetric to the vertices A, B, C with respect to their opposite sides. The lines d'_a, d'_b, d'_c pass through points A', B', C' respectively. If you change the picture by rotating lines d_a, d_b, d_c, then lines d'_a, d'_b, d'_c will rotate with identical angular velocities around A', B', C', respectively. The problem corresponds to Problem 3.5.2 (b) (a family of similar triangles circumscribed around triangle $A'B'C'$).

3.5.8. Let two cyclists ride with constant speeds, one from B to C and the other from D to A, so that they are at points B and D at one moment in time, and are at points C and A at another moment. Then they will get to E and F simultaneously. The desired common point of the circles is the center of rotation of the segment BC to DA (it is the point Z discussed in the commentary on Problem 3.5.3 (c)).

3.5.9. Let points D, E, F move along straight lines BC, CA, AB with equal speed so that the equalities from the statement of the problem are satisfied. Then circles BDF and CDE pass through fixed points X and Y respectively, located on the bisectors of BI and CI (see Problem 3.5.3 (c)). Considering triangle BIC and points D, X, Y on its sides, it is easy to show that P lies on the circle γ passing through X, I, Y.

In the case where D, E, F are the points of tangency of the excircles with the sides, $P = U$, where the point U is symmetric to I with respect to O; furthermore, in this case, circles BDF and CDE have BU and CU as diameters, so $UX \perp BI$ and $UY \perp CI$. Thus γ is a circle with IU as diameter; in particular, $|OP| = |OI|$.

3.5.10. It suffices to be able to solve the "dual" problem: for a given equilateral triangle find the largest of all circumscribing triangles with given angles (and construct it).

Comment. Using the notation of Problem 3.5.2 (a), the largest of the triangles ABC is the one for which, say, segment OA is greatest, that is, OA is the diameter (O is the common point of the three circles along which the vertices of the circumscribed triangle travel).

Note that for two families of equilateral triangles inscribed in triangle ABC, the corresponding points Z are the *Apollonius points*, points such that their pedal triangles are equilateral (see the suggestions for Problem 3.5.3 (c)).

3.5.11. *Answer*: The nail is closer to A and divides AC in the ratio $5 : 7$.

Nails P, Q, R on sides BC, CA, AB should be placed so that for the corresponding point O (see the suggestions for Problem 3.5.2) the triangle PQR is its pedal triangle. Otherwise, one can rotate with the same angular speed, either clockwise or counterclockwise, the lines BC, CA, AB around P, Q, R so that the triangle at the intersection of these lines (similar to ABC) increases in area. This would mean that this plywood triangle can be rotated (around point O) without taking it off the floor. Thus, it suffices to apply the Carnot principle (see suggestions for Problem 1.1.1 in Chapter 1).

3.5.12. It is possible to fix M and apply a homothety to continuously shrink T until each side of it contains at least one vertex of M. Let the sides BC, CA, AB contain the vertices P, Q, R, respectively. Then one can start to rotate (with the same angular speed) either clockwise or counterclockwise lines BC, CA, AB around P, Q, R so that the triangle at the intersection of these lines (similar to ABC) decreases in area. In this continuous process, the triangle always covers M. The process continues until one of the sides of M no longer lies on a side of the triangle.

6. Similarity (1)
By A. B. Skopenkov

A *similarity* is a transformation f of the plane that changes distances by the same nonzero factor: for any points $A \neq B$ and $A' \neq B'$, the ratio $\frac{|f(A)f(B)|}{|AB|} = \frac{|f(A')f(B')|}{|A'B'|} \neq 0$.

3.6.1. (Challenge.) Consider various geometric shapes and their properties and determine which are preserved and which are not preserved under similarities.

3.6.2. (a) On a rectangular map another map of the same region, on a smaller scale, is placed. Prove that one can pierce both maps with a needle so that the puncture point represents the same location on both maps.

(b) Same problem, but for an infinitely large map.

A similarity is called *proper* if it decomposes into a composition of a homothety and a proper isometry (i.e., an isometry that decomposes into a composition of an even number of axial symmetries).

3.6.3. (a) Any similarity is a composition of a homothety and an isometry.

(b) **Theorem.** Any proper similarity of a plane is either a translation or a rotational homothety.

(c)* The notion of proper similarity is well-defined, i.e., it does not depend on the specified decomposition.

7. Dilation to a line (2)
By A. Ya. Kanel-Belov

Let l be a straight line on a plane, and let $k \neq 0$. A *dilation* to line l with coefficient k is the transformation of the plane that maps an arbitrary point X to the point Y lying on the perpendicular XH to l ($H \in l$) such that $\overrightarrow{HY} = k\overrightarrow{HX}$.

The coefficient k is called the *dilation ratio*, and line l is called the *dilation axis*.

3.7.1. A dilation to a line

(a) maps any line to a line;

(b) maps parallel lines to parallel ones;

(c) preserves the ratio of segments lying on parallel lines;

(d) changes all areas by the factor $|k|$;

(e) maps lines that are perpendicular to the dilation axis to themselves.

3.7.2. Inscribe a rectangle of given area in a given triangle (two vertices of the rectangle should lie on the base of the triangle, and the two remaining vertices should lie on the triangle's other two sides).

3.7.3.* The composition of several dilations to lines can be represented as a composition of a single dilation and a similarity.

Let two perpendicular lines l_1 and l_2 be given. The composition of dilations to these lines with coefficients k_1 and $k_2 = 1/k_1$ is called a *hyperbolic rotation*.

3.7.4. (a) The notion of a hyperbolic rotation is well-defined; i.e., the composition of dilations does not depend on their order.

(b) A hyperbolic rotation with respect to the coordinate axes maps any hyperbola $y = k/x$ into itself.

When changing the dilation ratio, the hyperbola "slides" along itself, just as a circle slides along itself during an ordinary rotation around its center. This property explains the name "hyperbolic rotation."

3.7.5. The area of a triangle formed by a tangent to a hyperbola and intersects its asymptotes does not depend on the point of tangency.

3.7.6. A line crosses a hyperbola at points A and B, and intersects its asymptotes at points X and Y. Prove that $|AX| = |BY|$.

Dilation to a line l is a special case of an affine transformation (see the next section or [**Zas09a**]) which maps l and the lines perpendicular to it to themselves.

Suggestions, solutions, and answers

3.7.1. (a) If a given line intersects l, consider the point of intersection, and then apply Thales' Theorem.

(b, c) Use Thales' Theorem.

(d) For rectangles with a side parallel to the dilation axis, this is obvious. An arbitrary shape can be approximated from below and above by such rectangles (see section 5 "Pigeonhole principle and its applications in geometry" in [**SkoZa**]).

8. Parallel projection and affine transformations (2) *By A. B. Skopenkov*

Let α_1 and α_2 be two planes in space, and let l be a line not parallel to either plane. A *parallel projection* of plane α_1 onto plane α_2 in the direction l is a mapping which takes each point A in α_1 to the point of intersection of the line l_A (which passes through A and is parallel to l) with α_2.

3.8.1. (a) Any dilation to a line is a composition of a parallel projection and a spatial rotation.

(b) Any parallel projection is a composition of a dilation to a line and a spatial similarity.

3.8.2. Any parallel projection

(a) maps lines to lines;

(b) is a bijection;

(c) maps parallel lines to parallel lines;

(d) preserves the ratio of the lengths of parallel segments.

3.8.3. (a) Any triangle can be transformed into an equilateral triangle by a parallel projection.

(b) Let each side of a triangle be divided into three equal parts and join these trisection points by lines to the opposite vertices of the triangle. Prove that all the diagonals of the "inner" hexagon bounded by these lines are concurrent (cf. [**Pra95**, 29.12]).

3.8.4. (a) Prove that in any trapezoid, the midpoints of the bases, the intersection of the diagonals, and the intersection point of the legs (when extended) are collinear.

(b) If a pentagon has four diagonals parallel to their opposite sides, then the fifth diagonal is also parallel to its opposite side.

(c)* If in a hexagon opposite sides are parallel, then the three segments connecting the midpoints of opposite sides are concurrent.

3.8.5. Is it possible to use several parallel projections to transform

(a) any trapezoid into an isosceles one;
(b) any parallelogram into a square;
(c) any quadrilateral into any other quadrilateral;
(d) any trapezoid into any other trapezoid?

3.8.6. Do all parallel projections

(a) transform circles to circles;
(b) preserve angles;
(c) transform orthogonal lines into orthogonal lines;
(d) preserve distances?

3.8.7. (a) A parallel projection preserves the ratio of areas of polygons.

(b) Find the ratio of the area of the "inner" hexagon from Problem 3.8.3 (b) to the area of the whole triangle.

3.8.8. (a)* Give a definition of a dilation to a plane and of a parallel projection of space.

(b) Any tetrahedron can be transformed into any other tetrahedron by dilations to planes.

(c) The same is true for a parallelepiped and a cube.

(d) **Theorem about a cross-section of an affine-regular quadrangular pyramid.** Let points A_1, B_1, C_1, D_1 lie on the side edges of pyramid $SABCD$, whose base $ABCD$ is a parallelogram. Then A_1, B_1, C_1, D_1 are coplanar if and only if $\frac{1}{|SA_1|} + \frac{1}{|SC_1|} = \frac{1}{|SB_1|} + \frac{1}{|SD_1|}$.

(e) If $ABCD$ and $AB_1C_1D_1$ are parallelograms in space, then BB_1, CC_1, and DD_1 are all parallel to a plane.

3.8.9. (a) Dilation to a plane preserves the ratio of volumes.

(b) Any plane containing the midpoints of two skew edges of a tetrahedron cuts the tetrahedron into two solids of equal volume.

3.8.10. Let a transformation F of a plane be a composition of several parallel projections (such transformations are called *affine*).

(a) The transformation F may be decomposed into two parallel projections and an isometry.

(b) There exist two orthogonal lines with orthogonal F-images.

(c) The transformation F may be decomposed into a parallel projection and a spatial similarity.

3.8.11.* Prove that any transformation of a plane which maps a straight line to a straight line is

(a) uniquely determined by the images of three non-collinear points;

(b) affine.

Suggestions, solutions, and answers

3.8.4. (a) (Solution by K. Cherkasov.) Let $ABCD$ be the original trapezoid, let AD and BC be the trapezoid's bases, with $|AD| > |BC|$, let P be the intersection point of the trapezoid's legs when extended, and let O be the intersection point of the diagonals. From Problem 3.8.3 (a) we conclude that a parallel projection can transform triangle OAD into an isosceles triangle with AD as its base. Let points A, B, C, D, O, P map to A', B', C', D', O', P', respectively.

By symmetry, line $O'P'$ bisects the base of trapezoid $A'B'C'D'$. Since parallel projection preserves the ratio of parallel segments, line OP bisects the base of trapezoid $ABCD$.

3.8.5. *Answers*: (a)Yes; (b)Yes; (c)No; (d) No.

3.8.6. *Answers*: All "No."

3.8.11. (a) Let points A, B, C map to A', B', C'. Reflect triangles ABC and $A'B'C'$ to form parallelograms $ABCD$ and $A'B'C'D'$. Since the transformation is one-to-one and maps parallel lines to parallel lines, it maps D to D'. Likewise, the diagonals of $ABCD$ and their intersection point O are mapped to the diagonals of $A'B'C'D'$ and their intersection point O'. Consider the intersection of lines drawn through the vertices of $ABCD$ parallel to its diagonals with the extensions of the parallelogram's sides. These lines and points map to corresponding lines and points constructed on parallelogram $A'B'C'D'$. Continuing this process, we see that the grid of parallelograms generated by $ABCD$ is mapped to the grid generated by $A'B'C'D'$. Considering lines through the centers of the parallelograms drawn parallel to their sides, we see that the mapping holds for smaller and smaller grids.

To complete the argument, we need to show that our transformation is continuous. From the reasoning above, it follows that the midpoint of any segment is mapped to the midpoint of the segment joining the images of its endpoints. Consider a point Q on segment PR. Draw parallel lines p and q through P and Q and choose points P_1 and P_2 on p such that $|PP_1| = |PP_2|$. Next, choose points Q_1 and Q_2 on q such that $|QQ_1| = |QQ_2| < |PP_1|$. Let K and L, respectively, be the intersection points of the diagonals and legs of trapezoid $P_1P_2Q_2Q_1$. Then R lies on the line KL and we can choose the ratio $|QQ_1|/|PP_1|$ so that R is the midpoint of KL. The transformation will map K and L to K' and L', the intersections of the diagonals and legs of the image trapezoid, and will map R to the midpoint R' of $K'L'$, which lies outside segment $P'Q'$. Similarly we can prove that P' lies outside segment $Q'R'$, and therefore Q' lies on segment $P'R'$. Thus our transformation maps segments into segments. Therefore interior points of any parallelogram are mapped to interior points of the image parallelogram, which allows us to uniquely determine the image of any point of the plane. For more details see [**Zas09a**] and [**Yag09**].

9. Central projection and projective transformations (3) *By A. B. Skopenkov*

Let α_1 and α_2 be two planes in space, and let O be a point not lying on either plane. The *central projection* of α_1 onto α_2 with center O is the map that sends an arbitrary point A_1 of α_1 (not lying on the plane passing through O parallel to α_2) to the intersection point of line OA_1 with α_2.

3.9.1. (a) The central projection of α_1 onto a parallel plane α_2 with center O is a homothety.

(b) If the planes α_1 and α_2 intersect, then the central projection of α_1 onto α_2 with center O defines a one-to-one mapping of α_1 minus line l_1 to α_2 minus line l_2, where l_1 and l_2 are the lines of intersections of planes α_1 and α_2, respectively, with planes passing through O parallel to α_1 and α_2. Moreover, the mapping is not defined on l_1. (See [**Pra95**, 30.11].)

(c) The line on which the central projection is not defined and the line of points that do not have a preimage are called the *exceptional lines* of a given projection.

In a central projection, a line on α_1 (not parallel to the exceptional one on α_1) minus a point of intersection with the exceptional line is projected to a straight line minus a point. (See [**Pra95**, 30.12].)

3.9.2. (a) Draw a plane through a point on the side edge of a tetrahedral angle so that the resulting cross-section is a parallelogram.

(b) Any convex quadrilateral can be mapped to a parallelogram by a central projection.

3.9.3. (a) Let O be the intersection point of the diagonals of the convex quadrilateral $ABCD$. Let E be the intersection point of AB and CD, and let F be the intersection point of BC and AD. Line EO intersects sides AD and BC at points M and N, and line FO intersects sides AB and CD at points P and Q, respectively. Show that lines PM, NQ, and EF are concurrent or parallel. (See [**Pra95**, 30.25].)

(b) **Pappus's Theorem.** Let points C and C' lie on the lines AB and $A'B'$, respectively, and suppose that A, B, C, A', B', C' are distinct points.

If the lines AB' and $A'B$, BC' and $B'C$, and CA' and $C'A$ pairwise intersect, then the three intersection points are collinear. (See [**Pra95**, 30.27].)

(c) **Desargues' Theorem.** Let lines a, b, c have a common point O. Let $A_1B_1C_1$ and $A_2B_2C_2$ be triangles with vertices A_1 and A_2, B_1 and B_2, and C_1 and C_2 lying on a, b, and c, respectively. Assume that all six points are distinct and do not coincide with O. Let the intersection points of B_1C_1 and B_2C_2, A_1C_1 and A_2C_2, and A_1B_1 and A_2B_2 be A, B, and C, respectively. Then points A, B, and C are collinear. (See [**Pra95**, 30.26].)

(d) **Converse to Desargues' Theorem.** Let $A_1B_1C_1$ and $A_2B_2C_2$ be triangles with vertices A_1 and A_2, B_1 and B_2, and C_1 and C_2 lying on lines a, b, and c, respectively. Assume that all six points are distinct. Let the intersection points of B_1C_1 and B_2C_2, A_1C_1 and A_2C_2, and A_1B_1 and A_2B_2 be A, B, and C, respectively. If A, B, and C are collinear, then lines a, b, and c are concurrent or parallel.

3.9.4. (a) Let distinct points A, B, C, D lie on a line parallel to the exceptional line of a central projection, and let their respective images be A', B', C', D'. Prove that $\frac{|A'B'|}{|C'D'|} = \frac{|AB|}{|CD|}$. (See [**Pra95**, 30.14b].)

(b) If lines are parallel to the exceptional line of a central projection, then their images are also parallel.

3.9.5. (a) **The Brianchon construction.** Let $BC \parallel A_1A$, $O = A_1B \cap AC$, $B_i = AB \cap A_iC$, and $A_{i+1} = OB_i \cap A_1A$ $(i = 1, 2, \ldots)$. Prove that $A_nA = \frac{1}{n}A_1A$.

(b) Using only a straightedge, it is impossible to construct the midpoint of a given segment. (See [**Pra95**, 30.59].)

Cf. Problem 3.8.4 (a) above and Problem 6.3.1 in Chapter 6.

3.9.6.* Draw an image of your home using a central projection.

3.9.7. The *cross-ratio* $(A, B; C, D) := \frac{(\overrightarrow{AC}, \overrightarrow{BD})}{(\overrightarrow{AD}, \overrightarrow{BC})} = \frac{\overrightarrow{AC}}{\overrightarrow{AD}} : \frac{\overrightarrow{BC}}{\overrightarrow{BD}}$ (of an ordered 4-tuple of distinct points on a line) is preserved by the central projection. See the definition of the number $\overrightarrow{PQ}$ in section 2 of Chapter 4. (See also [**Pra95**, 30.2].)

A *projective transformation* is a composition of several central projections.

3.9.8. (a) The equality $(A, B; C, X) = (A, B; C, Y)$ holds if and only if $X = Y$. (See [**Pra95**, 30.3].)

(b) Any ordered triple of distinct points on a line can be transformed into any other by a unique projective transformation (cf. [**Pra95**, 30.4]).

(c) A transformation P of a line is projective if and only if in some coordinate system P is a linear-fractional transformation (i.e., there exist a, b, c, d such that $P(x) = \frac{ax+b}{cx+d}$ (cf. [**Pra95**, 30.7]).

(d) The transformation of the plane defined by the formula $P(x, y) = (1/x, y/x)$ is projective (cf. [**Pra95**, 30.22]).

3.9.9. (a*) There exists a central projection of the plane α_1 onto a non-parallel plane α_2 that takes a circle to a circle. (*Hint.* Use stereographic projection.)

(b*) There exists a central or parallel projection that takes a circle into a circle and takes a given point inside the circle to the center of the image. (See [**Pra95**, 30.16a].)

(c) There is a central or parallel projection that takes a circle into a circle for which a given line that does not intersect the circle is exceptional. (See [**Pra95**, 30.16b].)

3.9.10. Brianchon's Theorem. If the hexagon $ABCDEF$ circumscribes a circle, then AD, BE, and CF are concurrent (cf. [**Pra95**, 30.42]).

Note: Using the central projection, you can also get a proof of Pascal's Theorem different from the one proposed in section 10 ("Isogonal conjugation and the Simson line") of Chapter 1. Both Brianchon's and Pascal's theorems are true for an arbitrary conic, not just a circle (see section 3 "Conic sections" in Chapter 8).

3.9.11. (a) Any convex quadrilateral can be transformed into a square by the composition of central and parallel projections.

(b) Is it always possible to do this using only *one* projection?

For other properties of central projections, see [**Zas96**, **Fuk84**, **DS89**].

10. Inversion (2)
By A. B. Skopenkov

Let circle S with center O and radius R be given in plane Π. The transformation $f\colon \Pi - \{O\} \to \Pi - \{O\}$ that maps an arbitrary point A not coinciding with O to point A^* lying on ray OA at a distance $|OA^*| = R^2/|OA|$ from point O is called the *inversion with respect to circle* S. The inversion with respect to S is also called the *inversion with center* O *and degree* R^2, and circle S is called the *inversion circle*.

3.10.1. Triangle BOA is similar to the triangle A^*OB^*. (See [**Pra95**, 28.1].)

3.10.2. What are the images under inversion of

(a) points lying inside S;

(b) points lying outside S;

(c) lines and circles passing through O;

(d) other straight lines and circles?

(Cf. [**Pra95**, 28.2, 28.3].)

A *generalized circle* is a circle or a line. Parallel lines are considered to be *touching.*

Let two circles intersect at point A. The *angle between the circles* is the angle between the tangents to the circles at A. (It is clear that if the circles intersect at points A and B, then the angle between the tangents at point A is equal to the angle between the tangents at point B.) The angle between a line and a circle is determined similarly. The angle between parallel or coincident lines is considered to be zero.

3.10.3. (a) The tangency of generalized circles is preserved under inversion (cf. [**Pra95**, 28.4]).

(b) All possible pairs of externally tangent circles are inscribed in a circular segment (a region of a disc "cut off" by a chord). Find the locus of the points of tangency of these pairs of circles (cf. [**Pra95**, 28.22]).

(c) Radius XO of circle S with center O is perpendicular to line a. Two lines passing through X intersect a and S at points A, B, C, and D distinct from X. Prove that these four points are concyclic.

(d) Two circles intersect at points A and B. Through a point K on the first circle, draw lines KA and KB, which intersect the second circle at P and Q. Prove that chord PQ of the second circle is perpendicular to diameter KM of the first circle.

(e) Given circle ω and chord AB, two circles tangent to the chord and ω intersect at points C and D. Prove that CD halves arc AB (the arc opposite to the one that is tangent to the circles).

(f)* Points A and B lie on circle S. Point C is the midpoint of one of the arcs AB, and D lies on line segment AB. Circle S_1 is tangent to segments BD (at point B_1) and CD and to circle S. Circle S_2 is tangent to the ray AB (at point B_2), circle S (at K), and the ray CD. Prove that $\angle B_1KB_2$ is a right angle.

3.10.4. (a) Inversion preserves orthogonality and, in fact, all angles between generalized circles (cf. [**Pra95**, 28.5]).

(b) The angle between the circumscribed circles of triangles ABC and ABD is equal to the angle between the circumscribed circles of triangles ACD and BCD.

(c) Pairs of generalized circles α and β, β and γ, γ and δ, and δ and α intersect at distinct points A_1 and A_2, B_1 and B_2, C_1 and C_2, and D_1 and D_2, respectively. Prove that if points A_1, B_1, C_1, D_1 lie on a generalized circle, then the points A_2, B_2, C_2, D_2 also lie on a generalized circle. (See [**Pra95**, 28.31].)

3.10.5. The following equality holds: $|A^*B^*| = \frac{|AB|\cdot R^2}{|OA|\cdot|OB|}$.

From this equality we can obtain a proof of Ptolemy's Theorem (section 6 in Chapter 2).

3.10.6. *Apollonius's problem.* Construct a circle

(a) passing through two given points and tangent to a given circle;

(b) passing through a given point and tangent to two given circles;

(c)* tangent to three given circles (see [**Pra95**, 28.11]).

3.10.7.* Constructions using only a compass. Using only a compass (without a straightedge),

(a) bisect line segment AB (see [**Pra95**, 28.18]);

(b) construct the image A^* of a given point A under inversion with respect to a given circle (see [**Pra95**, 28.17]);

(c) construct a circle passing through three given points (see [**Pra95**, 28.20]).

See also the Mohr–Mascheroni Theorem in section 3 "Construction toolbox" of Chapter 6 and Problem 28.3 in [**Pra95**].

3.10.8. (a) Two non-intersecting circles S_1 and S_2 (or a circle and a line) can be transformed into a pair of concentric circles by inversion. (See [**Pra95**, 28.6].)

(b) **Steiner's porism.** Let circle R_1 lie inside circle R_2. If there is a chain of distinct circles $S_1, S_2, \ldots, S_n$, each of which is externally tangent to two neighbors (S_n is tangent to S_{n-1} and S_1) and also tangent to the circle R_1 externally and the circle R_2 internally, then there are infinitely many such chains. Namely, for any circle T_1 that is tangent to R_1 externally and R_2 internally, there is a similar chain of n circles tangent to circles $T_1, T_2, \ldots, T_n$. (See [**Pra95**, 28.39].)

(c) Formulate an analogue of Steiner's porism for the case where circles R_1 and R_2 are external to each other.

(d) Given two non-intersecting circles a and b, two circles c and d are externally tangent to them and, moreover, are tangent to each other at point A. Find the locus of these points of tangency.

3.10.9. A sphere with center in the base plane ABC of tetrahedron $SABC$ passes through the vertices A, B, and C and also intersects the edges SA, SB, and SC at points A_1, B_1, and C_1 respectively. The planes tangent to the sphere at A_1, B_1, and C_1 intersect at O. Prove that O is the center of the sphere that circumscribes tetrahedron $SA_1B_1C_1$.

3.10.10. Let ω be a sphere with unit radius and center at $E = (0, 0, 1)$. Define the *stereographic projection* $\omega - \{N\} \to \alpha$ to be the mapping of ω without point N onto the xy-plane $\alpha := \{z = 0\}$: the point A maps to the point A^* of the intersection of ray $\overrightarrow{NA}$ and α, where $N(0, 0, 2)$ is the point diametrically opposite to the point of tangency of ω and α. Prove that this transformation takes circles (on the sphere) to generalized circles.

3.10.11.* (a) Is it possible to "tile" space with, i.e., to construct a family of circles $\{\omega_\alpha, \alpha \in \mathbb{R}\}$ so that every point of the space lies on exactly one circle in this family?

(b) Construct a family of disjoint circles of equal radius which covers a cube with unit edge length.

Hint. Spatial inversion is useful for Problems 3.10.9–3.10.11.

3.10.12.* Find all flat curves (curves that lie in a plane) which transform into flat curves under spatial inversion with any center.

Other properties of inversion are discussed in [**Ur84**, **Sol90**]

Additional reading

For more information about geometric transformations, see, for example, [**Zas03**] (Chasles' Theorem in § 1.2, similarity and homothety in § 1.3, affine transformations in Ch. 2, projective transformations in Ch. 3, inversion in Ch. 4, complex interpretation of motions and similitude in § 6.1, and complex interpretation of inversion in § 6.2), [**Pra95**], and [**Yag09**].

We also recommend the following sources where particular geometric transformations are discussed: for classification of motions see [**Sol80**], for rotational homothety see [**Sp98**], for parallel projection and affine transformations see [**Zas09a**], for central projection and projective transformations see [**Zas96**, **Fuk84**, **DS89**], and for inversion see [**Ur84**, **Sol90**].

Chapter 4

Affine and projective geometry

Affine and projective transformations were studied in Chapter 3, sections 7–9. The branches of geometry that study the properties of figures that are preserved under these transformations are called affine and projective geometry respectively. In this chapter we explore a different approach to affine and projective geometry, not using directly the notion of transformation. Formally, this chapter can be studied independently of the previous one. However, the last sections of the previous chapter may be useful when studying sections 2 and 3 of this chapter. On the other hand, the material in section 1 can help in the study of affine transformations.

1. Mass points (2) *By A. A. Gavrilyuk*

A *mass point* is a pair (A, m) where A is a point in the plane and m is a real number (the mass at this point).

Let (A_1, m_1), ..., (A_n, m_n) be a system of mass points, with $m_1 + \ldots + m_n \neq 0$. The *center of mass* of this system is the point O in the plane such that $m_1\overrightarrow{OA_1} + \ldots + m_n\overrightarrow{OA_n} = \overrightarrow{0}$.

4.1.1. Let (A_1, m_1), ..., (A_n, m_n) be a finite system of mass points in the plane with nonzero total mass.

(a) Prove that the center of mass O exists and is unique.

(b) Prove that for any point X the following equality is true:

$$\overrightarrow{XO} = \frac{m_1\overrightarrow{XA_1} + \ldots + m_n\overrightarrow{XA_n}}{m_1 + \ldots + m_n}.$$

4.1.2. Mass grouping lemma. Given two mass point systems (X_1, a_1), ..., (X_n, a_n) and $(Y_1, b_1), \ldots, (Y_m, b_m)$, each with nonzero total mass and with centers of mass at X and Y, respectively, prove that the center of mass of these $m + n$ points coincides with the center of mass of the two-point system $(X, a_1 + \ldots + a_n)$ and $(Y, b_1 + \ldots + b_m)$.

4.1.3. Ceva's Theorem. Let ABC be a triangle with points A_1, B_1, C_1 lying on sides BC, AC, AB, respectively. Prove that segments AA_1, BB_1, CC_1 intersect at one point if and only if $\frac{|AC_1|\cdot|BA_1|\cdot|CB_1|}{|AB_1|\cdot|CA_1|\cdot|BC_1|} = 1$.

4.1.4. Suppose that in the previous problem, the three segments intersect at T. Prove that $\frac{|BT|}{|TB_1|} = \frac{|BA_1|}{|A_1C|} + \frac{|BC_1|}{|C_1A|}$.

4.1.5.° Equal masses are placed at the vertices of a triangle. What is the center of mass?

(a) The circumcenter;
(b) the incenter;
(c) the orthocenter;
(d) the intersection point of the medians.

4.1.6.° The lengths of the sides of a triangle are a, b, c. What masses must be placed at the vertices so that their center of mass is the incenter of the triangle?

(a) a, b, c; (b) a^2, b^2, c^2; (c) $b+c$, $c+a$, $a+b$.

4.1.7. Let K, L, M, N be the midpoints of the sides of quadrilateral $ABCD$, and let O be the intersection point of LN and KM. Prove that O bisects LN, KM, and the segment connecting the midpoints of the diagonals of $ABCD$.

4.1.8. Let $A_1, \dots, F_1$ be the midpoints of sides $AB, BC, \dots, FA$ of the hexagon $ABCDEF$. Prove that the points of intersections of the medians of triangles $A_1C_1E_1$ and $B_1D_1F_1$ coincide.

4.1.9. Three identical flies crawl along the perimeter of triangle ABC such that their center of mass stays fixed. It is known that one of the flies traversed the entire perimeter of the triangle. Prove that the center of mass of the flies is the centroid of triangle ABC.

4.1.10. A circle contains n points of unit mass. Through the center of mass of any $n-2$ points, draw a line perpendicular to the chord connecting the remaining two points. Prove that all such lines intersect at one point.

4.1.11. Inscribe pairwise tangent circles in the angles of triangle ABC. The tangency points A_1, B_1, C_1 of these circles lie respectively opposite the vertices of triangle ABC. Prove that the segments AA_1, BB_1, CC_1 are concurrent.

Suggestions, solutions, and answers

4.1.1. (a) We prove uniqueness. Suppose O_1 and O_2 are centers of mass. Then

$$m_1\overrightarrow{O_1A_1} + \ldots + m_n\overrightarrow{O_1A_n} = \overrightarrow{0},$$
$$m_1\overrightarrow{O_2A_1} + \ldots + m_n\overrightarrow{O_2A_n} = \overrightarrow{0}.$$

Subtracting the second equality from the first one yields

$$(m_1 + \ldots + m_n)\overrightarrow{O_1O_2} = \overrightarrow{0}.$$

Since $m_1 + \ldots + m_n \neq 0$, we have $O_1 = O_2$.

To prove existence, take an arbitrary point X and define O by the equality from part (b).

(b) We have

$$m_1\overrightarrow{XA_1} + \ldots + m_n\overrightarrow{XA_n}$$
$$= m_1(\overrightarrow{OA_1} + \overrightarrow{XO}) + \ldots + m_n(\overrightarrow{XA_n} + \overrightarrow{XO}) = (m_1 + \ldots + m_n)\overrightarrow{XO}.$$

4.1.2. Let O be the center of mass of the systems $(X, a_1 + \ldots + a_n)$ and $(Y, b_1 + \ldots + b_m)$. Then

$$(a_1 + \ldots + a_n)\overrightarrow{OX} + (b_1 + \ldots + b_m)\overrightarrow{OY} = \overrightarrow{0}.$$

Also,

$$a_1\overrightarrow{XA_1} + \ldots + a_n\overrightarrow{XA_n} = b_1\overrightarrow{YB_1} + \ldots + b_m\overrightarrow{YB_m} = \overrightarrow{0}.$$

Now subtract each of the above equalities from the first equality.

4.1.3. The point of intersection of these lines is the center of mass of the system (A, x), (B, y), (C, z), where $x : y = |BC_1| : |AC_1|$, $y : z = |CA_1| : |BA_1|$, and $z : x = |AB_1| : |CB_1|$.

4.1.4. Use the statement of Problem 4.1.2 and hints for Problem 4.1.3.

4.1.7. Place equal masses at the vertices of the quadrilateral and find the center of mass of the resulting system.

4.1.8. Place equal masses at the vertices of the hexagon and find the center of mass of the resulting system.

4.1.9. Investigate the location of the center of mass when one of the flies is at a vertex of the triangle.

4.1.10. Prove that all such lines intersect the line OM at one point, where O is the center of the circle and M is the center of mass of all the points.

A comprehensive discussion of the geometry of masses can be found in [**BB87**].

2. The cross-ratio (2)
By A. A. Gavrilyuk

The *cross-ratio* $(A, B; C, D)$ of four collinear points A, B, C, and D, where $A \neq D$ and $B \neq C$, is defined by $(A, B; C, D) := \frac{(\overrightarrow{AC}, \overrightarrow{BD})}{(\overrightarrow{AD}, \overrightarrow{BC})} = \frac{\overrightarrow{AC}}{\overrightarrow{AD}} : \frac{\overrightarrow{BC}}{\overrightarrow{BD}}$. Here it is understood that the line on which the points lie has a fixed direction. Thus if P and Q lie on it, $\overrightarrow{PQ}$ denotes $|PQ|$ if the vector points the same way as the direction of the line and $-|PQ|$ otherwise.

4.2.1.° Which statement is a consequence of the equality $(A, B; C, D) = 1$?
(a) $A = B$; (b) $C = D$; (c) $A = B$ or $C = D$;
(d) none of the above.

4.2.2.° Which of the following equations is always true?
(a) $(A, B; C, D) = (B, A; C, D)$;
(b) $(A, B; C, D) = (B, A; D, C)$;
(c) $(A, B; C, D) = (A, C; B, D)$.

4.2.3.° Let $(A, B; C, D) = k$. What is the value of $(B, A; C, D)$?
(a) $1 - k$; (b) $\frac{1}{k}$; (c) $1 - \frac{1}{k}$; (d) $\frac{1}{1-k}$.

4.2.4. Lines a, b, c, d pass through the point O. Lines l_1 and l_2 do not intersect O. They intersect a, b, c, d at A_1, B_1, C_1, D_1 and A_2 , B_2, C_2, D_2 respectively. Prove that $(A_1, B_1; C_1, D_1) = (A_2, B_2; C_2, D_2)$.

The *cross-ratio of four concurrent lines* a, b, c, and d in a plane, where $a \neq d$ and $b \neq c$, is defined by $(a, b; c, d) := (A, B; C, D)$, where A, B, C, D are the intersection points of lines a, b, c, d, respectively, with an arbitrary line that intersects them at distinct points.

The *cross-ratio of four arbitrary lines* a, b, c, and d in the plane, where $a \neq d$ and $b \neq c$, is defined as the cross-ratio of the four lines parallel to them drawn through one point. (Clearly, the cross-ratio is independent of the choice of the point through which these parallel lines are drawn.)

4.2.5. Let m and n be distinct lines intersecting at A. Let B_1, C_1, D_1 and B_2, C_2, D_2 lie on m and n, respectively, and assume that these points are distinct from A. Prove that the lines B_1B_2, C_1C_2, D_1D_2 are concurrent if and only if $(A, B_1; C_1, D_1) = (A, B_2; C_2, D_2)$.

4.2.6. Let $(A, B; C, D) = k$. Find the cross-ratios of the points A, B, C, D when the points are permuted.

4.2.7. In the convex quadrilateral $ABCD$, lines AB and CD intersect at G, AD and BC intersect at E, DB and EG intersect at H, and AC and EG intersect at F. Prove that $(E, G; F, H) = -1$.

We refer to four points or lines p, q, r, s satisfying $(p, q; r, s) = -1$ as being *in harmonic range.*

4.2.8. Prove that $(A, B; C, D) = (B, A; C, D)$ if and only if $(A, B; C, D)^2 = 1$.

4.2.9. (a) Let M_B be the midpoint of side AC of triangle ABC. Prove that $(AB, BC; BM_B, AC) = -1$.

(b) Let BL_B be an internal angle bisector of triangle ABC, and let BK_B be an external bisector. Prove that $(AB, BC; BL_B, BK_B) = -1$.

4.2.10. Lines l_1, l_2, l_3, l_4 in the plane are such that $(l_1, l_2; l_3, l_4) = -1$. Prove that

(a) if $O \in l_1, l_2, l_3$ and $O \notin l_4$, then l_1, l_2, l_3 cut equal segments on l_4;

(b) if the lines l_1 and l_2 are perpendicular, then they are parallel to the bisectors of the angles between l_3 and l_4.

4.2.11. Let A, B, C, D, X be five distinct concyclic points. Prove that the value of $(XA, XB; XC, XD)$ does not depend on the choice of point X.

Let A, B, C, D be points lying on a circle, and let X be an arbitrary point on the circle that is distinct from A, B, C, D. The expression

$$(A, B; C, D) := (XA, XB; XC, XD)$$

is called the *cross-ratio of four concyclic points* A, B, C, and D.

4.2.12. Let A, B, C, D lie on a circle. For any point X on this circle, let k_X denote the tangent to this circle through X.

(a) Prove that CD, k_A, k_B are concurrent or parallel if and only if $(A, B; C, D) = -1$.

(b) Prove that CD, k_A, k_B are concurrent or parallel if and only if AB, k_C, k_D are concurrent or parallel.

4.2.13. Let AB and CD be parallel chords of circle ω, and let point M be the midpoint of AB. Line CM intersects ω again at point K. Let P be the midpoint of DK. Prove that $\angle BPK = \angle KPA$.

4.2.14. For a non-isosceles triangle ABC, introduce the following notation:

H_B is the foot of the height drawn to AC;

K_B is the point of tangency of the incircle to AC;

L_B is the base of the angle bisector drawn to AC;

T_B is the point of tangency of an escribed circle to AC.

Points H_A, K_A, L_A, T_A are defined similarly.

Prove that

(a) $(T_B, K_B, L_B, H_B) = -1$;

(b) $|CT_B| = |AK_B| = \frac{|AC|+|AB|-|BC|}{2}$;

(c) $|CH_B| = \frac{|BC|^2+|AC|^2-|AB|^2}{2|AC|}$;

(d) $(C, H_B, T_B, K_B) = (C, H_A, T_A, K_A)$;
(e) lines T_AT_B, L_AL_B, K_AK_B, H_AH_B are concurrent.

Suggestions, solutions, and answers

4.2.4. We have the following equalities (segments, angles, and areas are oriented):

$$\frac{|A_1C_1|}{|B_1C_1|} = \frac{S_{OA_1C_1}}{S_{OB_1C_1}} = \frac{|OA_1|}{|OB_1|}\frac{\sin\angle A_1OC_1}{\sin\angle B_1OC_1}.$$

Apply these and similar equalities for the ratios $|B_1D_1|/|A_1D_1|$, $|B_2D_2|/|A_2D_2|$, and $|B_2C_2|/|A_2C_2|$.

4.2.5. If lines B_1B_2, C_1C_2, D_1D_2 intersect at point O, then the equality of cross-ratios follows from the statement of the previous problem applied to lines OA, OB_1, OC_1, OD_1. Conversely, let the cross-ratios be equal. If the lines B_1B_2 and C_1C_2 are not parallel, we denote by O their intersection point. By the previous problem, line OD_1 intersects B_2C_2 at point D_2. The case of $B_1B_2 \parallel C_1C_2$ is treated similarly.

4.2.6. The following equality is obvious:

$$(A, B; C, D) = (B, A; D, C) = (C, D; A, B) = (D, C; B, A),$$

i.e., 24 cross-ratios are divided into 6 groups of 4 equal ratios. Besides, $(B, A; C, D) = 1/k$. To find the remaining relations, assume that points A, B, C, D have coordinates a, b, c, d. Then

$$k = (A, B, C, D) = \frac{(a-c)(b-d)}{(a-d)(b-c)}, \quad (A, C, B, D) = \frac{(a-b)(c-d)}{(a-d)(c-b)}.$$

Rewrite the numerator of this fraction:

$$\begin{aligned}(a-b)(c-d) &= ((a-c)+(c-b))((b-d)+(c-b)) \\ &= (a-c)(b-d)+(c-b)(a-d).\end{aligned}$$

Thus, $(A, C, B, D) = 1 - k$. Repeating the above reasoning, we obtain that the remaining ratios are $1-1/k$, $1/(1-k)$, and $k/(k-1)$. From Problem 4.2.4, it follows that the same result is also true for cross-ratios of lines and points on a circle.

4.2.7. From Problem 4.2.4, it follows that the cross-ratios are invariant under central projections. Therefore, it is sufficient to transform quadrilateral $ABCD$ into a parallelogram (see Problem 3.9.11 in Chapter 3).

4.2.8. Use Problem 4.2.6.

4.2.9. Both statements can be proved either by direct calculation of the cross-ratio or by applying Problem 4.2.7.

4.2.10. Use the previous problem.

4.2.12. Apply the central projection that transforms the given circle into a circle and maps the intersection point of the chords AB and CD to its center (see Problem 3.9.9 in Chapter 3).

3. Polarity (2)
By A. A. Gavrilyuk and P. A. Kozhevnikov

Traditionally, polarity is studied using projective transformations. Instead, we will attempt to avoid projective geometry in our introduction to polarity and its applications.

Let point O and circle ω of radius R with center O be fixed in the plane.

For each point $X \neq O$ construct the point X' on ray $\overrightarrow{OX}$ such that $|OX| \cdot |OX'| = R^2$. (We say that X' and X are *inverse* with respect to the circle ω.) Through point X', draw line x perpendicular to OX'. Line x is called the *polar* of point X, and the point X is called the *pole* of line x. The correspondence $X \leftrightarrow x$ is a one-to-one correspondence between points distinct from O and lines that do not pass through O. We call this correspondence *polarity.*

Below we denote points other than O (poles) by upper-case Latin letters and denote their polars by the corresponding lower-case letters: $A \leftrightarrow a$, $B \leftrightarrow b$, $C \leftrightarrow c$, $\dots$

Fundamental properties and introductory problems

Prove the following two fundamental properties of polarity:

P1. Duality. The inclusion $A \in b$ holds if and only if $B \in a$; i.e., the polar of any point distinct from O is the locus of the poles of the lines passing through it.

P2.* Let two lines m and l intersect at an arbitrary point $A \notin \omega$, $A \neq O$, and suppose that m and n intersect ω at points M_1, M_2 and L_1, L_2, respectively. Then $M_1L_1 \cap M_2L_2 \in a$ or $M_1L_1 \parallel M_2L_2 \parallel a$.

Prove the following facts:

P3. If $A \in \omega$, then a is the tangent to ω drawn through A.

P4. If point A is located outside circle ω, then a passes through the points of tangency of the tangents to ω drawn through A.

P5. If O, A, B are not collinear, then $a \cap b \leftrightarrow AB$.

P6. Points $A, B, C \neq O$ are collinear if and only if $a, b, c \not\ni O$ are concurrent or are parallel.

Main problems

4.3.1.° Given a circle and a chord AB, $a \cap b$ is located
(a) inside the circle; (b) outside the circle; (c) on the circle.

4.3.2.° Let C be the midpoint of chord AB. Then c is
(a) parallel to AB;
(b) perpendicular to AB;
(c) tangent to the circle.

4.3.3.° The image of a triangle (viewed as the set of three vertices and three lines joining them) under polarity with respect to its incircle is
(a) the medial triangle;
(b) the orthotriangle;
(c) the triangle formed by the points of tangency of the sides of the original triangle with its incircle.

4.3.4. Let circle ω and line l have no common points. Let X be a point moving along l, and let XA and XB be the tangents to ω. Prove that chords AB have a common point.

4.3.5. The symmetric butterfly. (a) Let point A on the diameter BC of semicircle ω be fixed. Choose points X and Y on ω such that $\angle XAB = \angle YAC$. Prove that these lines XY go through a single point or are parallel.

(b) Points A and A' are inverse with respect to circle ω, and A' lies in the interior of ω. Chords XY are drawn through A' . Prove that the centers of the inscribed and one of the escribed circles of triangles AXY are fixed. (S. Markelov; see [**Sha82**].)

4.3.6. The main property of the symmedian. Let P be the intersection point of the tangents to the circumcircle of triangle ABC through B and C. Prove that AP is a symmedian (the line symmetric to the median AM with respect to the angle bisector of A).

4.3.7. The harmonic quadrilateral. Let quadrilateral $ABCD$ be inscribed in circle ω. It is known that the tangents to ω at points A and C intersect on line BD or are parallel to BD. Prove that the tangents to ω drawn at points B and D intersect on line AC or are parallel to AC.

In the following three problems, $ABCD$ is a given quadrilateral whose diagonals intersect at P. The extensions of sides AB and CD intersect at R, and the extensions of sides BC and DA intersect at Q.

4.3.8. Inscribed quadrilateral. Let quadrilateral $ABCD$ be inscribed in a circle with center O. Prove that the points O, P, Q, R are orthocentric (i.e., each point is the orthocenter of the triangle with vertices at the three remaining points).

4.3.9. Circumscribed quadrilateral. Let quadrilateral $ABCD$ be circumscribed about a circle. Let K, L, M, N be the points of tangency of sides AB, BC, CD, DA, respectively. Lines KL and MN intersect at S, and lines LM and NK intersect at T.

(a) Prove that Q, R, S, T are collinear.

(b) Prove that KM and LN intersect at P.

4.3.10. Inscribed-circumscribed quadrilateral. Quadrilateral $ABCD$ circumscribes circle ω with center I and inscribes circle Ω with center O.

(a) Prove that O, I, P are collinear.

(b) Fix ω and Ω and consider the various quadrilaterals $ABCD$ that circumscribe ω and inscribe Ω. Prove that for all such quadrilaterals, both P and QR do not change.

Note: By Poncelet's Theorem, if there exists at least one quadrilateral that circumscribes ω and inscribes Ω, then there exist infinitely many such quadrilaterals.

Additional problems

4.3.11. From point A, draw tangents AB and AC to a circle. Let B and C be the points of tangency, and let M be the midpoint of BC. A line passing through point A intersects the circle at points D and E and intersects segment BM. Prove that

(a) $|BM|^2 = |DM| \cdot |ME|$;

(b) $\angle DME = 2\angle DCE$.

4.3.12. Let MN be a chord in a circle. For each diameter AB of this circle, consider the point at which lines AM and BN intersect, and draw line l perpendicular to AB through it. Prove that all these lines l pass through one point. (E. Kulanin, *Tournament of Towns*, 1991.)

4.3.13. Given an angle with vertex Q and a point P inside it, draw pairs of straight lines through P that intersect the sides of the angle at four points lying on a circle. Prove that the centers of these circles lie on a line which passes through P.

4.3.14. In the acute triangle ABC, heights AH_a, BH_b, and CH_c intersect at H. Lines H_aH_c and AC intersect at point D. Prove that DH is perpendicular to the median drawn from vertex B.

In the following problems, triangle ABC has an inscribed circle with center I that is tangent to the sides at points A_1, B_1, C_1.

4.3.15. Line A_1C_1 intersects AC at B_2. Prove that $B_2I \perp BB_1$.

4.3.16. Prove that the projection of point C onto the bisector of angle ABC lies on B_1C_1.

4.3.17. A second tangent MK is drawn from the midpoint M of AC to the inscribed circle. Let K be the point of tangency, and let l be a line parallel to AC that passes through B. Prove that lines B_1K, l, and A_1C_1 are concurrent.

The problems in this section have simple solutions that use properties **P1** and **P2** (among other possible approaches). Other applications of polarity are discussed in [**Har86**, **Pon91**].

Suggestions, solutions, and answers

P1. Use the similarity of triangles OAB' and OBA', where A' and B' are the inverse points of A and B, respectively.

P2. Let the tangents to ω drawn through points M_1 and M_2 intersect at point M; i.e., $M \leftrightarrow m$. Let also $L \leftrightarrow l$, $P \leftrightarrow L_1M_1$, and $Q \leftrightarrow L_2M_2$. From property **P1** we get $ML = a$.

The proof follows by applying Menelaus's Theorem to triangles MLP and MLQ. (There are several different cases to consider for the location of points.)

Note: There is a more natural proof of property **P2**, using central projection in space from one plane to another or a projective transformation that preserves ω which maps A either into O (if A is inside ω) or to the point at infinity (if A is outside ω).

P3. The statement follows from the definition.

P4. If AK and AL are tangents (K and L are tangency points), then A' is the midpoint of KL. Note also that the statement of this property follows from property **P5** and can be considered to be the "limiting" case of **P2**.

P5. The statement follows from property **P1**.

P6. The statement follows from property **P1**.

4.3.4. Line AB is the polar of point X, so **P1** implies that it passes through the pole of line l.

4.3.5. If in **P2** the secants m and l are symmetric with respect to AO (and the intersection points with ω are labeled so that $M_1L_1M_2L_2$ is an equilateral trapezoid), then, on the one hand, the point $A' = M_1L_1 \cap M_2L_2$ lies on AO (by symmetry), and on the other hand it lies on the polar a of point A. Thus, A' is the inverse point of A. The statement of the problem can be reduced to consideration of the "symmetric butterfly" construction.

(a) Apply symmetry with respect to BC.

(b) Let chord X_1Y_1 be the reflection of chord XY with respect to AA'. The desired centers of the inscribed and escribed circles are the midpoints of arcs XX_1 and YY_1.

4.3.6. Note that the points M and P are mutually inverse. Thus AM and AP intersect the circumcircle ω for a second time at points that are symmetric about PM (completing the "symmetric butterfly").

4.3.7. The pole of line BD lies on AC, since a, c, and BD are concurrent or are parallel (property **P1**).

Note: You can solve the problem without using polars, by proving that the statement of the problem is equivalent to the equality $|AB| \cdot |CD| = |BC| \cdot |DA|$. An inscribed quadrilateral for which this equality holds is called *harmonic*; see also Problem 4.2.12.

4.3.8. Property **P2** implies that $R \leftrightarrow PQ$, $P \leftrightarrow QR$, and $Q \leftrightarrow RP$ (in other words, triangle PQR is *self-polar*).

Note: For an self-polar triangle PQR, it is possible to draw a secant through R that intersects the circle at points A and B, and then the construction can be extended up to four points A, B, C, D on the circle so that $R = AB \cap CD$, $Q = BC \cap DA$, and $P = AC \cap BD$.

4.3.9. (a) Let $P_1 = KM \cap LN$. According to Problem 4.3.8, we have $P_1 \leftrightarrow ST$. On the other hand, $Q \leftrightarrow LN \ni P_1$ and $R \leftrightarrow KM \ni P_1$. Property **P1** implies that $P_1 \leftrightarrow QR$. Thus lines ST and QR coincide.

(b) The above implies that $P_1 = P$.

4.3.10. (a) Problems 4.3.8 and 4.3.9 imply that RQ is the polar of P with respect to both circles ω and Ω. Therefore, $OP \perp RQ$ and $IP \perp RQ$.

(b) Further, let P' be the inverse point of P (relative to ω and to Ω). If we fix points O and $I \neq O$ and radii r and R of circles ω and Ω respectively, then the equalities $OP \cdot OP' = R^2$ and $IP \cdot IP' = r^2$ uniquely determine the pair of points P and P' lying on line OI.

4.3.11. This follows from the "symmetric butterfly" construction (see Problems 4.3.5 and 4.3.6).)

4.3.12. Use **P2** to show that $MN \cap AB \leftrightarrow l$. Then **P1** implies that all lines l pass through the pole of line MN.

4.3.13. Let A, B, C, D be the intersection points of lines with sides of the angle, so that $P = AC \cap BD$ and $Q = BC \cap DA$. Let $R = AB \cap CD$. It suffices to show that R lies on the fixed line l passing through O (sometimes this line is called the *polar of point P relative to the angle*), since the center O of circle $ABCD$ is the orthocenter of triangle PQR (cf. Problem 4.3.9). The required line l is the line such that l, QP, QA, QC are in harmonic range. The required locus of points O is contained in the perpendicular dropped from P to l.

4.3.14. Points A, C, H_a, H_c lie on a circle with center at the midpoint M of the segment AC. By Problem 4.3.9, points M, B, H, D form an orthocentric quadruple.

4.3.15. Line BB_1 is the polar of point B_2 relative to the inscribed circle.

4.3.16. Extend the construction so that it is symmetric with respect to the bisector of BI: let points A', B_1', and C' be respectively symmetric to points A, B_1, and C with respect to BI. Lines AB, BC, AC, $A'C'$ form the circumscribed quadrilateral that is symmetric with respect to BI. Then Problem 4.3.9 implies that point $X = C_1B_1 \cap A_1B_1'$ lies on CC', and point X also lies on BI due to symmetry.

4.3.17. Note that B is the pole of line A_1C_1, and M is the pole of line B_1K with respect to the inscribed circle. Let L be the pole of line l; then $IB_1 \perp l \parallel AC$ and **P1** implies that $L \in A_1C_1$; thus $L = IB_1 \cap A_1C_1$. It remains to prove that BL passes through M.

Draw a line parallel to AC through L, which intersects BA and BC at X and Y, respectively. Points I, C_1, L, X lie on a circle with diameter IX; therefore $\angle LXI = \angle LC_1I$. Similarly, $\angle LYI = \angle LA_1I$. Since $\angle LC_1I = \angle LA_1I$, we have $\angle LXI = \angle LYI$, and therefore $IX = IY$. Point L is the midpoint of XY (since $IL \perp XY$), so the homothety with center B that transforms triangle XBY into triangle ABC maps L into M.

Additional reading

For more information about mass points see [**BB87**], and about polarity see [**Pon91**, **Har86**].

Chapter 5

Complex numbers and geometry (3)

By A. A. Zaslavsky

This chapter is devoted to applications of complex numbers to geometric problems. To study this chapter, it suffices to know the definition of complex numbers and arithmetic operations with them (see section 5 of Chapter 3 in [**SkoA**]). The definitions and basic properties of isometries, similarities, and affine transformations will be helpful in understanding the first section of this chapter, and knowledge of inversion will be helpful for the second section. Some notable relevant topics not covered by this chapter can be found in [**BB73**, **Sko72**].

1. Complex numbers and elementary geometry

In the standard coordinate system of the plane, the complex number $z = x + yi$ corresponds to the point Z with coordinates (x, y). We define the *modulus* of the number z to be the distance from Z to the origin O, and we define the *argument* of z to be the oriented angle between the positive direction of the Ox axis and the vector $\overrightarrow{OZ}$, i.e., the angle by which the axis Ox should be rotated counterclockwise to align its positive direction with the direction of vector $\overrightarrow{OZ}$. The axes Ox and Oy are called the *real* and *imaginary* axes.

5.1.1. (Challenge.) Determine the geometric meaning of the addition of two complex numbers.

5.1.2. (a) What geometric transformation of the complex plane maps z to iz?

(b) (Challenge.) We write $e^{i\varphi} := \cos\varphi + i\sin\varphi$. What is the geometric meaning of multiplication by $e^{i\varphi}$ and by $re^{i\varphi}$, where r is a real number (see the definition of the trigonometric form of a complex number in section 5 of Chapter 3 in [**SkoA**])?

(c) Express the complex number w obtained by rotating z by angle φ counterclockwise with respect to center z_0 in terms of z, z_0, and φ.

(d) Prove that the composition of rotations of the plane (with different centers) is a rotation or parallel translation.

(e) Prove that the points z_1, z_2, z_3 are collinear if and only if the ratio $(z_3 - z_1)/(z_2 - z_1)$ is real.

Note: Problem 5.1.2 (b) can be easily solved with trigonometric addition formulas. However, you can do the opposite and solve this problem geometrically: prove that when multiplying complex numbers, their moduli are multiplied together, and the arguments are added, and then, using this result, prove the addition formulas without case-by-case considerations.

5.1.3.° Which transformation of the plane is performed by the mapping $z \mapsto 2z + 2$?

(a) Parallel translation by a vector of length 2;

(b) homothety with coefficient 2;

(c) dilation to a straight line with ratio 2.

5.1.4. Determine the geometric meaning of the transformation of the complex plane which maps the point z to

(a) $az + b$;

(b) $a\overline{z} + b$,

where a and b are arbitrary complex numbers with $a \neq 0$.

5.1.5. (a) For complex numbers a and z, find the image $H_a^{k,\varphi}(z)$ of z under the rotational homothety $H_a^{k,\varphi}$.

(b) Prove that all similarity transformations are given by one of the formulas in Problem 5.1.4.

5.1.6. Prove that affine transformations of the complex plane are exactly the mappings taking each point z to the point $z' = az + b\overline{z} + c$, where a, b, c are some complex numbers that satisfy the condition $|a| \neq |b|$.

Use complex numbers to get another proof of Napoleon's Theorem (see Problem 1.9.1 in Chapter 1), as well as the following more general fact.

5.1.7. On the sides of an affine-regular[1] n-gon, construct (internally or externally) regular n-gons. Prove that their centers form a regular n-gon.

[1] *Editor's note:* An affine-regular n-gon is the image of a regular n-gon under an affine transformation.

Suggestions, solutions, and answers

5.1.2. (a)*Answer*: Rotation counterclockwise by 90° about the origin.

(b)*Answer*: A rotational homothety with center at the origin, rotation angle φ, and coefficient r.

If $|a| = 1$, then $|az_1 - az_2| = |z_1 - z_2|$ holds for any two numbers z_1 and z_2. Consequently, multiplication by $e^{i\varphi}$ is an isometry. Since its only fixed point is $z = 0$, by Chasles Theorem (Problem 3.2.5 (c) in Chapter 3) this motion must be a rotation about the origin. Considering the image of 1 as a complex number, we see that the angle of rotation is φ. Similarly, we see that multiplication by $re^{i\varphi}$ is a rotational homothety with center at the origin of coordinates, angle of rotation φ, and coefficient r.

(c) It follows from part (b) that $w - z_0 = e^{i\varphi}(z - z_0)$. Therefore, $w = e^{i\varphi}z + z_0(1 - e^{i\varphi})$.

(d) By the above, each of the two rotations is given by a linear function of z. The composition of these functions is also linear. If the coefficient at z is 1, then we have a parallel translation; otherwise, we have a rotation.

5.1.4. (a) Similarly to the previous problem, we see that the transformation mapping z to az is a rotational homothety with center at the origin, coefficient equal to $|a|$, and angle of rotation $\arg a$. It is also obvious that the transformation that maps z into $z + b$ is a parallel translation. Composition of these two transformations is a similarity that preserves orientation, i.e., a translation or rotational homothety (see Problem 3.6.3 (b) in Chapter 3). With $a \neq 1$ the center of rotational homothety is its only fixed point, i.e., $b/(1 - a)$.

(b) Since conjugation reflects each point about the Ox axis, the transformation in question is an orientation-changing (improper) similarity.

5.1.5. (a) *Answer*: $H_a^{k,\varphi}(z) = a + k(\cos\varphi + i\sin\varphi)(z - a)$.

(b) Let the transformation in question map the points 0 and 1 to z_0 and z_1, respectively. Since there are exactly two similarities with these images, the transformation is either $az + b$ or $a\overline{z} + b$, where $a = z_1 - z_0$ and $b = z_0$.

5.1.6. To show that the map $z \mapsto az + b\bar{z} + c$ is affine it suffices to prove that a transformation of this type maps lines to lines (see Problem 3.8.11 in Chapter 3). Three different points z_1, z_2, z_3 are collinear if and only if the ratio $(z_2 - z_1)/(z_3 - z_1)$ is real. But this ratio is preserved under the transformation in question.

To show that each affine transformation is given by the required formula, we use the fact that the affine transformation is uniquely determined by the images of three non-collinear points. Let the points 0, 1, and i map to z_0, z_1, and z_2, respectively. Then this transformation is given by a formula of the required form, in which $c = z_0$, $a = ((z_1 - z_0) - (z_2 - z_0)i)/2$, and $b = ((z_1 - z_0) + (z_2 - z_0)i)/2$. Since z_0, z_1, z_2 are not collinear, it follows that $|a| \neq |b|$.

5.1.7. Let $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then $1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1}$ are the vertices of a regular n-gon. By the previous problem, we can assume that the vertices of the given affine-regular polygon are $a\varepsilon^k + b\varepsilon^{-k}, k = 0, 1, \dots, n-1$. Hence, the center z_k of the regular n-gon on side k satisfies the equality $a\varepsilon^{k+1} + b\varepsilon^{-k-1} - z_k = \varepsilon(a\varepsilon^k + b\varepsilon^{-k} - z_k)$. From this it is easy to show that the z_k form a geometric progression with common ratio ε, i.e., are vertices of a regular n-gon.

2. Complex numbers and Möbius transformations

A map of the plane into itself (defined everywhere except for possibly a single point) preserving generalized circles is called a *Möbius* transformation. Any Möbius transformation other than a similarity can be represented as the composition of an inversion and a motion.

5.2.1.° Four complex numbers z_1, z_2, z_3, z_4 satisfy $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = 2$. What can be said about the four points of the plane corresponding to the numbers z_1, z_2, z_3, z_4?

(a) They are the vertices of a parallelogram.

(b) They lie on the same line or on the same circle.

(c) The area of the triangle $0z_1z_2$ is equal to the area of the triangle $0z_3z_4$ (0 is, of course, the origin).

5.2.2. Prove that a map of the complex plane into itself (defined everywhere except for possibly a single point) is Möbius if and only if it is a linear fractional transformation of the form $f(z) = (az+b)/(cz+d)$ or $f(z) = (a\bar{z} + b/(c\bar{z}+d)$, where $ad - bc \neq 0$.

5.2.3. Prove that for any six different points A, B, C, A', B', C' there are exactly two Möbius transformations that map A to A', B to B', and C to C'.

The complex number $(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$ is called the *cross-ratio* of a, b, c, d, where $a \neq d$ and $b \neq c$.

5.2.4. Prove that for any eight distinct points A, B, C, D, A', B', C', D', a Möbius transformation that maps A to A', B to B', C to C', and D to D' exists if and only if the corresponding complex numbers satisfy $(a, b, c, d) = (a', b', c', d')$ or $\overline{(a, b, c, d)} = (a', b', c', d')$.

5.2.5. For any two triangles ABC and $A'B'C'$, show that there exists an inversion which maps ABC to a triangle congruent to $A'B'C'$.

5.2.6. For any quadrilateral, show that there exists an inversion that takes its vertices to the vertices of a parallelogram, and that all parallelograms obtained as a result of such inversions are similar.

See also Problem 3.10.4 (c) in Chapter 3.

Additional problems

5.2.7. (a) Let a, b, c be complex numbers corresponding to points A, B, C not lying on one line, and let $f(z) = (z - a)(z - b)(z - c)$. Prove that the two points corresponding to the roots of the derivative $f'(z)$ are isogonally conjugate with respect to triangle ABC.

(b)* The *Steiner ellipse* of a triangle is the ellipse of maximal area that lies inside the triangle. Prove that the foci of a Steiner ellipse correspond to the roots of the derivative $f'(z)$.

5.2.8. Let a, b, c be the complex numbers corresponding to points A, B, C, with $|a| = |b| = |c| = 1$. Prove that points Z_1 and Z_2 are isogonally conjugate with respect to the triangle ABC if and only if the corresponding complex numbers satisfy the relation

$$z_1 + z_2 + abc\bar{z}_1\bar{z}_2 = a + b + c.$$

5.2.9. For any triangle, Euler's formula $|OI|^2 = R^2 - 2Rr$ relates the distance between the circumcenter O and incenter I to the radii R and r of the respective circles (see Problem 1.4.10 in Chapter 1). Prove a generalization of Euler's formula: if an ellipse with foci F_1 and F_2 and shorter axis l is inscribed in a triangle, then

$$R^2 l^2 = (R^2 - |OF_1|^2)(R^2 - |OF_2|^2).$$

5.2.10. (A. Akopyan.) Given triangle ABC and point P, find the locus of points that are isogonally conjugate to point P with respect to all triangles that have the same circumscribed and inscribed circles as does ABC.

Suggestions, solutions, and answers

5.2.2. Notice that inversion with center z_0 and radius R is given by the relation $z' = R^2/(\overline{z - z_0}) + z_0$. Composition of such a transformation with a linear one obviously produces the composition of a linear fractional transformation with complex conjugation.

Consider the transformation defined by $z' = (az + b)/(cz + d)$, with $ad - bc \neq 0$ and $c \neq 0$. The composition of this transformation with an inversion with center at a/c will be a linear function, i.e., a similarity. So the original transformation is the composition of a similarity with an inversion.

If $c = 0$, then $a \neq 0$. Hence the transformation is a similarity and thus preserves circles.

5.2.3. Knowing the images of three points, it is possible to determine the coefficients of a linear fractional transformation up to a common factor.

5.2.4. Preservation of the cross-ratio follows from direct computation; the converse assertion follows the previous problem.

5.2.5. If the triangles are similar, then as the center of inversion we take the center of the circumscribed circle of triangle ABC. Otherwise, the statement of the problem follows from the existence of a Möbius transformation mapping A, B, C into A', B', C'.

5.2.6. For any complex number there exists a unique (up to similarity) parallelogram, the cross-ratio of the vertices of which is equal to the given number.

5.2.7. a) Let Z_1 and Z_2 be the points corresponding to the roots of the derivative $f'(z)$. Lines AZ_1 and AZ_2 are symmetric with respect to the bisector of angle A if and only if the quantity

$$\frac{z_1 - a}{b - a} \frac{z_2 - a}{c - a}$$

is real. It is easy to verify that this quantity is equal to $1/3$.

b) The Steiner ellipse is tangent to the sides of the triangle at their midpoints. Therefore, it is sufficient to verify that the midpoint of segment AB, the point symmetric to point Z_1 with respect to AB, and point Z_2 are collinear.

Additional reading

For more information about complex numbers and geometry see [**BB73**, **Sko72**].

Chapter 6

Constructions and loci

1. Loci (1)

By A. D. Blinkov

6.1.1.° Consider points $A(1,0)$ and $B(-1,2)$. Find the equations of the loci of points that are

(a) equidistant from A and B;
(b) at a distance of $|AB|$ from A.

6.1.2.° Determine, using equations or inequalities, the loci of points that are

(a) at a distance of more than 0.1 units from the origin;
(b) equidistant from the coordinate axes;
(c) at a distance of less than 1.5 units from the x-axis.

6.1.3.° The locus of points from which a given segment AB of length $\sqrt{2}$ subtends an angle of $45°$ is

(a) the union of two larger arcs of circles of radius 1 passing through the points A and B;
(b) a circle with diameter AB;
(c) the union of two smaller arcs of circles of radius 1 passing through the points A and B;
(d) impossible to determine.

6.1.4.° Given a circle with center O and radius R, the locus of the midpoints of all chords of length $R\sqrt{3}$ is

(a) a circle with center O and radius $\frac{R\sqrt{3}}{2}$;
(b) a chord of the circle at a distance of $\frac{R}{2}$ from the center;
(c) a circle with center O and radius $\frac{R}{2}$;
(d) impossible to determine.

6.1.5.° Given a circle with center O and radius R and a point M on this circle, the locus of the midpoints of all chords passing through M is

(a) a circle with center O and radius $\frac{R}{2}$;
(b) a circle with diameter OM;
(c) a semicircle with diameter OM;
(d) impossible to determine.

6.1.6. Let O be the center of rectangle $ABCD$. Find the locus of points M satisfying all the inequalities $|AM| \geq |OM|$, $|BM| \geq |OM|$, $|CM| \geq |OM|$, and $|DM| \geq |OM|$. (See **[Pra19]**.)

6.1.7. Let O be the center of gravity of the equilateral triangle ABC. Find the locus of points M such that any line drawn through M intersects either $[AB]$ or $[CO]$. (See **[Pra19]**.)

6.1.8. Two circles with equal radii intersect at two points, A and B. A line passing through B intersects these circles again at X and Y (distinct from A). Find the locus of the midpoints of the segments XY. (See **[Fom94]**.)

6.1.9. Find the locus of the midpoints of all segments whose endpoints lie on different diagonals of a square and are distinct from its center.

6.1.10. Let AB be a chord of a circle that is not a diameter. Consider all possible triangles ABC inscribed in this circle. Find the locus of the intersection points of the

(a) heights; (b) angle bisectors of ABC. (See **[Pra19]**.)

6.1.11. Let $ABCD$ be a square and let P be a point that is distinct from A and B. Lines AP and BD intersect at Q, and the line that passes through Q parallel to AC intersects line BP at M. Find the locus of M as P moves along the circumscribed circle of square $ABCD$. (See **[Pra19]**.)

6.1.12. Let A and B be distinct points in the plane and let α, β, $\gamma \in \mathbb{R}$. Find the locus of points X such that $\alpha|AX|^2 + \beta|BX|^2 = \gamma$.

6.1.13. Let $A_1, \ldots, A_n$ be points on the plane and let $\alpha_1, \ldots, \alpha_n, c \in \mathbb{R}$. Consider points X in the plane such that

$$\alpha_1|A_1X|^2 + \ldots + \alpha_n|A_nX|^2 = c.$$

Prove that their locus is of one of the following types, and classify the cases:

- circle;
- line;
- point;
- all points of the plane;
- empty set.

Suggestions, solutions, and answers

6.1.6. *Answer*: The boundary and interior of the rhombus $NKPL$, the sides of which lie on the perpendicular bisectors of OA, OB, OC, and OD (see Fig. 1).

The point M satisfies the required conditions if and only if it lies in the same half-plane as O with respect to each of the indicated perpendiculars.

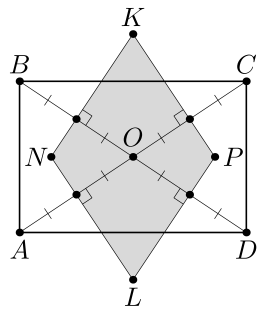


FIGURE 1

6.1.7. *Answer*: Quadrilateral $OKCP$, where K and P are the projections of O onto BC and AC, respectively (see Fig. 2).

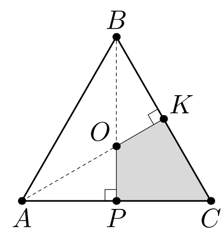


FIGURE 2

Let M lie in the shaded region. Then it lies inside or on the boundary of one of the triangles OCP and OCK. For example, if M lies in OCP and the line passing through it does not intersect the segment CO, then this line intersects CP and PO. Thus this line does not intersect AP, but it intersects side BP of triangle ABP and therefore must intersect side AB.

For any point outside the shaded region, it is easy to give an example of a line passing through it which intersects neither $[AB]$ nor $[CO]$.

6.1.8. *Answer*: The circle with diameter AB, excluding A and the midpoints of the chords cut on each of the circles by the tangent at point B to the other circle.

There are essentially two possible configurations for the points B, X, and Y: $B \in [XY]$ (see Fig. 3 *a*) or $X \in [BY]$ (see Fig. 3 *b*). In both cases $\triangle XAY$ is isosceles, since $\angle AXY = \angle AYX$. In the first case, these angles are inscribed and intercept identical arcs in congruent circles. In the second case, $\angle AXY = 180° - \angle AXB = \angle AYX$, since the inscribed angles AXB and AYX subtend complementary arcs.

Thus, M is the midpoint of $[XY]$ if and only if $\angle AMB = 90°$; equivalently, M lies on the circle with diameter AB.

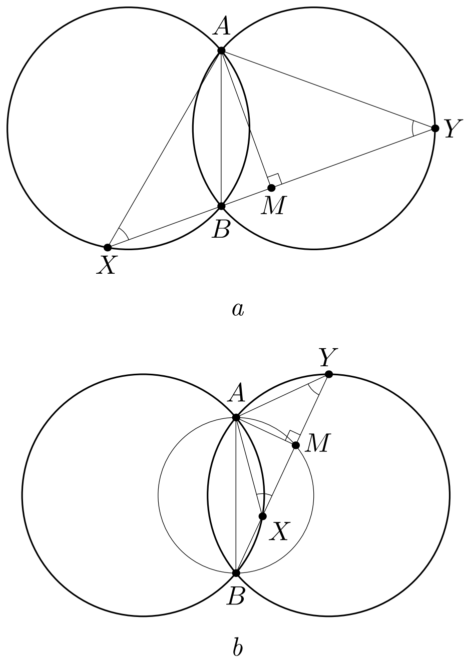


FIGURE 3

6.1.9. *Answer*: The boundary and interior of square $MKPL$ whose vertices are the midpoints of the sides of the given square, excluding the points lying on the diagonals of the given square (see Fig. 4 *a*).

Let $ABCD$ be the given square, with diagonals intersecting at O. Consider the Cartesian coordinate system defined by the base vectors $\overrightarrow{OA}$ and $\overrightarrow{OD}$ (see Fig. 4 *b*). Let $E \neq O$ lie on the segment AC; thus E has coordinates $(x;0)$, where $x \in [-1,1]$, $x \neq 0$. Similarly, if $F \neq 0$ belongs to segment BD, then F has coordinates $(0,y)$, where $y \in [-1,1]$, $y \neq 0$. If N is the midpoint of EF, then N has coordinates $(0.5x, 0.5y)$, and we have $-0.5 \leq 0.5x \leq 0.5$ and $-0.5 \leq 0.5y \leq 0.5$. These inequalities define a unit square with center O, the sides of which are parallel to the axes.

Let H be an arbitrary point of this square that does not lie on the coordinate axes (see Fig. 4 *c*). Draw ray OH and let G be a point on this ray satisfying $|GH| = |OH|$. If $G = (x;y)$, then $x \in [-1,1]$ and $y \in [-1,1]$. Drop perpendiculars GE and GF from G to the coordinate axes; then H is the midpoint of EF, where $|OE| \leq 1$ and $|OF| \leq 1$. Thus, H is the midpoint of the segment whose ends lie on the diagonals of the given square.

6.1.10. (a)*Answer*: A circle that is symmetric to the given one with respect to AB, excluding those points lying on lines passing through A or B and perpendicular to AB.

(b)*Answer*: Two arcs of circles lying inside the given circle, from which AB is visible at an angle of $90° + \frac{\alpha}{2}$ or $180° - \frac{\alpha}{2}$, where the smaller of arcs AB subtend the angle 2α.

First notice that for any location of the point C (except at A or B), angle ACB is inscribed in the circle and intercepts one of the arcs AB; therefore $\angle ACB = \alpha$ (see Fig. 5 *b*, *c*) or $\angle ACB = 180° - \alpha$ (see Fig. 5 *a*, *d*).

(a) The point symmetric to the orthocenter H with respect to AB lies on the circumcircle of triangle ABC (see Problem 1.5.6 in Chapter 1). Consequently, the orthocenter lies on a circle that is symmetric to the given one with respect to AB. This circle can be obtained from the given one also by translating by a vector perpendicular to AB of length $2R|\cos\alpha|$, where R is the radius of the given circle. Since the center of the given circle is the orthocenter of the medial triangle (see section 3 in Chapter 1), we have $CH = 2R|\cos\alpha|$; i.e., this translation takes C to H (see Fig. 5 *a*, *b*). Consequently, the desired locus contains any point on the circle that is symmetric to the given one with respect to AB, excluding the points lying on straight lines passing through A or B and perpendicular to AB.

(b) The measure of angle AOB between the angle bisectors BAC and ABC of the given triangle is seen to be (see Fig. 5 *a*, *d*)

$$\angle AOB = 180° - 0.5 \cdot (\angle BAC + \angle ABC) = 90° + 0.5\,\angle ACB,$$

so $\angle AOB = 90° + \frac{\alpha}{2}$ or $\angle AOB = 180° - \frac{\alpha}{2}$.

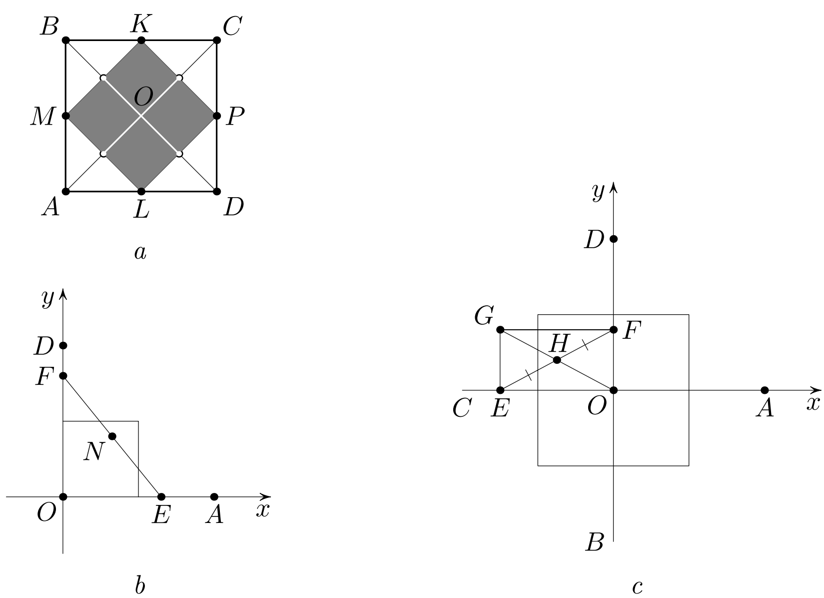


FIGURE 4

Therefore, the desired locus is the set of points from which segment AB is visible at these angles, i.e., all points of arcs of the corresponding circles, excluding points A and B.

6.1.11. *Answer*: Line DC, excluding the point symmetric to D with respect to C.

Let $O = AC \cap BD$ and let M be a point in the locus (see Fig. 6 a). Consider one possible configuration. Since $\angle BOC = 90°$ and $QM \parallel AC$, we have $\angle MQD = 90°$. Angle DPB is inscribed in the circle and subtends its diameter BD, so $\angle DPM = \angle DPB = 90°$. Therefore, points P and Q lie on a circle with diameter DM, so $\angle QDM = 180° - \angle QPM = \angle BPA = \angle BCA = 45°$. Thus, M lies on ray DC. For other configurations of points, M lies on the ray complementary to the ray $[DC)$ (see Fig. 6 b).

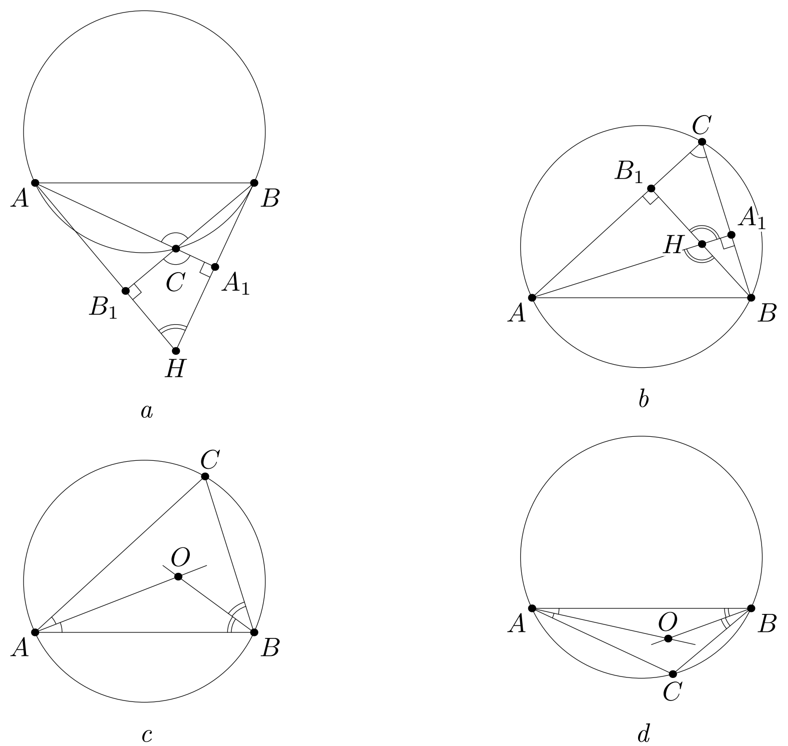
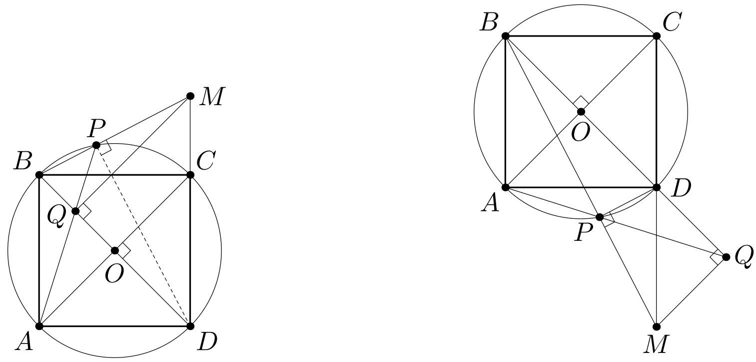


FIGURE 5

Conversely, let $M \in DC$, let the line BM intersect the given circle at point P, and let $AP \cap BD = Q$ (see Fig. 6 a, b). We will prove that $MQ \parallel AC$. Since $\angle BPA = \angle BCA = 45°$ or $\angle BPA = 135°$, we get that $\angle QPM = 135°$ (see Fig. 6 a) or $\angle QPM = 45°$ (see Fig. 6 b). In both cases, points P, M, D, and Q lie on the same circle, since $\angle QPM + \angle QDM = 180°$ or $\angle QPM = \angle QDM$, with $\angle DPM = \angle DPB = 90°$; therefore $[DM]$ is its diameter. So $\angle MQD = 90°$; i.e., $MQ \parallel AC$. This construction can be done for all points except the point symmetric to point D with respect to C.



a b

FIGURE 6

2. Construction and loci problems involving area (1) *By A. D. Blinkov*

When solving construction problems, it is important to clearly distinguish the construction itself (algorithm) from the proof that it works. In the problems below, try to avoid "computational" methods.

6.2.1.° Which segment divides an arbitrary triangle into two parts of equal area?
- (a) A midline;
- (b) an angle bisector;
- (c) a height;
- (d) a median;
- (e) any segment passing through a vertex;
- (f) impossible to determine.

6.2.2.° The diagonals of a convex quadrilateral cut it into four triangles with equal areas. Determine the type of this quadrilateral:
- (a) a parallelogram;
- (b) a rectangle other than a square;
- (c) a rhombus other than a square;
- (d) a square;
- (e) a trapezoid;
- (f) impossible to determine.

6.2.3.° Given segment AB, determine the locus of points M for which triangle AMB has a given area S:
- (a) a straight line parallel to AB;
- (b) a segment parallel to AB;
- (c) a circle with diameter AB;
- (d) the union of two segments parallel to AB;

(e) the union of two straight lines parallel to AB;
(f) impossible to determine.

6.2.4.° On the leg CD of trapezoid $ABCD$, point K is chosen so that the area of the triangle AKB is equal to half the area of the trapezoid. Then $|CK| : |KD|$ equals

(a) $1:2$; (b) $2:1$; (c) $1:1$; (d) $1:3$; (e) $3:1$;
(f) impossible to determine.

6.2.5. The lengths of the bases of a trapezoid are a and b. Find the length of the segment parallel to the bases of the trapezoid that divides it into two parts of equal area.

6.2.6. Given a point on a side of a triangle, construct a straight line passing through this point that divides the triangle into two parts of equal area. (See [**Got96a**].)

6.2.7. Determine the locus of points M inside triangle ABC such that $S_{ACM} + S_{BCM} = S_{ABM}$. (See [**Got96a**].)

6.2.8. (a) Determine the locus of points M lying in the plane of triangle ABC such that $S_{ACM} = S_{BCM}$. (See [**Got96a**].)

(b) Determine the locus of points M lying in the plane of triangle ABC such that $S_{ACM} = S_{BCM} = S_{ABM}$.

6.2.9. Prove that any line that halves both the area and the perimeter of a triangle or of a tangential polygon passes through the center of the inscribed circle. Describe a method for constructing such a line for a triangle. (See [**Pra19**].)

6.2.10. (a) Inside convex quadrilateral $ABCD$, determine at least one point M such that the piecewise-linear path AMC cuts the quadrilateral into two parts of equal area. (See [**Got96a**].)

(b) Find the straight line through a vertex of a convex quadrilateral cutting it into two parts of equal area. (See [**Got96a**].)

(c) A convex figure is bounded by angle ABC and arc AC. Find a line cutting it into two parts of equal area. (See [**Got96a**].)

6.2.11. Let P lie in the interior of parallelogram $ABCD$. Find the locus of all points Q on the boundary of the parallelogram such that the piecewise-linear path APQ cuts it into two parts of equal area. (See [**Got96a**].)

6.2.12. (a) Find points M in the interior of trapezoid $ABCD$ with bases AD and BC such that $S_{ADM} + S_{BCM} = S_{ABCD}/2$. (See [**Got96a**].)

(b) Find the locus of all points M in the interior of convex quadrilateral $ABCD$ such that $S_{ABM} + S_{CDM} = S_{ADM} + S_{BCM}$.

Suggestions, solutions, and answers

6.2.6. *Path to solution.* Let point K be given on side AB of a triangle ABC (see Fig. 7). Consider the median CM. If K coincides with M, then CM is the required line.

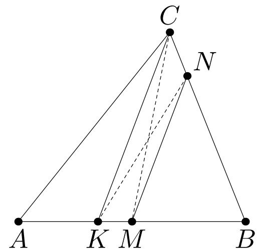


FIGURE 7

If K and M are distinct, then either $|AK| < |BK|$ or $|AK| > |BK|$. Suppose, for example, $|AK| < |BK|$; then $S_{AKC} < S_{BKC}$. To satisfy the condition of the problem, it is necessary to draw a line through point K such that the area of triangle CKM is "added" to the area of triangle AKC.

Let KN, where $N \in [BC]$, be the desired line. Then $S_{KCN} = S_{CKM} \iff MN \parallel CK$. Thus it is sufficient to draw a line through M parallel to (CK). In the case where $|AK| > |BK|$, the construction is carried out similarly but with $N \in [AC]$.

6.2.7. *Answer*: The midline of triangle ABC parallel to side AB, excluding its endpoints.

Since M lies inside triangle ABC, the given condition is equivalent to $S_{ABM} = 0.5S_{ABC}$. This holds if and only if the distance from M to AB is half the distance from C to AB (see Fig. 8).

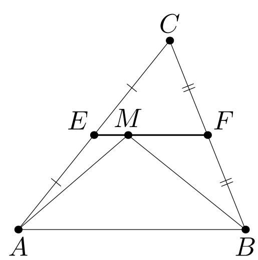
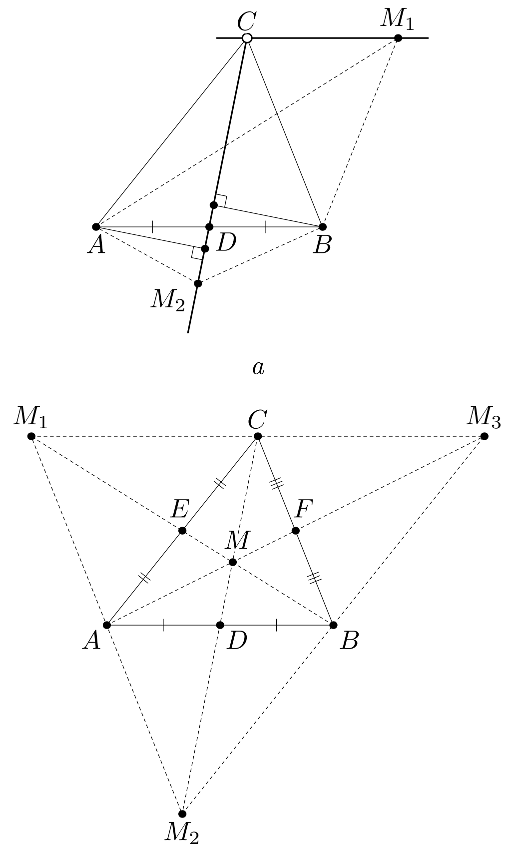


FIGURE 8

6.2.8. (a)*Answer*: The union of two lines passing through point C, one parallel to AB and the other containing median CD.

A point M satisfies the condition if and only if A and B are equidistant from CM. This is possible only in the two cases indicated in the answer



a

b

FIGURE 9

(see Fig. 9 *a*). The fact that this condition is not satisfied for the remaining points of the plane is easily proved by contradiction.

(b) *Answer*: The centroid of the triangle and the three points of pairwise intersection of lines passing through the vertices of the triangle parallel to the opposite sides.

We split the double equality of areas into three equalities and use the locations of points found in (a) (see Fig. 9 *b*). In order for a point lying in the plane of the triangle to satisfy the condition, it is necessary and sufficient to satisfy two equalities of areas. Therefore, all such points are pairwise intersections of the locations found.

6.2.9. Consider triangle ABC and a circle inscribed in it with center O and radius r (see Fig. 10). Suppose that line PQ dividing the area and the perimeter of the triangle in half does not contain point O (let $P \in [AC]$ and $Q \in [BC]$). Connect the center O with the vertices of the triangle and points P and Q. The piecewise-linear path POQ splits ABC into pentagon

$ABQOP$ and quadrilateral $PCQO$. Let us calculate their areas:

$$S_{ABQOP} = S_{AOB} + S_{BOQ} + S_{AOP} = \frac{1}{2}(|AB| + |AP| + |BQ|)r;$$
$$S_{PCOQ} = S_{COQ} + S_{COP} = \frac{1}{2}(|CP| + |CQ|)r.$$

Since PQ halves the perimeter of ABC, we have $|AB| + |AP| + |BQ| = |CP| + |CQ|$; therefore, $S_{ABQOP} = S_{PCOQ}$. Since $S_{ABQP} = S_{PCQ}$, we have $S_{POQ} = 0$, so $O \in PQ$, which was what was required to prove.

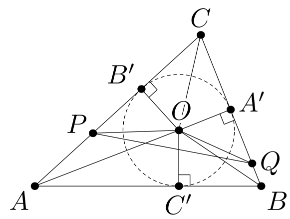


FIGURE 10

For the circumscribed polygon the proof is similar.

6.2.10. (a) *Answer*: For example, M is the midpoint of diagonal BD.

Let M be the midpoint of BD; then median AM divides the area of triangle ABD in half, and median CM divides the area of triangle BCD in half.

Note that any point of a line parallel to AC and passing through M that lies inside this quadrilateral also satisfies the condition.

(b) *Path to solution.* Let $ABCD$ be a convex quadrilateral. Draw diagonals AC and BD, and let O be their intersection point (see Fig. 11 a). Consider M, the midpoint of diagonal BD. If M coincides with O, then AC is the desired line.

If points O and M are distinct, then either $|BO| < |BM|$ or $|BO| > |BM|$. Suppose, for example, $|BO| < |BM|$; then $S_{ABC} < S_{ADC}$. In order for the condition of the problem to be satisfied, it is necessary to draw a line through point A so that the area of triangle ABC is "added" to the area of triangle AMC.

Let AN, where $N \in DC$, be the desired line. Then $S_{CNA} = S_{AMC}$ if and only if $MN \parallel AC$. Thus, it is sufficient to construct a line passing through point M parallel to AC. In the case where $|BO| > |BM|$, the construction is carried out similarly but with $N \in BC$ (see Fig. 11 a).

(c) Let D be the midpoint of arc AC. Connect it to points A and C (see Fig. 11 c). The resulting circular segments bounded by the chords AD and CD are equal; therefore, they have equal area. Then the problem is reduced to constructing a line passing through D that divides quadrilateral $ABCD$ into two parts of equal area.

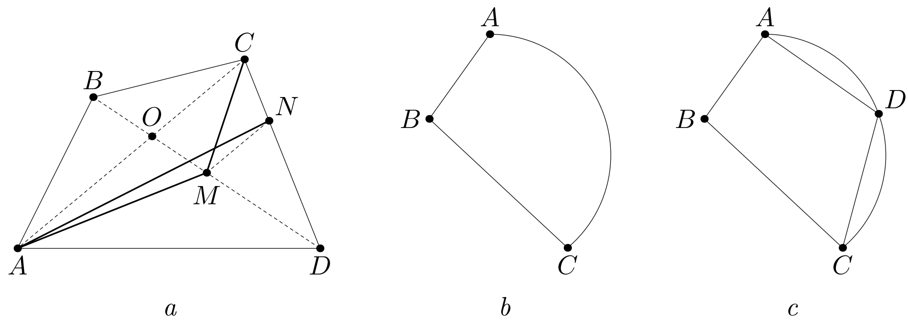


FIGURE 11

6.2.11. *Construction.* Draw line CP. If it intersects AD, denote the intersection point by K (see Fig. 12) and draw a line through A parallel to CK, which intersects BC at Q. If CP intersects AB, then construct similarly the point $Q \in CD$; if P lies on AC, then set $Q = C$. The point Q is the desired point.

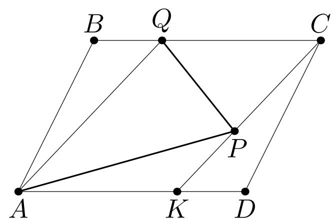


FIGURE 12

Proof. Quadrilateral $AKCQ$ is a parallelogram (degenerate for $P \in AC$); therefore $S_{AQP} = \frac{1}{2}S_{AKCQ}$. Since $\triangle ABQ$ and $\triangle CDK$ are congruent, they have equal area.

6.2.12. (a) *Answer*: The midline of the trapezoid, excluding its endpoints.

Let $|AD| = a$ and $|BC| = b$. The point M, which lies within the given trapezoid at distances m and n from AD and BC, respectively, satisfies the condition if and only if $0.5am + 0.5bn = 0.25(a+b)(m+n) \iff am + bn = an + bm \iff (a-b)(m-n) = 0$. Since $a \neq b$, we have $m = n$; i.e., M lies on the midline of the trapezoid (see Fig. 13).

(b) *Answer*: If $ABCD$ is a parallelogram, then any interior point; if $ABCD$ is not a parallelogram, then the segment with endpoints on the sides of the quadrilateral that passes through the midpoints of its diagonals, excluding those endpoints.

For the case of a parallelogram, the answer is obvious.

Now suppose that $ABCD$ is not a parallelogram. Let P and Q be the midpoints of diagonals AC and BD (see Fig. 13 *b*). Then $S_{ABP} + S_{CDP} = S_{ABQ} + S_{CDQ} = S_{ABCD}/2$.

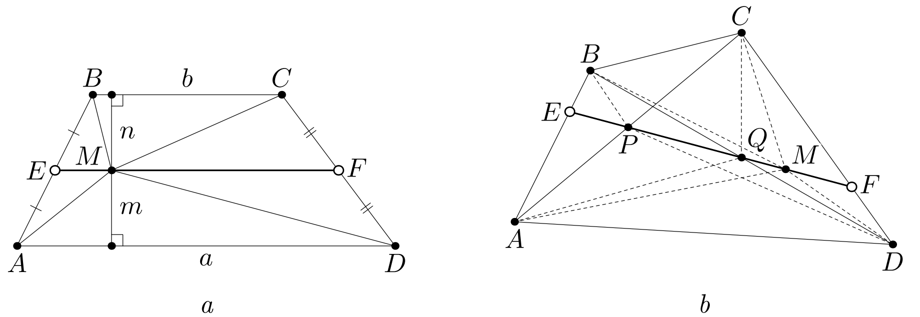


FIGURE 13

If M lies inside $ABCD$ on the line PQ, then $S_{APM} = S_{CPM}$ (since points A and C are equidistant from PM) and $S_{BPM} = S_{DPM}$ (since points B and D are equidistant from PM). Thus, $S_{ABM} + S_{CDM} = S_{ABP} + S_{CDP} + S_{APM} + S_{BPM} - S_{CPM} - S_{DPM} = S_{ABP} + S_{CDP} = S_{ABCD}/2 = S_{ADM} + S_{BCM}$.

If point M does not lie on this segment, then a similar analysis shows that the equality is not satisfied.

3. Construction toolbox (2)
By A. A. Gavrilyuk

For the problems in this section, it is a good idea to be familiar with the material of Chapter 2 ("Circle"), as well the literature recommended in it.

6.3.1. (a) Given two parallel lines and a segment on one of them, bisect this segment using only a straightedge.

(b) Given two parallel lines and a segment on one of them, double this segment using only a straightedge.

(c) Given two parallel lines and a segment on one of them, divide the segment into n equal parts using only a straightedge.

(See Problem 3.9.5 in Chapter 3.)

6.3.2. Given a circle ω, a diameter AB of ω, and a point X, drop a perpendicular from X to AB using only a straightedge, in the case where X lies

(a) not on the circle; (b) on the circle.

6.3.3. Suppose we are given a circle ω and a point X. Using only a straightedge, construct (all possible) tangents from X to the circle in the case where X lies

(a) outside the circle; (b) on the circle.

6.3.4. Using only a compass, locate the image of a given point X under inversion with respect to a given circle ω.

A *two-sided straightedge* is a construction tool that allows one to

- draw a line through two points;
- draw a line parallel to one drawn previously which is separated from the original line by a distance equal to the width of the straightedge;
- draw two parallel lines the width of the straightedge apart through two points separated by a distance not less than the width of the straightedge.

6.3.5. Given a circle in the plane, use a two-sided straightedge to locate its center.

6.3.6. Given line l and segment OA parallel to it, use a two-sided straightedge to locate the point of intersection of l with a circle of radius OA that is centered at O.

6.3.7. Using only a compass, construct a circle passing through three given points.

6.3.8. Apollonius's problem. Construct a circle tangent to three given ones, using only a compass and a straightedge.

In the following theorems, by *construction* we mean some sequence of the following elementary operations.

- Use a straightedge to draw a line through two given or previously constructed points.
- Use a compass to draw a circle with center A and radius BC, where A, B, C are given or previously constructed points.
- Find the point of intersection of two given or previously constructed lines or circles.

Note that no other operations are allowed in these theorems (in contrast to the previous problems, where, for example, the operation "take an arbitrary point of the already constructed set" is allowed). In particular, if fewer than two points are initially given, nothing can be constructed.

6.3.9.* Theorem. A segment of length a can be constructed with a compass and a straightedge, given a segment of length 1, if and only if the number a can be obtained from 1 by additions, subtractions, multiplications, divisions, and taking square roots (of positive numbers).

Note: Cf. the fundamental theorem on constructibility in subsection 1G of Chapter 8 in [**SkoA**].

6.3.10.* Theorem (Mohr–Mascheroni). Any construction that is possible using a compass and a straightedge can be done with just a compass (a straight line is considered to be constructed if two distinct points on it have been constructed). (See [**Fuk87**].)

6.3.11.* **Theorem** (Steiner). Any construction that is possible using a compass and a straightedge can be done with a just a straightedge, if one circle is drawn and its center marked (a circle is considered to be constructed if its center and a point lying on it have been constructed). (See [**Smo56**].)

The next problem will help you to consolidate these ideas.

6.3.12.° Using the Mohr–Mascheroni and Steiner theorems, determine what tools are needed to locate the center of a given circle.

1) A compass and a straightedge;
2) only a straightedge;
3) only a compass.

Suggestions, solutions, and answers

6.3.1. (a) Let AB be the given segment. Take point X outside the region bounded by the two parallel lines, and find the intersection points C and D of lines XA and XB with the line that does not contain AB. Let Y be the intersection of the diagonals of trapezoid $ABCD$. Then line XY divides the base of the trapezoid in half.

(b) Begin by taking an arbitrary segment on the other line and bisecting it.

(c) Take an arbitrary segment on the other line and increase it by n times, repeating the previous construction.

6.3.2. (a) If lines XA and XB also intersect the circle at B' and A', then the intersection point of AA' and BB' is the orthocenter of triangle XAB.

6.3.3. (a) If two lines passing through X intersect the circle at A and B and at C and D, then the line connecting the points of intersection of AC with BD and of AD with BC will be the polar of X.

6.3.4. Let O be center of the given circle. If a circle with center X and radius XO intersects the given circle at A and B, then our goal is the second intersection point of the circles with centers A and B and radii AO and BO, respectively.

6.3.9. It suffices to prove that points obtained as a result of each of the elementary operations satisfy the statement of the theorem.

6.3.10. It suffices to construct the intersection points of lines, each given by two points, with each other and with an arbitrary circle.

6.3.11. It suffices to construct the intersection points of circles, defined by their centers and radii, with each other and with an arbitrary line.

4. Auxiliary constructions (2^*)
By I. I. Shnurnikov

6.4.1. Prove that it is possible to place a circle of radius $\frac{S}{P}$ inside a convex quadrilateral of area S and perimeter P.

6.4.2. Given a convex polygon in which it is not possible to place any triangle with area 1, prove that this polygon can be placed in a triangle with area 4.

6.4.3. In acute triangle ABC, angle bisector AD, median BM, and height CH intersect at a single point. Prove that $\angle BAC > 45°$.

6.4.4. Prove that for any tetrahedron, there exist two planes such that the ratio of the areas of the tetrahedron's projections onto these planes is greater than $\sqrt{2}$.

6.4.5. The convex quadrilateral $ABCD$ is partitioned by its diagonals into four triangles. Prove that if the radii of all four circles inscribed in these triangles are equal to each other, then $ABCD$ is a rhombus.

6.4.6. In tetrahedron $ABCD$, edge AC is perpendicular to BC, and AD is perpendicular to BD. Prove that the cosine of the angle between lines AC and BD is less than $\frac{|CD|}{|AB|}$.

6.4.7. Four different points lie on a line, labeled A, B, C, D from left to right. Prove that for any point E not lying on line AD,

$$|AE| + |ED| + ||AB| - |CD|| > |BE| + |CE|.$$

6.4.8. A *channel* is the set obtained from the first coordinate quadrant by removing the set $\{(x, y) \colon x \geq 1, y \geq 1\}$. What is the maximum diameter of a raft that can make the turn in this channel? (The raft can be curved; its diameter is the maximum distance between two points on it.)

6.4.9. Given three vertices of a quadrilateral that is both inscribed and circumscribed, construct its fourth vertex. (See [**Pra19**].)

6.4.10. On a circle, two points A and B are fixed, and point M freely moves along the circle. Perpendicular KP is dropped from the midpoint K of segment MB to line MA. Prove that all such lines KP pass through one point.

6.4.11. Let ABC be an isosceles triangle with base AC. Drop perpendicular MH from the midpoint M of AC to side BC. Let P be the midpoint of MH. Prove that $AH \perp BP$.

6.4.12. The incircle of triangle ABC is tangent to side AC at point K. Prove that the line connecting the midpoint of AC with the incenter bisects segment BK.

6.4.13. Let O be the intersection point of the diagonals of convex quadrilateral $ABCD$. Prove that the line passing through the intersection points of the medians of triangles AOB and COD is perpendicular to the line passing through the intersection point of the heights of triangles BOC and AOD.

6.4.14. Point M lies in the interior of quadrilateral $ABCD$, and $ABMD$ is a parallelogram. Prove that if $\angle CBM = \angle CDM$, then $\angle ACD = \angle BCM$.

6.4.15. (a) In acute triangle ABC, the longest height AH has the same length as the median BM. Prove that angle ABC is not more than 60 degrees.

(b) In acute triangle ABC, height AH, median BM, and angle bisector CD have equal length. Prove that ABC is equilateral.

Suggestions, solutions, and answers

6.4.1. On each side of the quadrilateral, construct an interior rectangle whose other side has length $\frac{S}{P}$. The total area of these rectangles is S and they intersect, so there is a point not belonging to them inside the quadrilateral.

Note: This problem easily generalizes to the inequality $S \leq P^2/\pi$ for any convex polygon.

6.4.2. Choose the triangle of largest area among all the triangles whose vertices are vertices of the original polygon. Through each vertex of this triangle, draw a line parallel to the opposite side of the triangle. These three lines form the desired triangle.

6.4.4. (R. Devyatov.) Let us deduce the assertion from the following lemma.

Lemma 4.1. *Given segments XY and ZT in a plane, there exist two lines l_1 and l_2 such that the sum of the lengths of the projections XY and ZT onto l_1 differs from the corresponding sum for l_2 by a factor of at least $\sqrt{2}$.*

Let $ABCD$ be the given tetrahedron. Draw plane α parallel to lines AB and CD through point A. Draw plane $\beta \perp \alpha$ intersecting α along a line l. The area of the projection of tetrahedron $ABCD$ onto β is the product of the distance from the point C to α and the half-sum of the lengths of the projections of the segments AB and CD onto l. By the lemma, as β rotates, this half-sum will change by a factor of at least $\sqrt{2}$.

6.4.5. Reflect the triangle ABC with respect to the intersection point of the diagonals of $ABCD$.

6.4.7. Let $|AB| > |CD|$. Reflect points E and C with respect to the midpoint of segment AD and denote the reflected points by E_1 and C_1, respectively. Assume that the extension of BE intersects the segment E_1C_1 at F.

We have $|FC_1| < |FB| + |BC_1|$. Substituting $|BC_1| = ||AB| - |CD||$ we get

$$\begin{aligned} |BE| + |CE| &= |BE| + |FC_1| + |E_1F| \\ &< |BE| + |FB| + ||AB| - |CD|| + |E_1F| \\ &= ||AB| - |CD|| + |E_1F| + |FE| \\ &< ||AB| - |CD|| + |E_1A| + |AE|. \end{aligned}$$

6.4.8. *Answer*: The maximum raft diameter is $2 + 2\sqrt{2}$; for example, a 90° arc of a circle of radius $2 + \sqrt{2}$.

6.4.9. Let A, B, and C be the three given vertices of the quadrilateral $ABCD$ (see Fig. 14). Then the desired point D lies on the circumcircle of ABC. Furthermore, $|AD|+|BC| = |AB|+|CD|$. Without loss of generality,

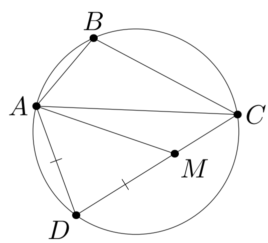


FIGURE 14

assume that $|AB| \le |BC|$; then $|AD| \le |CD|$, and there exists a point $M \in [CD]$ such that $|MD| = |AD|$. In triangle CAM we know the quantities $|AC|$, $|MC| = |CD|-|AD| = |BC|-|AB|$, and $\angle AMC = 90^\circ+\angle ADC/2 = 90^\circ+(180^\circ-\angle ABC)/2 = 180^\circ-\angle ABC/2$. Therefore, using these quantities, this triangle can be constructed (on two sides and an angle opposite to one of them), and the desired point D lies on ray $[CM)$.

Thus, D is the intersection point of the extension of the side CM of the auxiliary triangle CAM with the circumcircle of ABC.

6.4.10. The perpendicular drawn from point M to line AM intersects the circle at a fixed point.

6.4.11. If K is the projection of A onto BC, then AH and BP are medians in similar triangles AKC and BHM.

6.4.12. Connect vertex B to the point of tangency of the opposite side with the corresponding excircle.

6.4.13. Draw perpendiculars from A and C to BD and from B and D to AC. These lines form a parallelogram, similar to a parallelogram whose vertices are the midpoints of the sides of the quadrilateral.

6.4.14. The circumcircles of triangles ABC, BMC, CMD, and ADC are congruent.

6.4.15. (a) The assumptions of the problem imply that $\angle MBC = 30°$ and $AB \geq BC$.

Additional reading

For more information about constructions and loci see [**Fuk87**, **Smo56**].

Chapter 7

Solid geometry

Foreign land is akin to homeland
As much as open space adjoins a dead end.

I. Brodsky.

1. Drawing (2)
By A. B. Skopenkov

The subject of mathematics is that serious so that it is useful not to lose an opportunity to make it somewhat entertaining.

B. Pascal.

7.1.1. A cube with edge length 3 is divided into 27 unit cubes. Draw

(a) a hedgehog (i.e., the union of the central unit cube and the unit cubes having a common face with it);

(b) what is obtained when a hedgehog is removed from the original cube;

(c)* what is obtained by removing the corner unit cubes from the original cube.

7.1.2. Is it possible to fill space with non-intersecting hedgehogs?

7.1.3. The intersection of a cube with a plane is a regular polygon. How many sides can this polygon have?

7.1.4. (a) Draw the union of a cube with its image under rotation by $\pi/3$ about the long diagonal.

(b) Draw the union of a tetrahedron with its image under rotation by $\pi/2$ about a bimedian (a line connecting the midpoints of opposite edges).

7.1.5. On a plane stand a cube and the frame of a triangular pyramid whose height is greater than the height of the cube. Draw the shadow of the pyramid frame on the cube if the light beam is parallel to the line connecting the top vertex of the pyramid with the center of the top face of the cube.

7.1.6. (a) Draw a solid object whose projections onto three mutually orthogonal planes are a triangle, a square, and a circle. (All three images are "filled", i.e., two-dimensional.)

(b) The projections of a spatial figure onto two intersecting planes are straight lines. Is this figure necessarily a straight line?

7.1.7. (a, b) Figure 1 a and b show the views from the tops of two polyhedra (there are no invisible edges). Are such polyhedra possible?

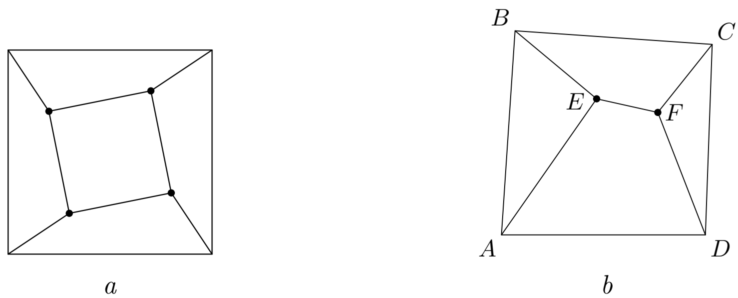


FIGURE 1

(c) In Fig 1b, what conditions must the points E and F inside convex quadrilateral $ABCD$ satisfy so that this can be a top view of some polyhedron?

Suggestions, solutions, and answers

7.1.3. *Answer*: 3, 4, or 6.

See Fig. 2 (drawings by O. Yafrakova).

2. Projections (2)
By M. A. Korchemkina

If you feel you are in a black hole,
don't give up. There's a way out.
S. Hawking.

2.A. Projections of figures constructed from cubes

A *toy* is a figure glued from cubes of the same size (glued face-to-face) such that from each cube one can get to any other by moving from cube to cube only through glued faces.

7.2.1. (a) Construct a toy given three projections: from the front (Fig. 3, upper left), from above (Fig. 3, lower left), and from the right (Fig. 3, upper right).

(b) (Challenge.) Is your construction unique?

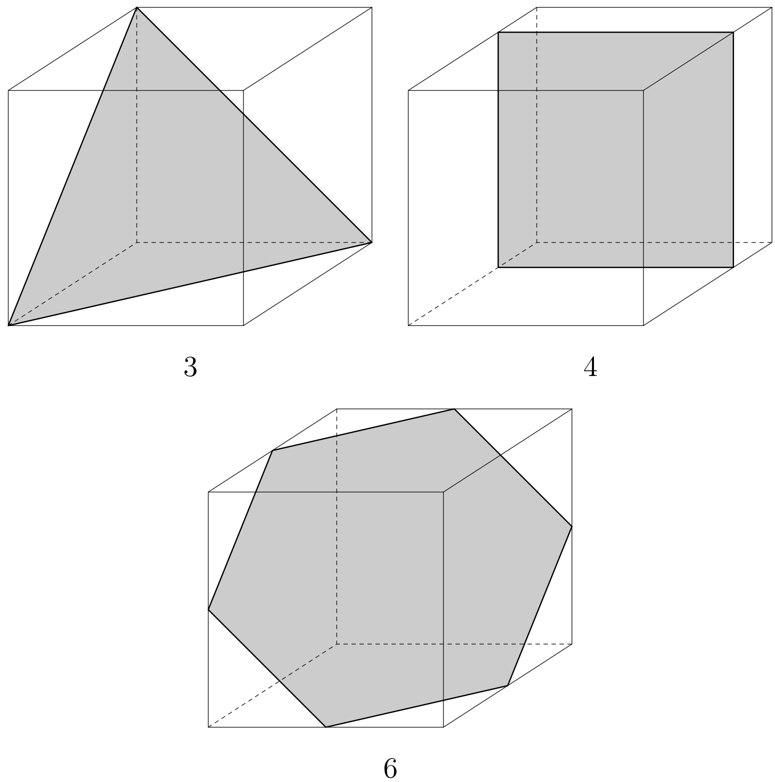


FIGURE 2

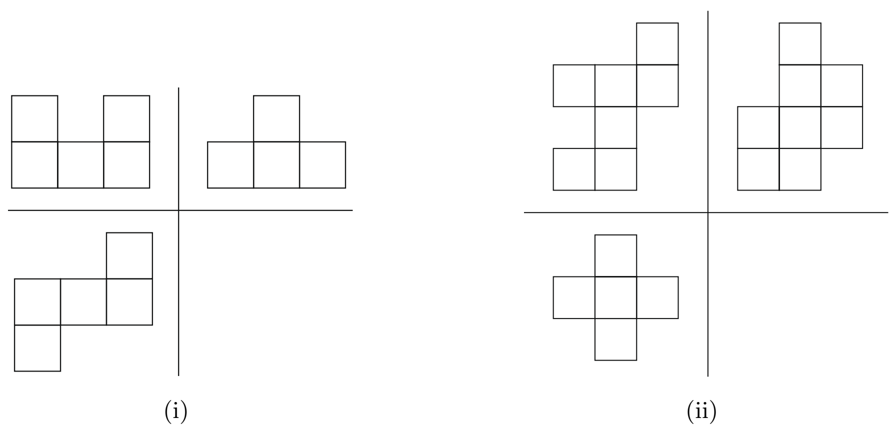


FIGURE 3

7.2.2. (a) Select three of the figures in Fig. 4 which can be projections (front, top, and right views) of one toy (the figures cannot be rotated), and draw the reconstructed solid.

(b) Find a triple of figures in Fig. 4 that cannot be obtained as projections of one toy. Explain.

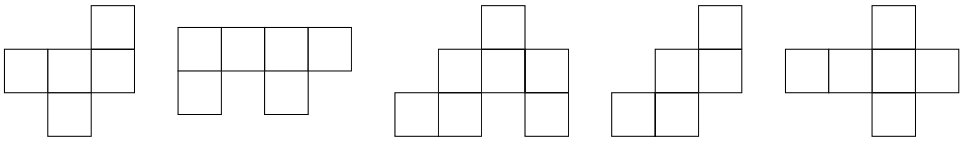


FIGURE 4

7.2.3. Does there exist a toy that is not a cube and whose three projections (front, top, and right views) are congruent squares?

2.B. Trajectories

7.2.4. (a, b) A line was drawn on the surface of a glass cube (Fig. 5). Imagine a snail crawling on the surface of the cube along this path and leaving a visible slime trace. Draw its front, top, and right views.

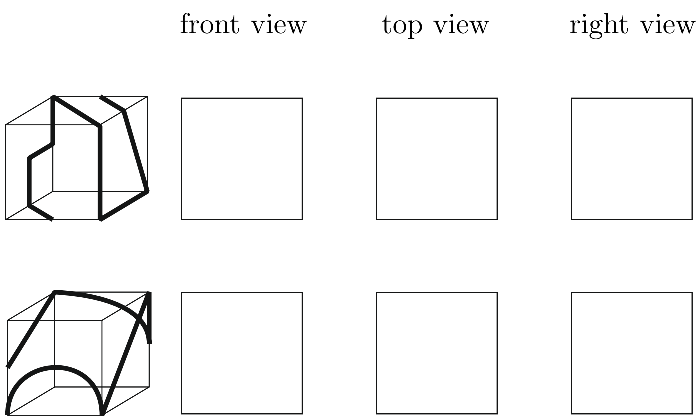


FIGURE 5

7.2.5. Given the front, top, and right views, reconstruct the line passing along the surface or inside the cube shown in Fig. 6. (Imagine a fish swimming in a cubic aquarium completely filled with water.)

7.2.6. Two different paths of a fish can have the same projections on the front and side faces of the aquarium (Fig. 7).

(a) Construct two other paths that yield the same projections.

(b) Reconstruct the trajectory of the fish inside the aquarium, given the projections shown in Fig. 8.

(c) Draw at least two different paths having the two projections shown in Fig. 9.

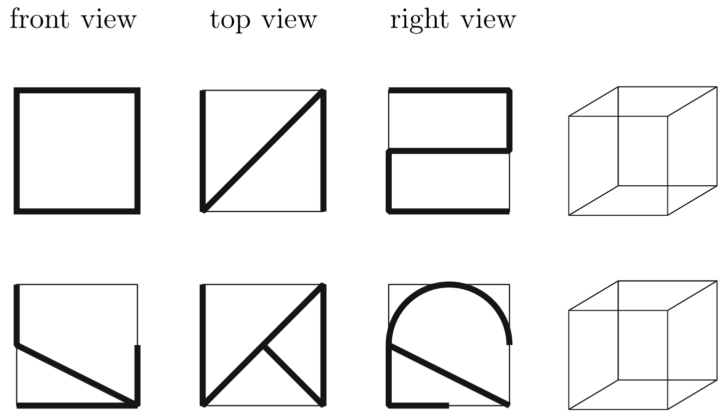


FIGURE 6

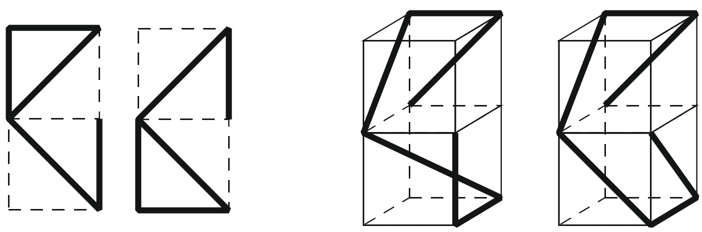


FIGURE 7

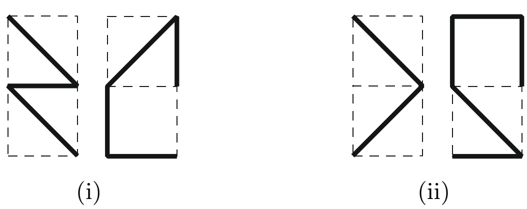


FIGURE 8

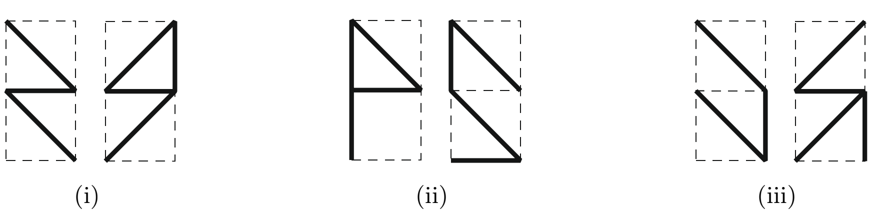


FIGURE 9

7.2.7. (a) (Challenge.) A snail crawls on the surface of a glass parallelepiped of size $1 \times 1 \times 2$, leaving a visible slime trail. Projections of this trail onto the front, right, and upper sides of the parallelepiped are drawn. Is it true that the snail's path can always be uniquely reconstructed by any pair of these projections?

(b) Does the answer change if in (a) the parallelepiped is replaced by a unit cube?

3. Regular polyhedra (3)

3.A. Inscribed and circumscribed polyhedra
By A. Ya. Kanel-Belov

A convex polyhedron is a bounded solid that is the intersection of a finite number of half-spaces (a *half-space* is a region of space lying on one side of a plane).

A regular polyhedron is a convex polyhedron for which all faces are congruent regular polygons and the polyhedral angles at all vertices are equal. It can be proven (see, for example, [**RSG$^+$16**, § 2.4]) that the following are the only possible regular polyhedra:

- the (regular) *tetrahedron*, bounded by four equilateral triangular faces;
- the *cube*, bounded by six square faces;
- the (regular) *octahedron*, bounded by eight equilateral triangular faces;
- the (regular) *dodecahedron*; bounded by twelve regular pentagonal faces;
- the (regular) *icosahedron*; bounded by twenty equilateral triangular faces.

The existence of a regular tetrahedron and a cube is obvious. The existence of the other regular polyhedra is discussed in this section.

The uniqueness of regular polyhedra of each of the five types up to similarity is discussed in this subsection.

7.3.1. (a) The octahedron exists.

(b) The centers of the faces of a regular polyhedron are the vertices of another regular polyhedron.

This new polyhedron is called the *dual* to the original one.

It is easy to see that the tetrahedron is dual to itself, and that the octahedron and cube are dual to each other.

7.3.2. (a) A regular tetrahedron can be inscribed[1] in the cube in two different ways; i.e., there are two different ways to select four vertices of the cube that form a regular tetrahedron. (Cube vertices are numbered from 1 to 8, and

[1] *Editor's note:* The term "inscribed" is usually applied to circles or spheres. Here it is used somewhat informally to indicate that one polyhedron lies inside another, with its vertices either coinciding with vertices or belonging to faces of the outer polyhedron. Conversely, the authors use the term "circumscribed" to indicate that one polyhedron encloses a smaller polyhedron, with the latter inscribed in the former.

tetrahedra are considered different if the sets of numbers of their vertices are different, even if one of them can be transformed to another by a rotation.)

(b) The intersection of these two tetrahedra is an octahedron.

In the problems below, the term "different" is understood in the same way as in Problem 7.3.2 (a).

7.3.3. (a) A regular tetrahedron can be circumscribed in two different ways around an octahedron so that all vertices of the octahedron lie on the faces of the tetrahedron.

(b) Both of these tetrahedra can be inscribed in a cube so that the midpoints of the cube faces are the vertices of our octahedron.

7.3.4. The faces of an octahedron are painted black and white so that any two adjacent faces are of different colors. Prove that the sum of the distances from any point inside the octahedron to the black faces is equal to the sum of the distances to the white ones.

7.3.5. (a) Let $ABCD$ be a square. Prove that there exist regular pentagons $AEFBX$ and $CFEDY$ in space.

(b) Prove that $\angle AED = 108°$.

(c) The dodecahedron exists.

(d) A cube can be inscribed in a dodecahedron in five different ways.

7.3.6. (a) The icosahedron exists.

(b) An octahedron can be circumscribed around an icosahedron in five different ways.

7.3.7. (a) A regular tetrahedron can be inscribed in a dodecahedron in 10 different ways.

(b) The intersection of these tetrahedra is the icosahedron.

7.3.8. (a) The faces of an icosahedron can be colored in two ways using five colors such that no face shares any point (including vertices) with a face of the same color.

(b) Extending the faces that are the same color in (a) will form a regular tetrahedron.

(c) The sum of the distances from each point inside the icosahedron to the faces of the same color does not depend on the choice of color.

7.3.9. The dodecahedron and the icosahedron are dual to each other.

7.3.10. The five regular polyhedra are unique up to similarity.

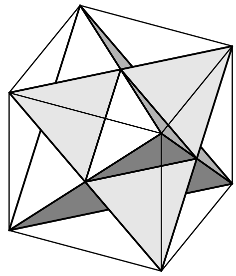


FIGURE 10

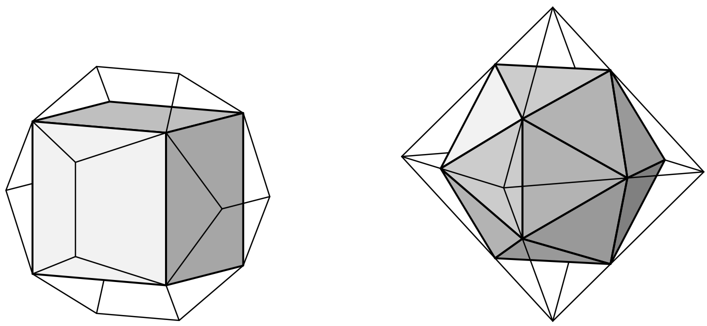


FIGURE 11

Suggestions, solutions, and answers

7.3.2. (a) See Fig. 10.

7.3.5. (d) For each edge of the cube, one can construct a regular pentagon for which this edge is a diagonal; these twelve pentagons are the faces of the dodecahedron (Fig. 11, left).

7.3.6. (a) On each edge of the octahedron, one can choose a point so that these twelve points are the vertices of the icosahedron; see Fig. 11, right. For ideas towards another solution see [**ST04**, Fig. 7a].

3.B. Symmetries
By A. B. Skopenkov

Werde der du bist.
J. W. Goethe.

Become yourself.
A. A. Tarkovsky.

7.3.11. Find all rotations and reflections that transform the set of vertices into itself for

(a) a 1×2 rectangle;

(b) a square;

(c) an equilateral triangle;
(d) a regular n-gon.

A *symmetry* of a figure (i.e., of a set of points) M is an isometry f for which $f(M) = M$.

7.3.12. (a) The number of symmetries (of the set of vertices) of an equilateral triangle is no more than six.

(b) Construct a one-to-one correspondence (bijection) between the set of symmetries of a regular triangle and the set of permutations of the three-element set. Prove that this is a bijection and that it "preserves the composition."

7.3.13. (a) Find all translations of space that transform a given cube into itself.

(b) Do the same for central symmetries.
(c) Do the same for mirror symmetries.
(d) Do the same for axial symmetries.
(See the definitions in sections 1 and 3 of Chapter 3.)

7.3.14. (a) The axis of any spatial rotation that transforms a regular polyhedron to itself passes either through a vertex, through the center of an edge, or through the center of a face.

(b) Find all spatial rotations (see the definition in section 3 of Chapter 3) that transform a given cube into itself.
(c) Do the same for a regular tetrahedron.
(d) Do the same for the a regular octahedron.
See also Problem 3.1 in section 3 in Chapter 4 of [**SkoA**].

7.3.15. Let A, B, C, D be non-coplanar points in space, and let f and g be motions.

(a) If $f(A, B, C, D) = (A, B, C, D)$, then f is the identity function.
(b) If $f(A, B, C, D) = g(A, B, C, D)$, then $f = g$.

7.3.16. (a) Construct a one-to-one correspondence between the set of symmetries of a regular tetrahedron and the set of permutations of the four-element set. Prove that this is a bijection and that it "preserves the composition."

(b)* Do the same for the union of edges and for the union of faces of a regular tetrahedron.

7.3.17. For regular polyhedra, their duals have the same number of symmetries.

7.3.18. Find all symmetries of the set of vertices of
(a) a cube;
(b) a rectangular parallelepiped different from a cube;

(c) a pyramid whose base is a regular n-gon;
(d) a regular octahedron;
(e)* a regular dodecahedron;
(f)* a regular icosahedron.

For each polyhedron and type of symmetry, draw a picture (beautiful color images are welcome!), specify the permutation of the vertices of the polyhedron "realized by" this symmetry, and state the number of symmetries of this type.

7.3.19. For each of (a), (b), (c), (d), (e*), (f*) in Problem 7.3.18, construct a bijection that preserves the composition between each set of symmetries and some subset of permutations of the n-element set.

7.3.20. (a) Find two rotations of a regular dodecahedron whose compositions can be used to obtain any other.

(b) Construct a bijection preserving the composition between the set of rotations of a dodecahedron and the set of even permutations of a set of five elements.

4. Higher-dimensional space (4*)
By A. Ya. Kanel-Belov

4.A. Simplest polyhedra in higher-dimensional space
By Yu. M. Burman and A. Ya. Kanel-Belov

It is well known that every point of the plane can be associated with a pair of numbers, i.e., its Cartesian coordinates (for this you need to first select a coordinate system, that is, an origin and a set of axes). Thus, the plane can be understood simply as the set of all possible pairs (x_1, x_2) of real numbers. Similarly, three-dimensional space can be regarded simply as the set of all possible triples (x_1, x_2, x_3). By imposing various restrictions on these numbers, we can describe various subsets of the plane and space.

7.4.1.° Consider the following three sets of conditions on numbers x_1, x_2, x_3:
1) $x_1 = x_2 = 2x_3$;
2) $x_1 + 2x_2 + 3x_3 = 0$, $3x_1 + 2x_2 + x_1 = 1$;
3) $x_1^2 + x_3^2 - 2x_3 = -1$.

which of these defines a line in three-dimensional space?

When there are more than three dimensions, the coordinate approach dominates: it is convenient to *define*, say, four-dimensional space as the set of all possible collections (x_1, x_2, x_3, x_4) of four real numbers.

In this subsection, the (unit) *interval* is defined to be the set $[-1, 1] = \{x\colon |x| \leq 1\}$ of numbers with absolute value not exceeding 1; the *square* is the set $[-1, 1]^2 = \{(x_1, x_2)\colon |x_1|, |x_2| \leq 1\}$ of pairs of numbers each with absolute value not exceeding 1; the *cube* is the set $[1, 1]^3 = \{(x_1, x_2, x_3)\colon |x_1|, |x_2|, |x_3|$

$\leq 1\}$; and the *four-dimensional cube* is the set of 4-tuples of numbers whose absolute value does not exceed 1, etc. See Fig. 12; also cf. [**DR86**, **Gal15**].

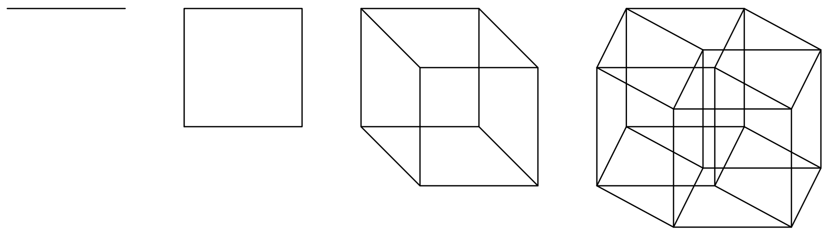


FIGURE 12. Cubes in dimensions 1, 2, 3, and 4

7.4.2. (a) Label the coordinates of the vertices in each figure.

(b) How many vertices does the n-dimensional cube have?

(c) Which vertices of the n-dimensional cube are connected by an edge, and which are not?

A convex polytope with $n+1$ vertices such that all pairwise distances between vertices are equal is called the regular *n-dimensional tetrahedron* or *simplex*, and a convex polytope with $2n$ vertices at points $(x_1 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_n = 0, x_i = \pm a), i = 1, \ldots, n$, is called a *regular n-dimensional octahedron*. By a *face* of a convex polytope we mean the intersection of the polytope with a *hyperplane* $a_1x_1 + \cdots + a_nx_n + b = 0$, or the polytope itself. Notice that faces can have arbitrary dimension from 0 to n; vertices and edges are now treated as particular cases of faces.

7.4.3. For an n-dimensional cube,

(a) determine the number of edges;

(b) determine the number of two-dimensional faces;

(c) determine the number of k-dimensional faces, where k is an arbitrary number from 0 to n;

(d) determine the total number of faces of all possible dimensions.

7.4.4. Same questions, but for the n-dimensional simplex.

7.4.5. Same questions as in Problem 7.4.3, but for the n-dimensional octahedron.

A point on a line is given by an equation of the form $x = p$. A line in the plane is given by an equation of the form $a_1x_1 + a_2x_2 = p$, where a_1 and a_2 are not both equal to zero. A plane in space is given by an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = p$, where at least one of a_1, a_2, and a_3 is nonzero. A similar equation involving four variables defines a three-dimensional subspace in four-dimensional space, etc.; in general (in n-dimensional space) such an object is called a *hyperplane*. We will be interested in hyperplanes determined by the equations $x_1 + x_2 = 0$, $x_1 + x_2 + x_3 = 0$, etc.

An *orthant* is the part of n-space consisting of all points such that each coordinate has a fixed sign. In the plane we would more precisely say "quadrant" and in three-dimensional space "octant." For example, on the line there are two orthants, the rays $\{x \mid x > 0\}$ and $\{x \mid x < 0\}$; in the plane there are four quadrants, etc. Clearly n-dimensional space contains 2^n orthants.

7.4.6.° How many orthants intersect with the hyperplane $x_1 + \ldots + x_n = 0$?
(a) All;
(b) all but two;
(c) exactly half.

7.4.7. The intersection of the line $x_1 + x_2 = 0$ and the square is the segment connecting two vertices (a diagonal). The intersection of the plane $x_1 + x_2 + x_3 = 0$ and the cube is a polygon. Which polygon, specifically? Draw it and list the coordinates of its vertices.

It should be expected that the intersection of the four-dimensional cube $[-1, 1]^4$ and the three-dimensional hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ is a three-dimensional polyhedron; we denote it by Q_3.

7.4.8. (a) List all vertices of the polyhedron Q_3.
(b) List all its edges. Which vertices are connected by an edge, and which are not? Draw a "frame" polyhedron Q_3 consisting of vertices and edges.
(c) List all faces of the polyhedron Q_3. How many vertices belong to each face and which vertices belong to the same face? Draw Q_3.

7.4.9. Prove that the cross-section of a three-dimensional cubic lattice cut by a plane perpendicular to the main diagonal of a cube and passing through its center will be a tessellation of equilateral triangles and regular hexagons.

7.4.10. Prove that the cross-section of a four-dimensional cubic lattice cut by a hyperplane perpendicular to the main diagonal of the cube and passing through its center will be a tessellation of regular tetrahedra and octahedra.

7.4.11. Is it possible to tile space with semi-regular bodies, such as truncated octahedra?

Here is a method that allows us to approach the above problems. In a polyhedron, define a *flag* to be the ordered triple (v, e, f) where v is a vertex ("nail"), e is an edge ("pole"), and f is a face ("panel"), such that v is one of the endpoints of e and e is one of the sides of f. A polyhedron Q is called *regular* if for any two of its flags (v_1, e_1, f_1) and (v_2, e_2, f_2), there is an isometry of space mapping Q to itself which also maps v_1 to v_2, e_1 to e_2, and f_1 to f_2. This definition is equivalent to the one in section 3 of this chapter.

7.4.12. Prove that for any two vertices v_1 and v_2 of Q_3, there is an isometry of four-dimensional space that takes Q_3 into itself and v_1 into v_2.

7.4.13. (a) Let e_1 and e_2 be two edges of Q_3 with the common vertex $v = (1, 1, -1, -1)$. Prove that there is an isometry of four-dimensional space that takes Q_3 to itself, v to itself, and e_1 to e_2.

(b) Let f_1 and f_2 be two faces of Q_3 with common edge $e = \{(-t, 1, t, -1) \mid -1 \le t \le 1\}$. Prove that there is an isometry of four-dimensional space that takes Q_3 to itself, edge e and vertex v (from (a)) to themselves, and f_1 to f_2.

7.4.14. Show that Q_3 is a regular polyhedron.

7.4.15. Test the method you have learned with a simple example: following Problems 7.4.12, 7.4.13, and 7.4.14 show that the polygon in Problem 7.4.7 is regular.

7.4.16. What regular polyhedra can be obtained by intersecting a four-dimensional cube with a three-dimensional hyperplane?

7.4.17.* Apply the above method to a more complex example: Consider the four-dimensional polytope Q_4 obtained by intersection of the five-dimensional cube $[-1, 1]^5$ and the hyperplane $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. How many vertices does it have? How many edges? How many two-dimensional faces? How many three-dimensional faces? Is it regular?

Two faces (of arbitrary dimensions) of a polytope are called *incident* if the face of larger dimension contains the face of smaller dimension. Two polytopes are called *combinatorially equivalent* if between their vertices, edges, and faces of all dimensions one can establish a one-to-one correspondence that preserves incidence.

7.4.18. Prove that any n-dimensional cross-section of an $(n+1)$-dimensional cube perpendicular to the main diagonal and passing through an arbitrary vertex is combinatorially equivalent to a "zone" in an n-dimensional cube between two similar cross-sections. More precisely, let $L_n(a, b) := \{(x_1, \ldots, x_n) \mid x_i \in [0, 1], a \le x_1 + \ldots + x_n \le b\}$. Prove that $L_{n+1}(k, k)$ is combinatorially equivalent to $L_n(k-1, k)$. Start with the case of $n = 3$ and $k = 2$.

7.4.19. Consider the (hyper)plane $\sum_1^n a_i x_i = c$, where $n > 2$ and the a_i are positive integers with no common factor. Show that when this (hyper)plane intersects a unit lattice consisting of n-dimensional cubes, the number of different (up to parallel translations) pieces in the intersection is equal to $\sum_{i=1}^n a_i$.

The dimension n is called a *Hadamard number* if one can specify a set of n pairwise orthogonal vectors with all the coordinates being ± 1.

7.4.20. Prove that 1, 2, 4, 8 are Hadamard numbers, while the numbers 3, 5, 6 are not.

7.4.21. Prove that all Hadamard numbers greater than 2 are multiples of 4.

7.4.22. Prove that all powers of two are Hadamard numbers.

7.4.23. Prove that 12 is a Hadamard number.

7.4.24. Prove that 20 is a Hadamard number.

A very important open problem is the following: Is it true that all multiples of 4 are Hadamard numbers? For more about this problem see [**RSG**$^+$**16**, p. 7.2] and references cited there.

4.B. Multi-dimensional volumes

Calculation of the "volume" of an n-dimensional polytope is similar to finding the area of a figure in a plane (see the section "The pigeonhole principle and its application in geometry" in the book [**SkoZa**]).

The *n-dimensional volume* (or *volume* for brevity) of an n-dimensional polyhedron is a nonnegative function V defined on the set of polytopes that satisfies the following conditions.

- If the polytope M_1 can be transformed into the polytope M_2 by an isometry, then $V(M_1) = V(M_2)$.
- $V(M_1 \cup M_2) = V(M_1) + V(M_2) - V(M_1 \cap M_2)$.
- The volume of any subset of an $(n-1)$-dimensional hyperplane is zero.
- The volume of the n-dimensional cube with edge length a is a^n.

Using these properties and, if necessary, upper and lower estimates, one can find the volume of any polytope. For example, the n-dimensional volume of the n-dimensional pyramid is given by the formula $V = \frac{1}{n}Sh$, where S is the $(n-1)$-dimensional volume of the base of the pyramid and h is its height. One can also find the volumes of some n-dimensional bodies (i.e., of bounded subsets of n-dimensional space) that are not polytopes.

7.4.25. For a 100-dimensional watermelon (ball), the radius is 1 meter and the thickness of the rind is 1 cm. What percentage of its volume does the flesh take up?

7.4.26. Prove that the main building of Moscow State University[2] can be placed in a unit cube of sufficiently large dimension; i.e., there exists a three-dimensional plane whose intersection with the cube can completely contain this building.

[2] *Editor's note:* This is one of the tallest buildings in Moscow, with a very large "footprint." It is about half the height of the Empire State Building, but its base is roughly the size of the Empire State Building if it were horizontal.

7.4.27. Find n such that a circle of radius R can be placed inside the n-dimensional unit cube.

7.4.28. Find n such that a (three-dimensional) ball of radius R can be placed in the n-dimensional unit cube.

7.4.29. Find n such that one can place an n-dimensional ball of radius R in the n-dimensional unit cube.

7.4.30. What is the limit of the volume of an n-dimensional ball of radius 2015 as $n \to \infty$?

It is known that the volume of an n-dimensional ball of radius R is

$$B_n = \frac{\pi^{n/2} R^n}{\Gamma(n/2+1)},$$

where $\Gamma(z) := \int_0^\infty y^z e^{-y} dy$, $z > 0$, is the famous *gamma function* of Euler (see, e.g., [**Zas10**]). It extends the definition of factorial to the complex plane: $\Gamma(k) = (k+1)!$ for an integer k and $\Gamma(z) = \Gamma(z-1)z$. The last equality allows us to define $\Gamma(z)$ also for $\Re(z) < 0$. It is known that $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi z)$; in particular, $\Gamma(1/2) = \sqrt{\pi}/2$.

7.4.31. Find the surface area of an n-dimensional ball of unit volume.

7.4.32. Find the volume of the n-dimensional simplex with unit edge length. Find the length of the edge of an n-dimensional simplex with unit volume. (The definitions of the n-dimensional simplex and the octahedron are given in subsection 4.A.)

The *diameter* of a bounded subset M of n-dimensional space is defined to be $\sup\{\mathrm{dist}(X,Y),\ X,Y \in M\}$, where $\mathrm{dist}(X,Y)$ is the distance between points X and Y.

7.4.33. Find the volume of an n-dimensional octahedron with unit edge length. Find the diameter of the n-dimensional simplex with unit volume.

4.C. Volumes and intersections

Define $x_+ = \max(x,0) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

An *open half-plane* is a set of points of a plane that lie strictly on one side of some line. A *closed half-plane* is the union of an open half-plane and its boundary line. The straight line defined by the equation $ax + by + c = 0$ splits the plane into two half-planes (closed and open), the coordinates of the points of which satisfy the inequalities $ax + by + c \geq 0$ and $ax + by + c < 0$. Open and closed *half-spaces* in n-dimensional space are defined similarly.

In the problems of this subsection, the words "open" and "closed" will be omitted, since it does not matter which kind of half-space is considered.

7.4.34. Let $S(a,b,d)$ denote the area of intersection of the unit square $K = \{(x,y)\colon 0 \le x, y \le 1\}$ with the half-plane $ax + by \le d$, where $a, b > 0$ with $a^2 + b^2 = 1$. Prove that

$$S(a,b,d) = \frac{1}{2ab}\big(d_+^2 - (d-a)_+^2 - (d-b)_+^2 + (d-a-b)_+^2\big).$$

7.4.35. Let $V(a,b,c,d)$ denote the volume of the intersection of the unit cube $K = \{(x,y,z)\colon 0 \le x, y, z \le 1\}$ with the half-space $ax + by + cz \le d$, where $a, b, c > 0$ with $a^2 + b^2 + c^2 = 1$. Prove that

$$\begin{aligned} V(a,b,c,d) = \frac{1}{6abc}\big(d_+^3 &- (d-a)_+^3 - (d-b)_+^3 - (d-c)_+^3 \\ &+ (d-a-b)_+^3 + (d-b-c)_+^3 + (d-c-a)_+^3 \\ &- (d-a-b-c)_+^3\big). \end{aligned}$$

7.4.36. Let $V(\vec{a}, d)$, where $\vec{a} = (a_1, \dots, a_n)$, denote the volume of the intersection of the n-dimensional unit cube $K = \{(x_1, \dots, x_n)\colon 0 \le x_i \le 1; i = 1, \dots, n\}$ with the half-space $\sum a_i x_i \le d$, where $a_i > 0$, $i = 1, \dots, n$, with $\sum a_i^2 = 1$. Prove that

$$V(\vec{a}, d) = \frac{1}{n! \prod_{i=1}^n a_i}\left(\sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|}\left(d - \sum_{i \in I} a_i\right)_+^n\right).$$

4.D. Research problems

This subsection provides a number of difficult research problems, some still open.

We define the *distance* $\operatorname{dist}(A,B)$ between sets A and B by

$$D = \operatorname{dist}(A,B) = \inf_{X \in A, Y \in B} \operatorname{dist}(X,Y).$$

7.4.37. Let $\varphi_n \subset \mathbb{R}^n$ be the ball of unit volume, and let $V_n(\varepsilon)$ be the supremum of the distances between its subsets of volume ε. Is the sequence $V_n(\varepsilon)$, where $n = 1, 2, \dots$ and ε is fixed, bounded?

7.4.38. Same question as above, but now let $V_n(\varepsilon)$ be restricted to *convex* subsets of volume ε.

Problems 7.4.37 and 7.4.38 are special cases of the following.

Problem 1. Let $\varphi_n \subset \mathbb{R}^n$, $n = 1, 2, \dots$, be a family of bodies of unit volume, and let $V_n(\varepsilon)$ be the supremum of the distance between volume-ε subsets of φ_n. For which families φ_n is the sequence $V_n(\varepsilon)$ bounded?

Problem 2. Let $\varphi_n \subset \mathbb{R}^n$, $n = 1, 2, \dots$, be a family of bodies of unit volume, and let $V_n^{\text{conv}}(\varepsilon)$ be the supremum of the distance between volume-ε *convex* subsets of φ_n. For which families φ_n is the sequence $V_n^{\text{conv}}(\varepsilon)$ bounded?

Begin investigating these problems by considering cubes, simplices, and octahedra.

7.4.39. Prove that for Problem 2, it is sufficient to consider only subsets of φ_n which are formed by intersecting it with half-spaces.

4.E. Partitions into parts of smaller diameter
By A. M. Raigorodsky

Recall that the *diameter* of the set $\Omega \subset \mathbb{R}^n$ is defined to be

$$\operatorname{diam} \Omega = \sup_{\vec{x}, \vec{y} \in \Omega} |\vec{x} - \vec{y}|,$$

where $|\vec{x} - \vec{y}|$ denotes standard Euclidean distance and "sup" is the "supremum," or the least upper bound. Since the cases below deal with closed balls or finite sets, for simplicity, we can assume that the supremum is an ordinary maximum.

7.4.40. (a) Given the set of points $\mathcal{A} = \{\vec{x}_1, \ldots, \vec{x}_n\} \subset \mathbb{R}^2$, prove that the number of pairs $(\vec{x}_i, \vec{x}_j)$ for which $|\vec{x}_i - \vec{x}_j| = \operatorname{diam} \mathcal{A}$ does not exceed n.

(b) Show that any finite set in the plane can be divided into three parts of smaller diameter.

7.4.41. (a) Show that any three-dimensional ball can be divided into four parts of smaller diameter.

(b) (Challenge.) Try to ensure that the diameters of all the parts in (a) are as small as possible.

(c) Any n-dimensional ball can be divided into $n+1$ parts of smaller diameter.

7.4.42. (a) For any n, there exists M such that any bounded set in $\mathbb{R}^n$ can be divided into M parts of smaller diameter.

(b) For any n, there is a bounded set in $\mathbb{R}^n$ that cannot be divided into n parts of smaller diameter.

7.4.43. (a) For any n, there exists M such that all points of space $\mathbb{R}^n$ can be colored in M colors so that the distance between any two points of the same color is not equal to 1.

(b) For any $n \geq 2$ it is impossible to color all points of space $\mathbb{R}^n$ in $n+1$ colors so that the distance between any two points of the same color is not equal to 1.

Suggestions, solutions, and answers

7.4.2. (a) See Fig. 13.

(b) The vertices of a cube have coordinates $(\pm 1, \pm 1, \ldots, \pm 1)$. Each coordinate is either 1 or -1. Since there are n coordinates, the total number of vertices is $2 \times 2 \times \ldots \times 2 = 2^n$.

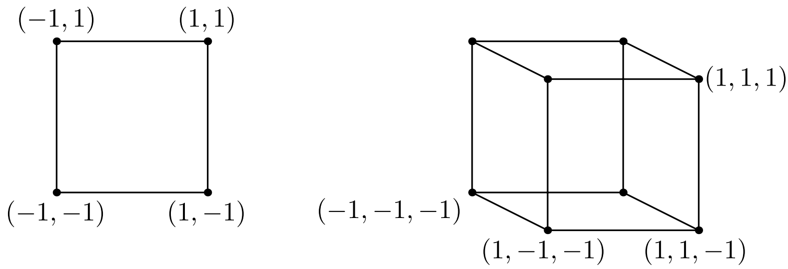


FIGURE 13

(c) The vertices are connected by an edge if and only if the corresponding sets of coordinates differ in exactly one position.

7.4.3. (a) *Answer*: $n2^{n-1}$.

A point on an edge of an n-dimensional cube is an ordered n-tuple where all coordinates except, say, the ith are equal to ± 1 and the ith coordinate is a real number between -1 and 1. Therefore, to determine an edge, you must first fix the index i and then indicate which numbers (1 or -1) are in the remaining $n-1$ places. The first choice can be made in n ways and the second in 2^{n-1} ways. Therefore, the total number of cube edges is $n2^{n-1}$.

(b) *Answer*: $n(n-1)2^{n-3}$.

Similarly, each point in a two-dimensional face is an ordered n-tuple, with two coordinates having values between -1 and 1 and the remaining coordinates equal to ± 1. To determine a two-dimensional face, you need to select two indices for the "free" coordinates (this can be done in $\binom{n}{2} = n(n-1)/2$ ways) and indicate which numbers ($+1$ or -1) are in the remaining coordinates (which can be chosen in 2^{n-2} ways). Therefore, the total number of two-dimensional faces is $2^{n-2}\binom{n}{2}$.

(c) *Answer*: $2^{n-k}\binom{n}{k}$.

The proof is similar to that of (b).

(d) A point in a face of dimension k is an ordered n-tuple in which $n-k$ coordinates are ± 1 and the remaining k coordinates are "free" values between -1 and 1. Therefore, each of the n places is either $1, -1$, or a "free" value. Thus, the total number of possible faces is $3 \times 3 \times \ldots \times 3 = 3^n$. One of these faces has dimension n (all numbers are free): the cube itself.

7.4.12–7.4.14. To solve these problems, it is not necessary to know *all* isometries of four-dimensional space, just a few crucial ones. For example, for every permutation i_1, i_2, i_3, i_4 of the numbers $1, 2, 3, 4$ there exists an isometry—a permutation of coordinates $(x_1, x_2, x_3, x_4) \mapsto (x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})$. This obviously preserves the cube $[-1, 1]^4$ and the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ (recall that the word "permutation" means that among i_1, i_2, i_3, i_4 each number $1, 2, 3, 4$ occurs exactly once). Therefore, this isometry also preserves the intersection of Q_3 with the hyperplane. If you solved Problem 7.4.8, you can easily see that permutations of coordinates can take any vertex of the polyhedron Q_3 to any other.

Another useful isometry that transforms Q_3 into itself is the map

$$(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4).$$

7.4.40. (b) Use part (a).

Additional reading

For more information about drawing see [**Sha74**, **Fuk84**], and about higher-dimensional space see [**DR86**, **Zas10**, **Gal15**].

Chapter 8

Miscellaneous geometry problems

1. Geometric optimization problems (2)
By A. D. Blinkov

8.1.1.° Village A is connected to village B by a straight road of length 3 km. There are 50 schoolchildren in village A and 100 in village B. Where should a school be located on the road so that the total distance traveled by all students to the school is minimized?

(a) In village A.
(b) One kilometer from A.
(c) Equidistant from the villages.
(d) One kilometer from B.
(e) In village B.
(f) Impossible to determine.

8.1.2.° The lengths of two sides of a triangle are 6 and 12. What is the possible range of the length x of the third side?

(a) $6 < x < 12$; (b) $6 \leq x \leq 12$; (c) $6 < x < 18$;
(d) $6 \leq x \leq 18$; (e) $0 < x < 18$; (f) impossible to determine.

8.1.3.° Which of the following types of triangles with given sides b and c has the largest area?

(a) Acute-angled;
(b) right, in which the right angle lies between these sides;
(c) right, in which the right angle lies opposite one of these sides;
(d) obtuse, in which the obtuse angle lies between these sides;
(e) obtuse, in which the obtuse angle lies opposite one of these sides;
(f) impossible to determine.

8.1.4.° Which of the following types of parallelograms with a given area has the smallest perimeter?

(a) A rhombus other than a square;
(b) a rectangle different from a square;
(c) a square;
(d) impossible to determine.

8.1.5.° Specify the interior point of an equilateral triangle for which the sum of its distances to the sides is maximal.

(a) The orthocenter (the point of intersection of the heights);
(b) the point of intersection of the medians;
(c) the center of the inscribed circle;
(d) any internal point;
(e) the center of the circumscribed circle;
(f) such a point does not exist.

8.1.6. Among all triangles with a given fixed angle and a fixed

(a) side opposite to the given angle; (b) perimeter

find the triangle of largest area.

8.1.7. Among all triangles inscribed in a given circle, find a triangle such that the sum of the squares of its sides is maximized.

8.1.8. Perpendiculars PA', PB', and PC' are dropped from point P lying inside triangle ABC to sides BC, CA, and AB or to their extensions, respectively. Find the position of point P for which the product $|PA'|\cdot|PB'|\cdot|PC'|$ is maximized. Generalize this problem to quadrilaterals.

8.1.9. From point P lying inside triangle ABC, drop perpendiculars PA', PB', and PC' to lines BC, CA, and AB respectively. Find the position of point P for which the sum $\frac{|BC|}{|PA'|} + \frac{|CA|}{|PB'|} + \frac{|AB|}{|PC'|}$ is minimized.

8.1.10. Given the side lengths of a quadrilateral, specify which type of quadrilateral has the largest area.

8.1.11. Point P lies inside angle AOB. Construct segment MN with endpoints on the sides of the angle containing point P such that the sum $OM + ON$ is minimized.

8.1.12. On side AB of triangle ABC construct a square external to ABC. Let O be the center of the square. Let points M and N be the midpoints of sides AC and BC respectively, and let the lengths of these sides be fixed and equal to b and a respectively. Find the maximal value of $OM + ON$ as the angle ACB is varied.

Optimization problems involving area and perimeter are discussed in [**Gas85**, **Tr85**]. A comprehensive introduction to geometric optimization problems can be found in [**Pro05**].

Suggestions, solutions, and answers

8.1.6. *Answer*: An isosceles triangle with base opposite the given angle.

Let triangle ABC be given, in which $\angle BAC = \alpha$, $|BC| = a$, and the perimeter P_{ABC} is equal to $2p$.

(a) Consider all triangles with fixed side BC and fixed angle A. They are inscribed in a circle of fixed radius $R = \frac{a}{2\sin\alpha}$ so that the vertices opposite the side BC lie in the same half-plane relative to BC (see Fig. 1 a). Since $S_{ABC} = 0.5ah_a$, the largest area is achieved with the largest height, i.e., when the triangle is isosceles.

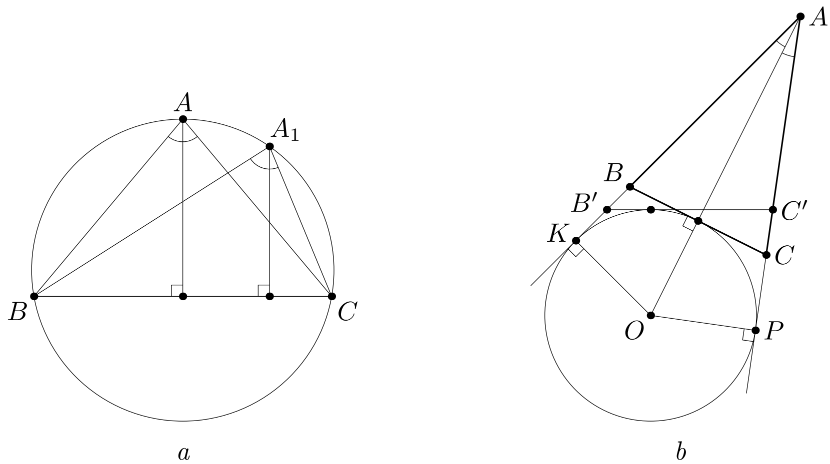


FIGURE 1

(b) Consider all triangles with a fixed perimeter and fixed angle A. There is a fixed circle with center O that is a common excircle of all these triangles, with point of tangency on side a. The radius of the excircle is given by $r_a = |AK| \cdot \tan\frac{\alpha}{2} = p \cdot \tan\frac{\alpha}{2}$, where K is the tangency point of the excircle with the extension of side AB (see Fig. 1 b). Since $S_{ABC} = (p-a)r_a$, the largest area is achieved with the smallest possible value of a, that is, when tangent BC to the excircle is perpendicular to the bisector AO (this can be proved rigorously by contradiction). Therefore, the required triangle ABC is isosceles.

8.1.7. *Answer*: An equilateral triangle.

Consider triangle ABC inscribed in a circle of radius R with center O (see Fig. 2). Let be $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$, Then

$$|AB|^2 + |BC|^2 + |CA|^2 = (\vec{b}-\vec{a})^2 + (\vec{c}-\vec{b})^2 + (\vec{a}-\vec{c})^2$$
$$= 2(\vec{a}^2 + \vec{b}^2 + \vec{c}^2) - 2((\vec{a},\vec{b}) + (\vec{b},\vec{c}) + (\vec{c},\vec{a})).$$

Since $(\vec{a}+\vec{b}+\vec{c})^2 = \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$, we obtain

$$|AB|^2+|BC|^2+|CA|^2 = 3(\vec{a}^2+\vec{b}^2+\vec{c}^2) - (\vec{a}+\vec{b}+\vec{c})^2 \le 3(\vec{a}^2+\vec{b}^2+\vec{c}^2) = 9R^2,$$

and equality is achieved if and only if $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, i.e., when point O coincides with the center of mass of the triangle. This implies that triangle ABC is equilateral.

Alternatively, fix one side of the triangle and show, using the law of cosines, that of all triangles inscribed in a given circle with this fixed side, the isosceles triangle will maximize the sum of the squares of the sides.

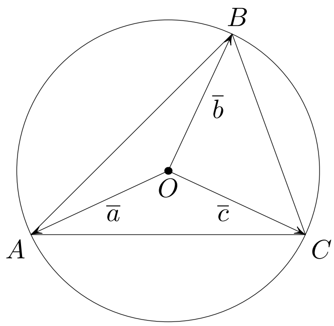


FIGURE 2

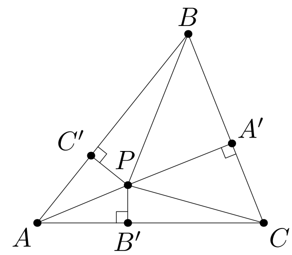


FIGURE 3

8.1.8. *Answer*: The centroid (center of gravity) of the triangle.

Consider triangle ABC and point P inside it. Drop perpendiculars PA', PB', and PC' to the sides of the triangle and connect P with the vertices (see Fig. 3). Then

$$|PA'| \cdot |PB'| \cdot |PC'| = \frac{2S_{BPC}}{a} \cdot \frac{2S_{CPA}}{b} \cdot \frac{2S_{APB}}{c}.$$

Since the sides of the triangle are fixed, the product of the distances will be maximal if and only if the product of the areas of the triangles with vertex at point P is maximal. The sum of these areas does not depend on the location of the point P and is S_{ABC}; thus their product will be maximal if $S_{BPC} = S_{CPA} = S_{APB}$, i.e., if P is the intersection point of the medians of triangle ABC.

8.1.9. *Answer*: The incenter of the triangle.

Consider triangle ABC and point P inside it. Drop perpendiculars PA', PB', and PC' to the lines extending the sides of the triangle, and connect P with the vertices (see Fig. 3). Letting $PA' = x$, $PB' = y$, and $PC' = z$, we get $ax + by + cz = 2S_{ABC}$.

Then

$$\begin{aligned}
&\Big(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\Big)\cdot 2S_{ABC}\\
&\quad=\Big(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\Big)\cdot(ax+by+cz)\\
&\quad=a^2+b^2+c^2++ab\Big(\frac{x}{y}+\frac{y}{x}\Big)+bc\Big(\frac{y}{z}+\frac{z}{y}\Big)+ca\Big(\frac{z}{x}+\frac{x}{z}\Big)\\
&\quad\geq a^2+b^2+c^2+2ab+2bc+2ca=(a+b+c)^2,
\end{aligned}$$

and equality is achieved if and only if $x=y=z$. This means that P is the center of the inscribed circle.

8.1.10. *Answer*: The quadrilateral inscribed in a circle.

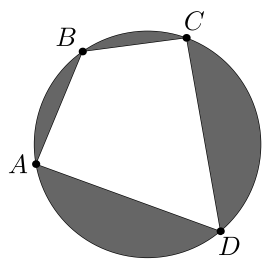


FIGURE 4

Consider quadrilateral $ABCD$ with the given sides, inscribed in a circle (see Fig. 4). Suppose its area is not maximal. Then "glue" the segments of the circle to the sides $ABCD$ and put "hinges" at its vertices. Flex at the hinges to produce a quadrilateral with greater area. The resulting figure (with the glued segments attached) has the same perimeter as before, but has a larger area than the circle. This contradicts the isoperimetric property of a circle (see [**Pro05**]).

8.1.11. *Path to solution.* Let MN be the desired segment. Draw lines parallel to the sides of the given angle through point P, intersecting rays OA and OB at points K and L, respectively. Since $|OM|+|ON|=(|OK|+|OL|)+(|KM|+|LN|)$, this quantity will be minimized if and only if $|KM|+|LN|$ is minimized.

Since triangles KMP and LPN are similar, we have $\frac{|KM|}{|PL|}=\frac{|KP|}{|LN|}$; i.e., $|KM|\cdot|LN|=|KP|\cdot|PL|$. Consequently,

$$|KM|+|LN|\geq 2\sqrt{|KM|\cdot|LN|}=2\sqrt{|KP|\cdot|PL|}=2\sqrt{|OK|\cdot|OL|},$$

and equality is achieved if and only if $|KM|=|LN|=\sqrt{|OK|\cdot|OL|}$.

Thus, the desired construction is achieved by marking segments equal to the geometric mean of the sides of the parallelogram on the extensions of sides OK and OL of parallelogram $OKPL$.

Note that the sought segment is unique, and the smallest value of the sum $|OM|+|ON|$ is $(\sqrt{a}+\sqrt{b})^2$, where a and b are the sides of the parallelogram $OKPL$.

Note: The equality $|KM|\cdot|LN|=|KP|\cdot|PL|$ can be proved in another way. Through points M and N draw lines parallel to the sides of angle AOB. Let Q be their intersection point, and let E and F be the intersection points of the lines LP and KP with MQ and NQ, respectively. Then the parallelograms $OKPL$ and $PEQF$ have the same area and the same angles.

8.1.12. *Answer*: $\frac{1+\sqrt{2}}{2}(a+b)$.

Let $ABDE$ be the square constructed on side AB. Then $[OM]$ is a midline of $\triangle ADC$, and $[ON]$ is a midline of $\triangle BEC$ (see Fig. 5). On segments AC and BC, construct squares $AKLC$ and $BTPC$ external to the triangle. Draw $[BK]$ and $[AT]$. Then $\triangle ABT \cong \triangle DBC$ and $\triangle BAK \cong \triangle EAC$ (by SAS). Therefore, the length of $|DC|$ is maximized if and only if $|AT|$ is maximized, i.e., $T \in (AC)$. Similarly, the length of $|EC|$ is maximized if and only if $|BK|$ is maximized, i.e., $K \in (BC)$. To fulfill these conditions, it is necessary and sufficient that $\angle ACB = 135°$.

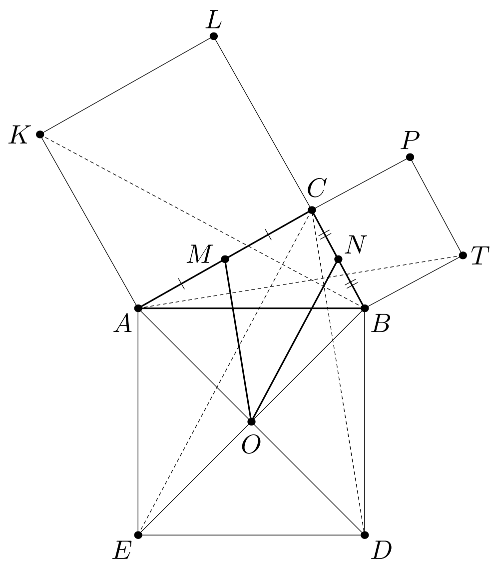


FIGURE 5

Then

$$|OM|+|ON| = \frac{1}{2}(|DC|+|EC|) = \frac{1}{2}(b+a\sqrt{2}+a+b\sqrt{2}) = \frac{1+\sqrt{2}}{2}(a+b).$$

Note: The congruence of the triangles can also be proved by rotations with centers at B and A, respectively.

2. Area (2)
By A. D. Blinkov

8.2.1.° Find the angle between the side and the smaller diagonal of a rhombus if the side length is $2\sqrt{6}$ m and the area of the rhombus is 12 m^2.

8.2.2.° Find the area of an isosceles trapezoid with larger base of length 15 m, legs of length 4 m, and diagonals of length 13 m.

8.2.3.° Find the area of an isosceles trapezoid with obtuse angle α and inscribed circle of radius r.

8.2.4.° A line parallel to the base of a triangle divides its area in the ratio $p : q$ (top : bottom). By what ratio does it divide the legs?

8.2.5.° The sides of a triangle are 35, 29, and 8 cm. Find
(a) the longest height of the triangle;
(b) the radii of the circumscribed and inscribed circles.

8.2.6.° A triangle has a side of length 10, and the median opposite to it has length 9. If another median has length 6, find the area of the triangle.

8.2.7. Prove that the area of a regular octagon is equal to the product of the lengths of the largest and smallest of its diagonals.

8.2.8. Let point O lie in the interior of a convex quadrilateral of area S. Consider the points that are symmetric to O with respect to the midpoints of the sides of the quadrilateral. Find the area of the new quadrilateral formed by these points.

8.2.9. In a convex quadrilateral $ABCD$, let E be the midpoint of CD, F the midpoint of AD, and K the intersection of AC and BE. Prove that the area of triangle BKF is half the area of triangle ABC.

8.2.10. Let PQ be the diameter of a circle, and let chord MN be perpendicular to PQ, intersecting it at point A. Let C lie on the circle, and let B be in the interior of the circle, with $BC \parallel PQ$ and $|BC| = |MA|$. Drop perpendiculars AK and BL from points A and B to CQ. Prove that triangles ACK and BCL have equal area.

8.2.11. Each diagonal of a convex pentagon cuts off a triangle of unit area. Find the area of the pentagon.

8.2.12. Diagonal BD of cyclic quadrilateral $ABCD$ bisects angle ABC. Find the area of the quadrilateral, given that $|BD| = 6$ and $\angle ABC = 60°$.

8.2.13. The extensions of sides AD and BC of the convex quadrilateral $ABCD$ intersect at point E; M and N are the midpoints of sides AB and CD, respectively; P and Q are the midpoints of diagonals AC and BD, respectively. Prove that

(a) $S_{PMQN} = \frac{1}{2}|S_{ABD} - S_{ACD}|$; (b) $S_{PEQ} = \frac{1}{4}S_{ABCD}$.

8.2.14. Let $ABCD$ be a convex quadrilateral of area S. Let α be the angle between lines AB and CD and let β be the angle between lines AD and BC. Prove that

$$\frac{|AB|\cdot|CD|\cdot\sin\alpha + |AD|\cdot|BC|\cdot\sin\beta}{2} \le S \le \frac{|AB|\cdot|CD| + |AD|\cdot|BC|}{2}.$$

Other interesting problems involving area can be found in [**Pra86**, **To84**]. It is all right to use the common, non-rigorous, understanding of area in solving all of the above problems. A rigorous definition is given in the section "Pigeonhole principle and its applications in geometry" in the book [**SkoZa**].

Suggestions, solutions, and answers

8.2.7. Let $ABCDEFGH$ be a regular octagon with area S.

First method. Let K and L be the projections of D onto lines AC and GE, respectively, and let M and N denote the projections of H onto these lines (see Fig. 6). Then $S = S_{KLNM} = |KL| \cdot |KM| = |CE| \cdot |DH|$.

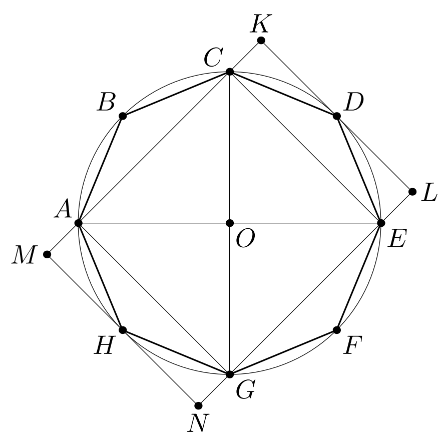


FIGURE 6

Second method. The interior angle of the regular octagon is 135°. Let $|CD| = a$ and $AO = R$, where O is the center of the octagon. Applying the law of cosines to triangle CDE yields

$$|CE|^2 = 2a^2 - 2a^2\cos 135^\circ \iff |CE| = a\sqrt{2+\sqrt{2}};$$
$$|AE| = |CE|\sqrt{2} = a\sqrt{2}\cdot\sqrt{2+\sqrt{2}};$$
$$S = 8S_{\triangle AOB} = 4R^2\cdot\sin 45^\circ = 2R^2\sqrt{2} = \frac{1}{2}|AE|^2\sqrt{2} = |AE|\cdot|CE|.$$

8.2.8. *Answer*: $2S$.

Let $ABCD$ be the given quadrilateral; let E, F, G, and H be the midpoints of its sides; let E', F', G', and H' be the symmetric images of O about these midpoints (see Fig. 7).

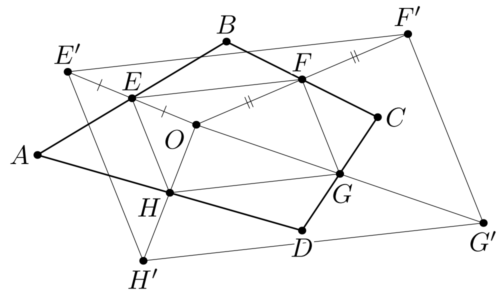


FIGURE 7

Since $[EF]$ is the midline of triangle $E'OF'$, we have $S_{E'OF'} = 4S_{EOF}$. Similarly, $S_{F'OG'} = 4S_{FOG}$, $S_{G'OH'} = 4S_{GOH}$, and $S_{H'OE'} = 4S_{HOE}$. Thus, $S_{E'F'G'H'} = 4S_{EFGH}$. By Varignon's Theorem, $S_{EFGH} = S_{ABCD}/2$; hence $S_{E'F'G'H'} = 2S$.

8.2.9. *First method*. Draw the midline EF of triangle ADC (see Fig. 8 a). Then $\frac{S_{\triangle BKF}}{S_{\triangle BEF}} = \frac{|BK|}{|BE|}$, since the two triangles have the same height from F. Since $EF \parallel AC$, the lengths of the perpendiculars dropped from B to lines EF and AC are in the ratio $|BE| : |BK|$; therefore

$$\frac{S_{\triangle BEF}}{S_{\triangle ABC}} = \frac{|EF|\cdot|BE|}{|AC|\cdot|BK|} = \frac{1}{2}\cdot\frac{|BE|}{|BK|}.$$

Multiplying the two equations yields $\frac{S_{\triangle BKF}}{S_{\triangle ABC}} = \frac{1}{2}$.

Second method. Let a, c, f, and d be the lengths of the perpendiculars dropped to line BE from points A, C, F, and D, respectively (see Fig. 8 b). Then $c = d$ and $f = \frac{a+d}{2}$. Therefore,

$$\begin{aligned} S_{\triangle ABC} &= S_{\triangle ABK} + S_{\triangle KBC} \\ &= \frac{1}{2}|BK|\cdot(a+c) = \frac{1}{2}|BK|\cdot(a+d) = |BK|\cdot f = 2S_{\triangle BKF}. \end{aligned}$$

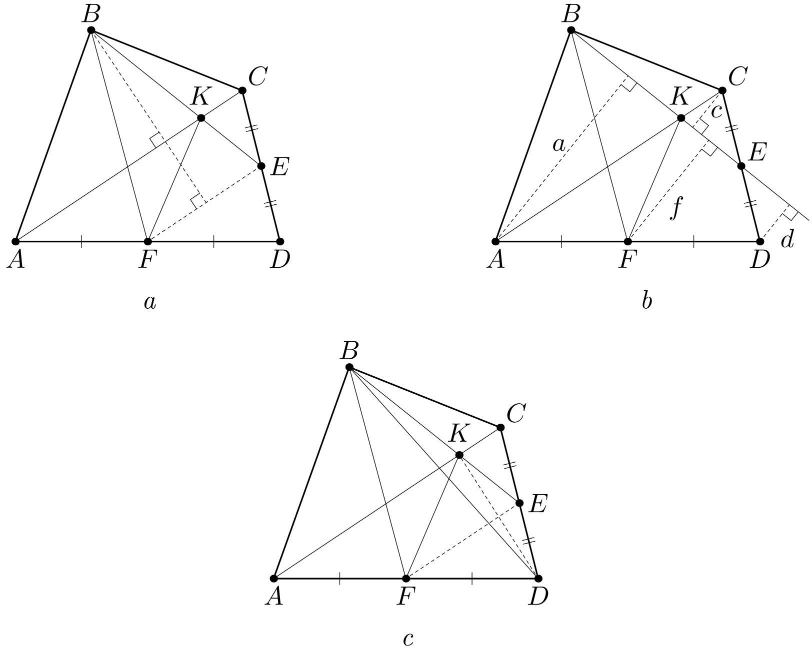


FIGURE 8

Third method. Draw segments BD and EF (see Fig. 8 c). The median of a triangle halves its area, implying $S_{\triangle BAF} = \frac{1}{2}S_{\triangle BAD}$ and $S_{\triangle BCD} = \frac{1}{2}S_{\triangle BCE}$. Also, since EF is the midline of triangle ADC, we have $S_{\triangle DEF} = S_{\triangle EFK} = \frac{1}{4}S_{\triangle ACD}$. Therefore,

$$\begin{aligned} S_{\triangle BKF} &= S_{ABCD} - S_{\triangle ABF} - S_{\triangle BCE} - S_{DFKE} \\ &= \frac{1}{2}(S_{\triangle ABD} + S_{\triangle BCD} - S_{\triangle ACD}) = \frac{1}{2}S_{\triangle ABC}. \end{aligned}$$

8.2.10. Consider the case shown in Fig. 9 (other cases are similar). Let $\angle BCQ = \angle PQC = \alpha$. Since $\angle PCQ = 90^\circ$, we have $\angle CPQ = 90^\circ - \alpha$ (see Fig. 9). Draw $AD \parallel CQ$ so that $D \in [CP]$. Since $DCKA$ is a rectangle, it follows that $|AD| = |CK|$. Next, we have

$$\begin{aligned} S_{ACK} &= \frac{1}{2}|CK| \cdot |AK| = \frac{1}{2}|AD| \cdot |AK| \\ &= \frac{1}{2}|AP| \cdot \sin(90^\circ - \alpha) \cdot |AQ| \cdot \sin\alpha = \frac{1}{2}|AP| \cdot |AQ| \cdot \sin\alpha\cos\alpha \\ &= \frac{1}{2}|AM|^2 \sin\alpha\cos\alpha = \frac{1}{2}|BC|^2 \sin\alpha\cos\alpha = \frac{1}{2}|CL| \cdot |BL| = S_{BCL}. \end{aligned}$$

8.2.11. *Answer*: $\frac{5+\sqrt{5}}{2}$.

Let $ABCDE$ be the pentagon, and let $P = AC \cap BD$ (see Fig. 10). Since $S_{AED} = S_{CED} = 1$, points A and C are equidistant from the line DE;

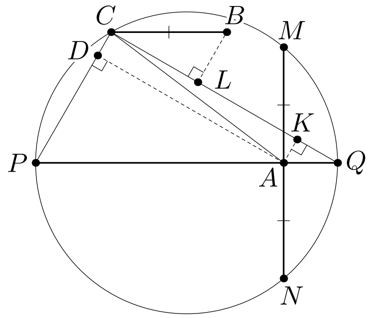


FIGURE 9

i.e., $AC \parallel DE$. Similarly, $BD \parallel AE$; i.e., $APDE$ is a parallelogram, so $S_{APDE} = 2$.

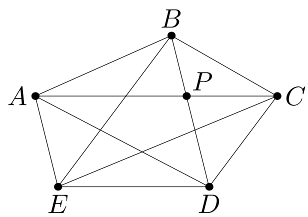


FIGURE 10

Let $S_{ABP} = x$. Then $S_{ABCDE} = S_{APDE} + S_{ABP} + S_{BCD} = 3 + x$. Since $\frac{S_{ABP}}{S_{CBP}} = \frac{|AP|}{|CP|} = \frac{S_{ADP}}{S_{CDP}}$, we have

$$\frac{x}{1-x} = \frac{1}{1-(1-x)} \iff x^2 + x - 1 = 0 \iff x = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $x > 0$, we get $S_{ABCDE} = 3 + \frac{\sqrt{5}-1}{2} = \frac{5+\sqrt{5}}{2}$.

8.2.12. *Answer*: $9\sqrt{3}$.

Since equal arcs of a circle are cut off by equal chords, we have $|AD| = |DC|$.

First method. Consider reflection with respect to BD (see Fig. 11 a). Since D is equidistant from the sides of angle ABC, the image of height DK of triangle ABD will be height DK' of triangle BCD. Then right triangles AKD and $CK'D$ are congruent (by SH), so

$$S_{ABCD} = S_{BKDK'} = 2S_{\triangle BKD} = |BK| \cdot |KD| = 3\sqrt{3} \cdot 3 = 9\sqrt{3}.$$

Second method. Consider clockwise rotation with center D by the angle ADC (see Fig. 11 b). Under this rotation, the images of A and B are C and B', respectively; i.e., the image of triangle ABD is the congruent triangle $CB'D$. Since the quadrilateral $ABCD$ is cyclic, we have $\angle BCD + \angle BAD =$

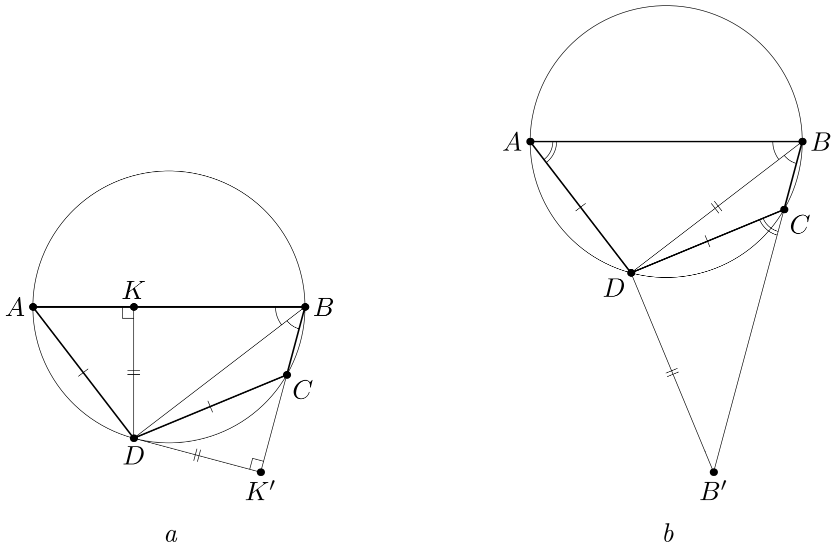


FIGURE 11

180°, which implies that B, C, and B' are collinear. Thus, quadrilateral $ABCD$ and triangle BDB' have equal area.

Since the triangle BDB' is isosceles, we get $|DB'| = |DB| = 6$ and $\angle B' = \angle B = 30°$, so

$$S_{ABCD} = S_{\triangle BDB'} = \frac{1}{2}|BD| \cdot |B'D| \cdot \sin 120° = 9\sqrt{3}.$$

8.2.13. (a) Quadrilateral $PMQN$ is a parallelogram. Let $\alpha = \angle PMQ = \angle CED$. Then

$$S_{PMQN} = |MP| \cdot |MQ| \cdot \sin\alpha = \frac{1}{4}|BC| \cdot |AD| \cdot \sin\alpha.$$

Suppose that $E = CB \cap DA$ (see Fig. 12). Then $S_{ABD} < S_{ACD}$.

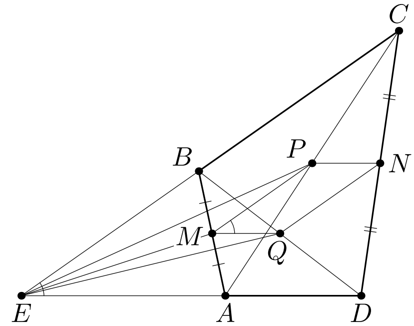


FIGURE 12

The lengths of the heights of triangles ABD and ACD drawn from vertices B and C are equal, respectively, to $|BE| \cdot \sin\alpha$ and $|CE| \cdot \sin\alpha$, so

$$S_{ACD} - S_{ABD} = \frac{1}{2}(|CE| - |BE|) \cdot |AD| \cdot \sin\alpha = \frac{1}{2}|BC| \cdot |AD| \cdot \sin\alpha.$$

If rays BC and AD intersect at E, then $S_{ABD} > S_{ACD}$. Similarly, we see that $S_{ABD} - S_{ACD} = \frac{1}{2}|BC| \cdot |AD| \cdot \sin\alpha$.

Thus, $S_{PMQN} = \frac{1}{2}|S_{ABD} - S_{ACD}|$.

(b) As in (a) above, suppose that $E = CB \cap DA$. Since $PM \parallel CE$ and $QM \parallel DE$, the point M lies in the interior of triangle PEQ (see Fig. 12), so $S_{PEQ} = S_{PMQ} + S_{PME} + S_{QME}$. By (a) it follows that

$$S_{PMQ} = \frac{1}{2}S_{PMQN} = \frac{1}{4}(S_{ACD} - S_{ABD}).$$

By Thales' Theorem, line PM bisects EA, which implies that $S_{PME} = S_{AMP} = \frac{1}{4}S_{ABC}$. Likewise, $S_{QME} = S_{BMQ} = \frac{1}{4}S_{ABD}$.

Consequently,

$$\begin{aligned} S_{PEQ} &= \frac{1}{4}S_{ACD} - \frac{1}{4}S_{ABD} + \frac{1}{4}S_{ABC} + \frac{1}{4}S_{ABD} \\ &= \frac{1}{4}(S_{ACD} + S_{ABC}) = \frac{1}{4}S_{ABCD}. \end{aligned}$$

8.2.14. Without loss of generality, assume that rays BA and CD intersect at P, and that rays BC and AD intersect at Q. Then $\angle BPC = \alpha$ and $\angle BQA = \beta$ (see Fig. 13).

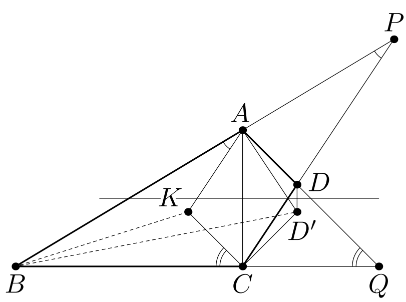


FIGURE 13

Let us prove the first inequality. Construct parallelogram $ADCK$ with diagonal AC. Thus K lies inside $ABCD$. Therefore,

$$\begin{aligned} S \geq S_{ABCK} = S_{ABK} + S_{BCK} &= \frac{1}{2}|AB| \cdot |AK| \cdot \sin\alpha + \frac{1}{2}|CB| \cdot |CK| \cdot \sin\beta \\ &= \frac{1}{2}(|AB| \cdot |CD| \cdot \sin\alpha + |CB| \cdot |AD| \cdot \sin\beta). \end{aligned}$$

Let us prove the second inequality. Let m be the perpendicular bisector of $[AC]$ and let D' be the reflection of D with respect to m. Then

$$\begin{aligned} S = S_{ABCD'} &= S_{ABD'} + S_{BCD'} \\ &\leq \frac{1}{2} \cdot |AB| \cdot |AD'| + \frac{1}{2} \cdot |CB| \cdot |CD'| \\ &= \frac{1}{2} \cdot (|AB| \cdot |CD| + |BC| \cdot |AD|). \end{aligned}$$

3. Conic sections (3^*)
By A. V. Akopyan

The editors thank A. Terteryan for helpful comments.

Let F_1 and F_2 be two points in the plane. The *ellipse with foci F_1 and F_2* is defined to be the set of points P such that the sum of the distances from P to F_1 and F_2 is a constant greater than $|F_1F_2|$. This constant is called the *major axis* of the ellipse.

Let F_1 and F_2 be two different points in the plane. Define the *hyperbola with foci F_1 and F_2* to be the set of points P such that the absolute value of the difference of the distances from P to F_1 and F_2 is a positive constant less than $|F_1F_2|$.

The set of lines passing through the center of the hyperbola (the midpoint of F_1F_2) and not intersecting it fill two vertical angles. The two lines forming the sides of these angles are called the *asymptotes* of the hyperbola. A hyperbola with perpendicular asymptotes is called *equilateral* or *rectangular*.

Let point F and line l not passing through it be given. Define the *parabola with focus F and directrix l* to be the set of points equidistant from F and l.

Ellipses, hyperbolas, and parabolas are called *second-order curves*, *conic sections*, or *conics*.

A *tangent* to an ellipse is a line that intersects it at exactly one point. A tangent to a parabola is a line that is not parallel to its axis of symmetry and has exactly one intersection point with the parabola. A tangent to a hyperbola is a line having exactly one intersection point with it that is also not parallel to its asymptotes.

8.3.1. Given two circles, each of which lies outside the other, find the locus of the centers of circles that are

(a) tangent to the two given circles in the same sense (externally or internally);

(b) externally tangent to one of the given circles and internally tangent to the other.

8.3.2. (a) Prove that the sum of the distances from any point in the interior of an ellipse to its foci is less than the length of its larger axis. Conversely, show that this sum is greater than the length of the larger axis if the point lies outside the ellipse.

(b) Formulate and prove similar statements for a hyperbola and parabola.

8.3.3. (a) **Optical property.** Let line l be tangent to an ellipse at point P. Prove that l is the external bisector of angle F_1PF_2.

(b) State and prove similar optical properties for parabolas and hyperbolas.

(c) Let points F_1 and F_2 be given. Prove that any hyperbola and ellipse with foci F_1 and F_2 intersect each other at right angles (i.e., the tangents to the ellipse and to the hyperbola are perpendicular at their point of intersection).

8.3.4. Do there exist

(a) two ellipses that are not similar;
(b) two hyperbolas that are not similar;
(c) two parabolas that are non-homothetic;
(d) two parabolas that are not similar?

Consider a circle in space and a point distinct from its center that lies on the line passing through the center and perpendicular to its plane. A *right circular cone* is the union of lines in space passing through this given point and intersecting the circle. These lines themselves are called the *generators* of the cone, and this point is called the *vertex* of the cone. A *circular cylinder* is the union of lines in space intersecting a given circle and perpendicular to its plane. These lines are called the *generators* of the cylinder.

8.3.5. The Dandelin spheres. (a) Suppose that a plane α intersects the central axis of a cylinder at exactly one point (thus, α intersects the cylinder in a circle or an ellipse). Consider two congruent spheres inside the cylinder, tangent to it along circles b_1 and b_2, respectively, and tangent to α on opposite sides at points F_1 and F_2, respectively. Prove that the intersection of the cylinder with plane α is in fact an ellipse with foci F_1 and F_2 for which the length of the major axis is equal to the distance between the planes of circles b_1 and b_2.

(b) Formulate and prove similar statements for intersections of a right circular cone with planes not passing through its vertex.

8.3.6. Derive the equations of the ellipse, parabola, and hyperbola in the coordinate system of your choice.

8.3.7. Given a point F, a line l, and a number $e > 0$, prove that the set of points X for which $|XF| = e\,\mathrm{Dist}(X, l)$ is an ellipse if $e < 1$ and a hyperbola if $e > 1$. (The expression $\mathrm{Dist}(X, l)$ denotes the distance from the point X to the line l.)

8.3.8. Prove that the midpoints of parallel chords of

(a) an ellipse; (b) a parabola; (c) a hyperbola

are collinear.

8.3.9. Let a conic and one of its chords AB be given. Let the tangents at points A and B intersect at point P.

(a) Prove that if AB contains F_1, then PF_1 and AB are perpendicular.

(b) Prove that $\angle F_1PA = \angle F_2PB$ (if the conic is not a parabola).

(c) Prove that $\angle AF_1P = \angle PF_1B$.

8.3.10. Find the locus of points from which the ellipse is visible at right angles, i.e., the set of points P such that the tangents drawn from P to the ellipse are perpendicular.

8.3.11. Let a parabola be tangent to three lines containing the sides of a triangle.

(a) Show that its focus lies on the circumcircle of this triangle.

(b)* Show that (a) implies that the circumcircles of triangles formed by four tangent lines have a common point (called the *Miquel point*).

8.3.12. Let a parabola be tangent to three lines containing the sides of a triangle.

(a) Show that its directrix passes through the orthocenter of the triangle.

(b)* Show that (a) implies that the orthocenters of triangles formed by four tangent lines are collinear. (The common line is called the *Aubert line.*)

8.3.13. Prove that if the vertices of a triangle lie on an equilateral hyperbola, then the orthocenter of the triangle also lies on this hyperbola.

8.3.14.* (Challenge.) A loop of string is wrapped around ellipse α and is pulled taut with a pencil. Prove that as the pencil moves around α, keeping the string taut, it will draw another (larger) ellipse with the same foci as α.

8.3.15.° We are given a circle and a line that does not intersect it. What is the locus of the centers of circles tangent to both the given circle and the line?

(a) An ellipse; (b) a hyperbola; (c) a parabola; (d) two parabolas.

Using the elementary methods introduced in this section, one can develop the geometry of conics quite far; see [**AkZa07a**] and [**Ben86**].

Suggestions, solutions, and answers

8.3.1. *Answer*: (a) A hyperbola with foci at O_1 and O_2 such that the difference of distances to them is $|r_1 - r_2|$, where O_1 and O_2 are the centers of the given circles and r_1 and r_2 are their radii.

(b) A hyperbola with foci O_1 and O_2 such that the difference of distances to them is $r_1 + r_2$.

If a circle with center O and radius r is tangent to both circles externally, then $|OO_1| = r + r_1$, $|OO_2| = r + r_2$, and, therefore, $|OO_1| - |OO_2| = r_1 - r_2$; i.e., O lies on one of the branches of the hyperbola with foci O_1 and O_2. If the circle is tangent to both circles internally, then its center lies on the other branch of this hyperbola.

If a circle is externally tangent to one circle and internally tangent to the other one, then the difference between the lengths of OO_1 and OO_2 is equal to $r_1 + r_2$; i.e., the point O describes another hyperbola with the same foci.

8.3.2. (a) Denote the foci of the ellipse by F_1 and F_2, and the point under consideration by X. The intersection point of ray F_1X with the ellipse is denoted by Y. First, suppose that X lies inside the ellipse. By the triangle inequality, $|F_2X| < |XY| + |YF_2|$, so that $|F_1X| + |XF_2| < |F_1X| + |XY| + |YF_2| = |F_1Y| + |F_2Y|$ (see Fig. 14).

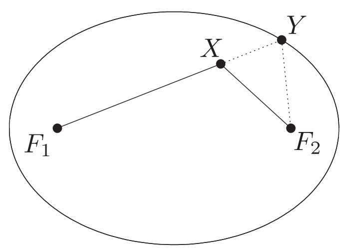


FIGURE 14

But $|F_1Y| + |F_2Y|$ equals the major axis of the ellipse. Arguing similarly, if X lies outside the ellipse, we get $|F_2Y| < |XY| + |XF_2|$. Thus $|F_1X| + |XF_2| = |F_1Y| + |YX| + |XF_2| > |F_1Y| + |F_2Y|$.

(b) For points lying on the same side of the parabola as its focus, the distance to the focus is shorter than the distance to the directrix; for points lying on the other side, it is longer.

For hyperbolas, the statement is formulated as follows: Let the difference of the distances from any point on the hyperbola to the foci F_1 and F_2 be d. Let Γ be the branch of the hyperbola closest to F_1. If the segment XF_1 intersects Γ, the value $|XF_2| - |XF_1|$ is less than d; otherwise, it is greater.

8.3.3. (a) Let X be an arbitrary point on line l other than P (see Fig. 15). Since X lies outside the ellipse, by the previous problem we have $|XF_1| + |XF_2| > |PF_1| + |PF_2|$; i.e., among all points in l, the sum of the distances to F_1 and F_2 is minimized at the point P. But, as you know, this implies that the angles formed by the lines PF_1 and PF_2 with l are equal.

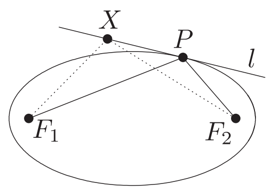


FIGURE 15

(b) For a parabola, the optical property is formulated as follows: if line l is tangent to a parabola at point P, and P' is the projection of the point P onto the directrix, then l is the angle bisector of FPP'.

For hyperbolas, the formulation is as follows: if line l is tangent to a hyperbola at point P, then l is the angle bisector of F_1PF_2, where F_1 and F_2 are the foci of the hyperbola.

(c) Let the ellipse and hyperbola with foci F_1 and F_2 intersect at P. Then the tangents to them at this point will be bisectors of the external and internal angles F_1PF_2, respectively. Consequently, they will be perpendicular.

8.3.5. (b) The intersection will be an ellipse, parabola, or hyperbola, depending on how many generators of the cone are parallel to the plane.

8.3.9. (a) By the optical properties of conics, PA and PB are bisectors of the exterior angles of triangle F_2AB. Therefore P is the center of its excircle. The point of tangency F_1' of the excircle with the side along with the opposite vertex F_2 divide the perimeter of the triangle in half; i.e., $|F_1'A| + |AF_2| = |F_2B| + |BF_1'|$. But F_1 also satisfies this condition, and there is only one such point. So F_1' and F_1 coincide. Thus F_1 is the tangent point of the excircle with the side, so PF_1 and AB are perpendicular.

(b) Let F_1' and F_2' be points symmetric to F_1 and F_2 with respect to PA and PB respectively (see Fig. 16).

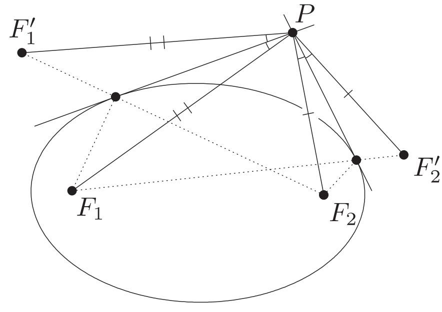


FIGURE 16

Then $|PF_1'| = |PF_1|$ and $|PF_2'| = |PF_2|$. Also, points F_1, B, and F_2' are collinear (by the optical property). Likewise, F_2, A, and F_1' are collinear. Thus $|F_2F_1'| = |F_2A| + |AF_1| = |F_2B| + |BF_1| = |F_2'F_1|$. Consequently, triangles PF_2F_1' and PF_1F_2' are congruent (by SSS). Therefore,

$$\angle F_2PF_1 + 2\angle F_1PA = \angle F_2PF_1' = \angle F_1PF_2' = \angle F_1PF_2 + 2\angle F_2PB.$$

This implies that $\angle F_1PA = \angle F_2PB$.

(c) Since triangles PF_2F_1' and PF_1F_2' are congruent, angles $PF_1'F_2$ and PF_1F_2' are equal. Thus

$$\angle PF_1A = \angle PF_1'F_2 = \angle PF_1F_2' = \angle PF_1B.$$

8.3.10. Denote the foci of this ellipse by F_1 and F_2. Let the tangents to the ellipse at points X and Y intersect at point P. Let F_1' be the reflection of F_1 with respect to PX. By the previous problem, $\angle XPY = \angle F_1'PF_2$ and $|F_1'F_2| = |F_1X| + |F_2X|$; i.e., the length of the segment $F_1'F_2$ is equal to the length of the major axis of the ellipse. Angle $F_1'PF_2$ is a right angle if and only if $|F_1'P|^2 + |F_2P|^2 = |F_1'F_2|^2$ (by the converse to the Pythagorean Theorem). Therefore, angle F_1PF_2 is a right angle if and only if $|F_1P|^2 + |F_2P|^2$ is equal to the square of the length of the major axis of the ellipse.

It is easy to show that this means that point P belongs to a circle. Indeed, let F_1 and F_2 respectively have coordinates (x_1, y_1) and (x_2, y_2). Then the coordinates (x, y) of P will satisfy

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 = C,$$

where C is the square of the major axis. Since the coefficients of x^2 and y^2 are equal (namely, to 2) and the coefficient of xy is 0, the set of points satisfying this equation is a circle. By symmetry, it is easy to see that the center of this circle is the midpoint of segment F_1F_2, and its radius can be found by drawing tangents parallel to the axes of the ellipse; the radius is equal to $\sqrt{a^2 + b^2}$, where a and b are the major and minor semiaxes of the ellipse.

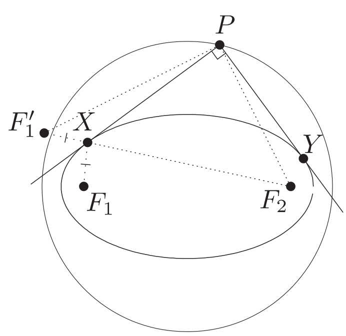


FIGURE 17

8.3.11. (a) By the optical property of the parabola, the reflection of the focus with respect to any tangent will lie on the directrix. This clearly implies that the projection of the focus onto tangents to the parabola will always fall on the line that is tangent to the parabola at its vertex. It suffices to then use Problem 1.10.1 in Chapter 1.

(b) *First method* (A. Terteryan). Consider a point of intersection of the circumscribed circles of two of the triangles, other than a vertex. Its Simson lines with respect to the two triangles coincide. A parabola whose focus is this point and whose directrix is this Simson line is tangent to all four lines. The focus of this parabola will be the Miquel point.

Second method. For any five lines in general position (i.e., such that no three of them are concurrent and no two are parallel), there exists a conic tangent to each of these lines. If we assume that one line is at infinity, then we see that for any four lines in general position there is a parabola that is tangent to them. The focus of this parabola will be the Miquel point.

8.3.12. (a) This is a direct consequence of Problem 8.3.11 (a) and Problem 1.10.3 in Chapter 1.

(b) The solution is similar to that of Problem 8.3.11.

8.3.13. Let ABC be a given triangle, H its orthocenter, and X and Y the asymptotes of the hyperbola. Draw lines through A and B parallel to X, and through C and H parallel to Y. Let UV be the diagonal of the rectangle formed by these lines, and let B' be the foot of the height of the triangle dropped from the vertex B (see Fig. 18).

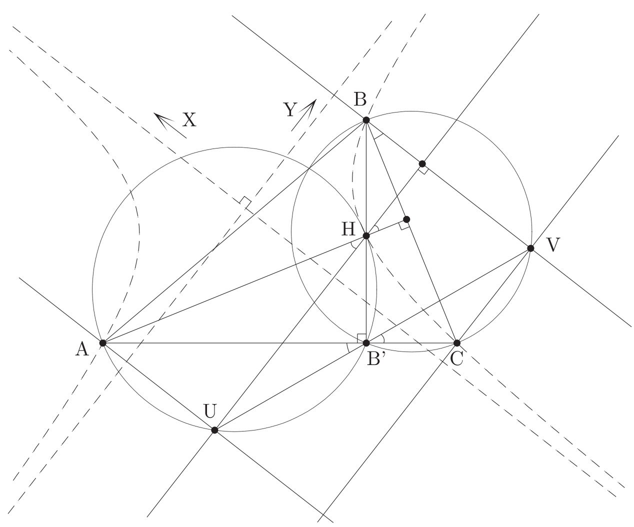


FIGURE 18

Quadrilaterals $BB'CV$ and $AUB'H$ are inscribed in circles with diameters BC and AH; therefore, $\angle AB'U = \angle AHU$ and $\angle VB'C = \angle VBC$.

But $\angle AHU$ and $\angle VBC$ are equal as angles with orthogonal sides, which implies that points U, B', V are collinear. Then, arguing similarly to the proof of the converse of Pascal's Theorem (or just applying the theorem for the hexagon $AXBHYC$ in the *projective plane*, where X and Y denote the infinitely distant points of the asymptotes X and Y), we obtain that the hexagon $AXBHYC$ is inscribed in a conic; i.e., the equilateral hyperbola $ABCXY$ passes through H.

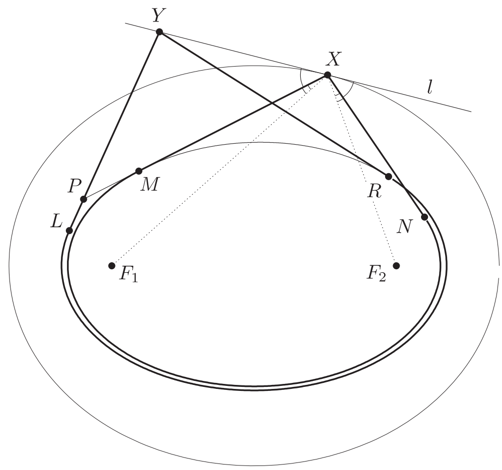


FIGURE 19

8.3.14. Let us present a visual heuristic argument. Obviously, the resulting curve (call it α_1) is smooth. We will show that at every point X on α_1 the tangent will coincide with the external angle bisector of F_1XF_2.

Let XM and XN be tangents to α. Then $\angle F_1XN = \angle F_2XM$, and so the external angle bisector of NXM will coincide with the external angle bisector of F_1XF_2. Call it l.

Let Y be an arbitrary point on l, and let YL and YR be tangents to α drawn from the same sides as XM and XN, as shown in Fig. 19. Without loss of generality, assume that Y lies "to the left" of X.

Let P be the intersection of XM and YL. It is easy to see that $|YN| < |YR| + \smile RN$ and $\smile LM < |LP| + |PM|$ (the arcs denote the curved shape that the string takes). Furthermore, since l is the external angle bisector of NXP, we have $|PX| + |XN| < |PY| + |YN|$, so

$$\begin{aligned}|MX| + |XN| + \smile NM &< MX + XN + \smile NL + |LP| + |PM| \\ &= |PX| + |XN| + \smile NL + |LP| < |PY| + |YN| + \smile NL + |LP| \\ &= |LY| + |YN| + \smile NL < |LY| + |YR| + \smile RN + \smile NL \\ &= |LY| + |YR| + \smile RL.\end{aligned}$$

Therefore, Y will lie in the exterior of α_1. This is true for any point Y on the line l. It turns out that α_1 contains a single point on the line l, i.e., is tangent to this line. It also immediately follows from the above that the resulting curve is convex.

Thus, when we move along the boundary of the figure α_1, the rate of change of the sum of the distances to the foci F_1 and F_2 at any moment in time is equal to zero. Therefore, this sum is constant, and thus the pencil path coincides with the ellipse.

4. Curvilinear triangles and non-Euclidean geometry (3*) *By M. B. Skopenkov*

The purpose of this collection of problems[1] is to show by a simple example how some ideas of non-Euclidean geometry naturally arise from Euclidean geometry problems. We assume that the reader is familiar with the concept of inversion and its simplest properties (see, for example, section 10 "Inversion" in Chapter 3).

Motivating example. Consider the image of an equilateral triangle under all possible compositions of reflections with respect to its sides. This will produce a tessellation of the plane by equilateral triangles.

A *curvilinear triangle* is a figure bounded by three circular arcs a, b, and c with no common points other than endpoints. The arcs a, b, and c are the *sides* of the curvilinear triangle. A *reflection* with respect to a side of a curvilinear triangle is the inversion with respect to the circle containing this side. The *angle* of a curvilinear triangle is the angle between the tangents to its sides at their common endpoint.

The main question. What will be the image of a *curvilinear* triangle $\triangle$ under all possible compositions of reflections with respect to its sides?

We call a curvilinear triangle *regular* if it invariant under rotation by 120° around some point.

8.4.1.$^\circ$ Describe all circles that remain invariant under inversion with respect to a circle a.

(a) There are no such circles;
(b) the circle a itself;
(c) circles perpendicular to the circle a;
(d) circles perpendicular to the circle a, and the circle a itself;
(e) all circles.

8.4.2.$^\circ$ Which of the following statements are true for any curvilinear triangle? Indicate all correct answers.

(a) There is an inversion transforming one of its sides to a straight line segment.

(b) There is an inversion transforming two of its sides into straight line segments.

---

[1]The author is grateful to F. K. Nilov for Problems 8.4.14–8.4.16

(c) There is an inversion transforming all three of its sides into straight line segments.

8.4.3. Answer the main question in the case where $\triangle$ is a regular curvilinear triangle with three angles of $90°$ (draw the resulting picture).

8.4.4. Let $\triangle$ be a regular curvilinear triangle with zero angles (that is, a triangle made up of three pairwise tangent arcs). Prove that its image under all possible compositions of reflections with respect to its sides lies inside its circumscribed circle, that is, the circle passing through the three common endpoints of the sides.

8.4.5. Let $\triangle$ be a regular curvilinear triangle, with all three angles less than $60°$.

(a) Prove that there exists a circle d orthogonal to all three sides a, b, and c of $\triangle$.

(b) Prove that any composition of reflections with respect to a, b, and c leaves the circle d invariant.

(c) Prove that the image of triangle $\triangle$ under any composition of reflections with respect to its sides lies inside d.

8.4.6. Let $\triangle$ be a curvilinear triangle with the sum of its angles less than $180°$. Prove that the union of its images under all possible compositions of reflections with respect to its sides will not cover the entire plane.

8.4.7. Let $\triangle$ be a curvilinear triangle with angle sum $180°$.

(a) Prove that the three circles containing the sides of this triangle have a common point.

(b) Prove that the plane cannot be covered by a *finite* number of this triangle's images under compositions of reflections with respect to its sides.

(c) Prove that *all* of the triangle's images under compositions of reflections with respect to its sides cover the plane (except for one point).

Define a *bisector* of two intersecting circles to be a circle passing through both their points of intersection which divides one of the angles between them into equal angles.

8.4.8. Prove that three bisectors of a curvilinear triangle with angle sum $180°$ intersect at one point.

8.4.9.° How many bisectors does a given pair of intersecting circles have?
1) 1; 2) 2; 3) 3; 4) infinitely many.

8.4.10. Let $\triangle$ be an eighth of a sphere (i.e., a section of unit sphere contained in one coordinate octant). What will its image be (on the sphere) under reflections with respect to the planes that contain the sides of the "triangle" $\triangle$? Compare the result with the answer to Problem 8.4.3.

8.4.11. Let $\triangle$ be a curvilinear triangle whose angle sum exceeds $180°$, and let a, b, and c be its sides.

(a) Prove that under stereographic projection onto a sphere tangent to the plane at the radical center of the three circles a, b, and c, these circles transform into great circles of the sphere.

(b) Prove that the plane is covered by a finite number of images of triangle $\triangle$ under compositions of reflections about its sides.

8.4.12. Prove that there exist three bisectors of a curvilinear triangle with angle sum exceeding $180°$ that have a common point.

Additional problems

8.4.13.* Define the *height* h_a of a curvilinear triangle to be a circle passing through *both* intersection points of circles b and c which is perpendicular to circle a. Likewise, we define the other two heights h_b and h_c. (We exclude from consideration curvilinear triangles with zero angles or two right angles.) Prove that if two heights of a curved triangle intersect at some point, then the third height passes through this point.

8.4.14.* Let the heights h_a, h_b, and h_c of a curvilinear triangle intersect arcs a, b, and c at points H_a, H_b, and H_c, respectively. We call the circle passing through the points H_a, H_b, and H_c a *nine-point circle*. Let M_a, M_b, and M_c be the second intersection points of the nine-point circle with circles a, b, and c respectively. Define a *median* m_a of a curvilinear triangle to be the circle passing through both intersection points of circles b and c and M_a. Define the medians m_b and m_c in the same way. Prove that if two medians of a curvilinear triangle intersect at some point, then the third median also passes through this point.

8.4.15.* Define the *inscribed* circle of a curvilinear triangle to be the circle that is externally tangent to all three circles a, b, c. Let G_a, G_b, G_c be tangent points of an inscribed circle with circles a, b, c respectively. Draw a circle g_a through the point G_a and both intersection points of circles b and c. Define circles g_b and g_c similarly. Prove that if two of the circles g_a, g_b, g_c intersect at some point, then the third one passes through this point.

8.4.16.* Is it true that the nine-point circle (see Problem 8.4.14) is tangent to the inscribed circle (see Problem 8.4.15)?

Investigate other Euclidean theorems; which remain valid for curvilinear triangles, and which do not?

Different elementary introductions to non-Euclidean geometry can be found in [**Sab84**, **SU84**, **Bol91**].

Suggestions, solutions, and answers

8.4.3. See Fig. 20.

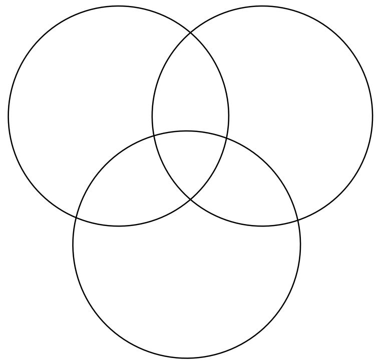


FIGURE 20

8.4.4. Note that the circumscribed circle d of a regular curvilinear triangle with zero angles is perpendicular to the circles a, b, and c. Consequently, the interior of the circle bounded by d remains fixed upon inversion with respect to any of the circles a, b, and c. This means that it remains fixed under any composition of these inversions. Since the original curvilinear triangle lies inside the circle d, the same is true for its image under such compositions.

8.4.13. See [**Sko07**, theorem about heights of curvilinear triangle].

Additional reading

A comprehensive introduction to geometric optimization problems can be found in [**Pro05**]. For more information about optimization problems involving area and perimeter see [**Gas85**, **Tr85**]. Other interesting problems involving area can be found in [**Pra86**, **To84**]. More results on conics obtained by the elementary methods introduced in this chapter can be found in [**AkZa07a**] and [**Ben86**]. Elementary introductions to non-Euclidean geometry (different from that given in this chapter) can be found in [**Sab84**, **SU84**, **Bol91**].

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This book is a translation from Russian of Part II of the book *Mathematics Through Problems: From Olympiads and Math Circles to Profession*. Part I, *Algebra*, was recently published in the same series. Part III, *Combinatorics*, will be published soon.

The main goal of this book is to develop important parts of mathematics through problems. The authors tried to put together sequences of problems that allow high school students (and some undergraduates) with strong interest in mathematics to discover and recreate much of elementary mathematics and start edging into more sophisticated topics such as projective and affine geometry, solid geometry, and so on, thus building a bridge between standard high school exercises and more intricate notions in geometry.

Definitions and/or references for material that is not standard in the school curriculum are included. To help students that might be unfamiliar with new material, problems are carefully arranged to provide gradual introduction into each subject. Problems are often accompanied by hints and/or complete solutions.

The book is based on classes taught by the authors at different times at the Independent University of Moscow, at a number of Moscow schools and math circles, and at various summer schools. It can be used by high school students and undergraduates, their teachers, and organizers of summer camps and math circles.

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